From magnetically twisted to hyperkähler

Symplectic Zoominar

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26. January 2024

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Riemannian manifold (N, g)

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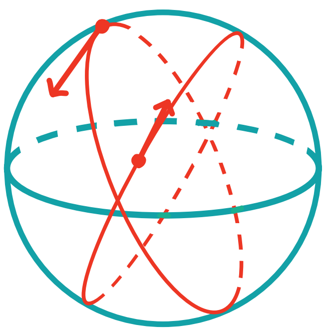

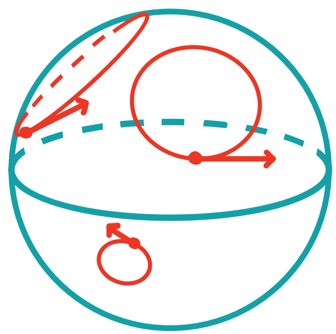

Hyperkähler Structure

* (M, G) Riemannian manifold
+ I, J, K integrable complex structures, that are Kähler w.r.t. G and satisfy $I^2 = J^2 = K^2 = IJK = -1$

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Let (N, g, j) be a real-analytic Kähler manifold. Then there exists a hyperkähler metric in a neighborhood of the zero section of the tangent bundle (TN, G, I, J, K).

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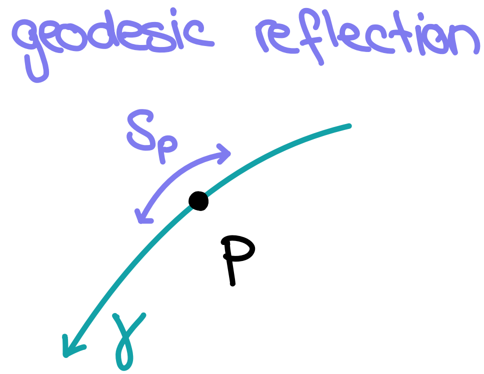

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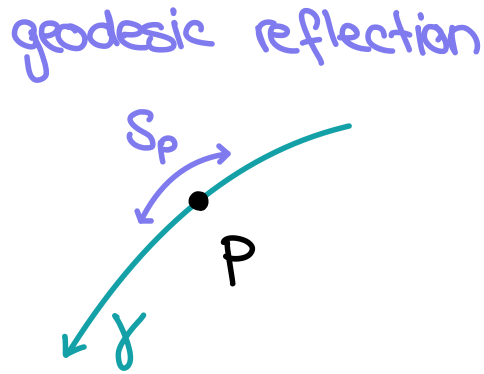

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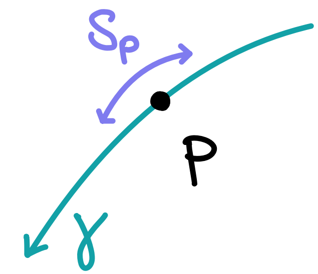

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Let N be an Hermitian symmetric space, then there is a unique G-invariant hyperkähler metric on

$$UN := \{(x,v) \in TN \mid g(jv, R(w, jw)v) \leq \|w\|^2 \ \forall w \in T_xN\}.$$

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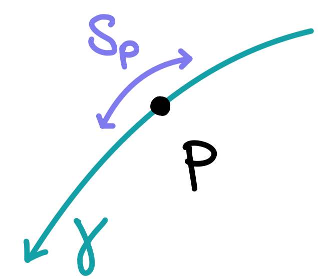

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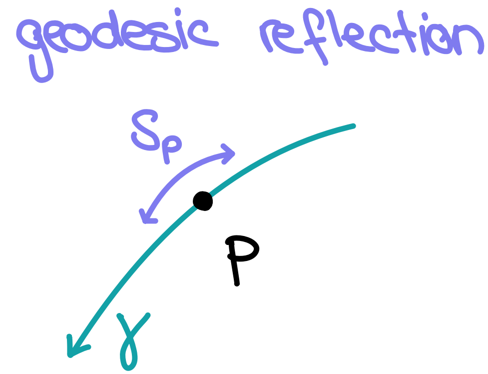

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with $\upsilon(x,v) = g_x(F(jR_{jv,v})v, v)$, $F(y) = \frac{1}{y}\left(\sqrt{1+y} - 1 - \ln \frac{1+\sqrt{1+y}}{2}\right)$.

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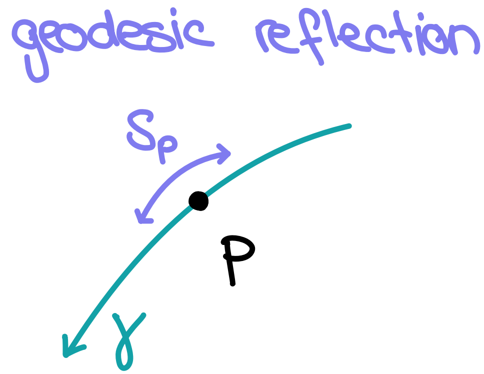

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$jR_{jv,v} : T_xN \to T_xN; \quad w \mapsto jR(jv,v)w \quad$ self-adjoint

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Lagrangian fibration

symplectic fibration

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Lemma: If $\psi : UN \to UN$ is a smooth equivariant bijection, s.t.

$$\begin{tikzcd} UN \arrow[rr, "\psi"] \arrow[dr, "\mu_1"'] & & UN \arrow[dl, "\mu_2"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes, then ψ is a symplectomorphism.

Polyspheres / Polydiscs

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Fact: Denote d the rank of N. Then every two points lie on a copy of $\overbrace{\mathbb{CP}^1 \times \ldots \times \mathbb{CP}^1}^{d}$ resp. $\overbrace{\mathbb{CH}^1 \times \ldots \times \mathbb{CH}^1}^{d}$.

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* Use symmetry to reduce to polysphere / polydisc

compact type:

$$\begin{tikzcd}
(T\mathbb{CP}^1)^d \arrow[rrr, hook, "d\iota"] \arrow[dr, "\Sigma_i \mu_1^i"] \arrow[dd, "\varphi \times \ldots \times \varphi"'] & & & TN \arrow[dl, "\mu_1"'] \arrow[dd, "\tau"] \\
& SU(2)^d \arrow[r, hook, "affine"'] & \mathfrak{g} & \\
(T\mathbb{CP}^1)^d \arrow[rrr, hook, "d\iota"] \arrow[ur, "\Sigma_i \mu_2^i"'] & & & TN \arrow[ul, "\mu_2"]
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Polyspheres / Polydiscs

Fact: Denote d the rank of N. Then every two points lie on a copy of $\underbrace{\mathbb{C}P^1 \times \dots \times \mathbb{C}P^1}_{d}$ resp. $\underbrace{\mathbb{C}H^1 \times \dots \times \mathbb{C}H^1}_{d}$. This copy is a totally geodesic, complex submanifold.

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non-compact type:

$$\begin{tikzcd}
(D\mathbb{C}H^1)^d \arrow[rrrr, hook, "d\iota"] \arrow[ddd, "\varphi \times \dots \times \varphi"'] \arrow[drr, "\Sigma_i \mu_1^i"] & & & & UN \arrow[dl, "\mu_1"'] \arrow[ddd, "\tau"] \\
& & \mathfrak{sl}(2,\mathbb{R})^d \arrow[r, hook, "\text{affine}"'] & \mathfrak{g} & \\
& & & & \\
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$\mathbb{C}P^1$

$\mathbb{R}^3 \cong SU(2)$

$\cdot x \cdot \leftrightarrow [\cdot \, ; \cdot]$

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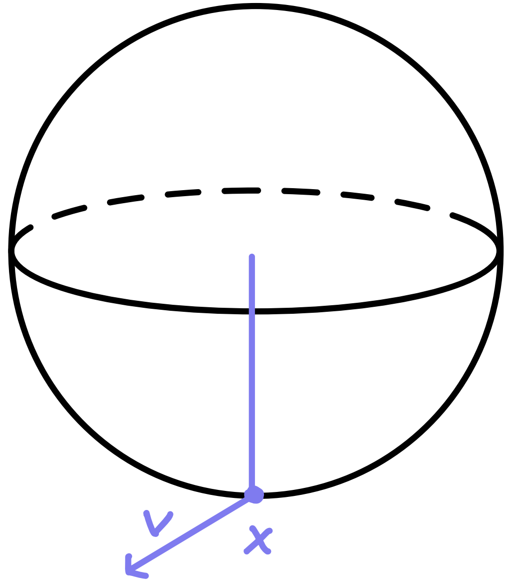

$\mathbb{C}P^1$

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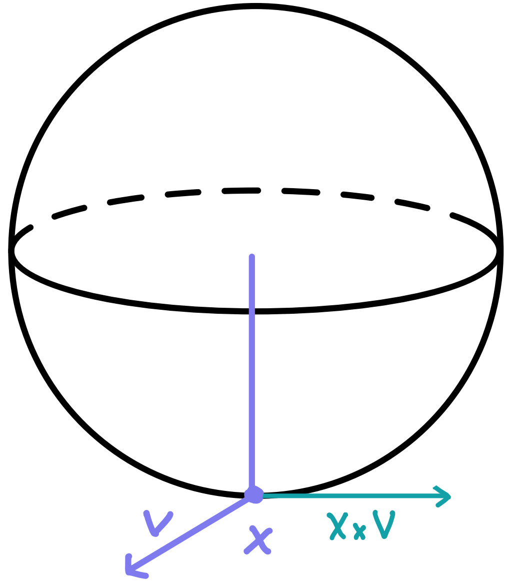

moment maps:

$$\mu_1(x,v) = x \times v + x$$

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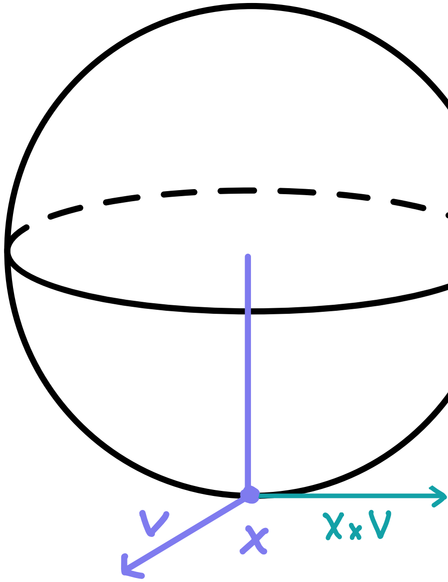

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$$\overset{!}{=}$$

$$\mu_2(\underbrace{\varphi(x,v)}_{(y,w)}) = \sqrt{1+|w|^2}\, y$$

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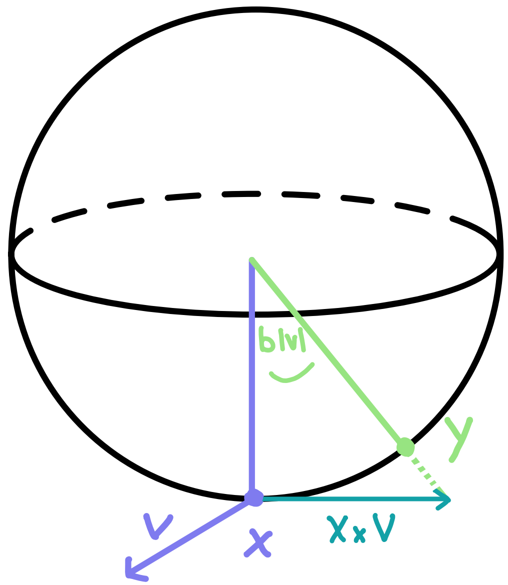

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$$\overset{!}{\parallel}$$

$$\mu_2(\underbrace{\varphi(x,v)}_{(y,w)}) = \sqrt{1+|w|^2}\; y$$

$$\Rightarrow \varphi(x,v) = \left(\exp_x(-jb(|v|^2)v),\, P_\gamma v(1)\right); \qquad b(|v|^2)|v| = \arctan(|v|)$$

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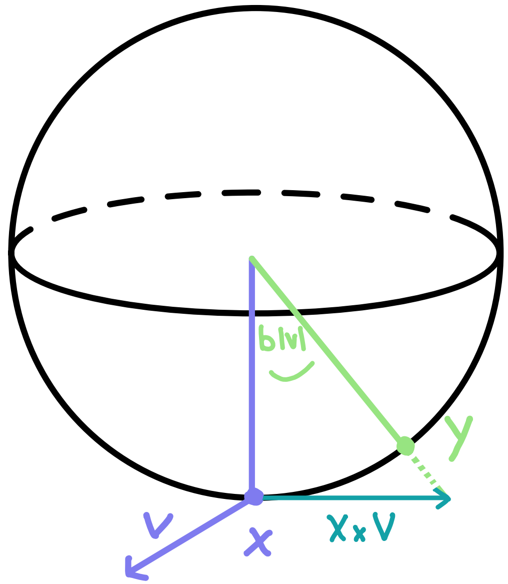

moment maps :

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$$\parallel !$$

$$\mu_2(\underbrace{\varphi(x,v)}_{(y,\omega)}) = \sqrt{1+|\omega|^2}\, y$$

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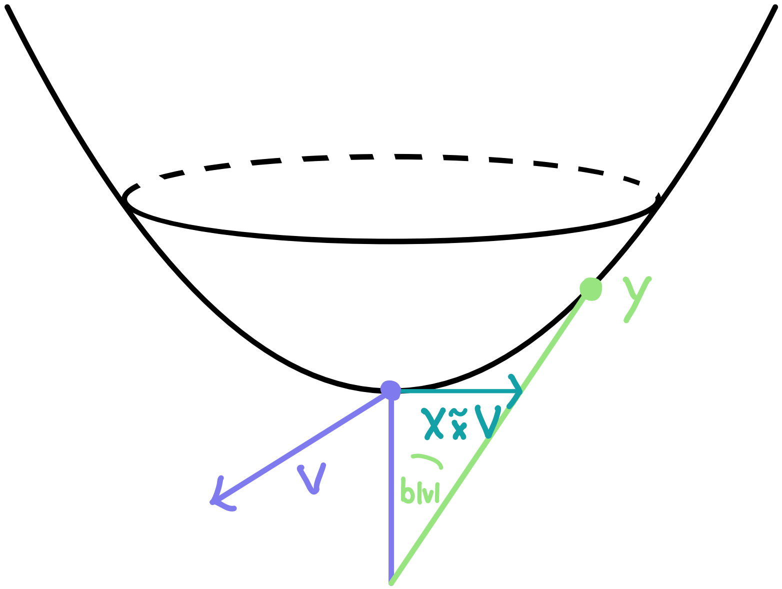
$$\Rightarrow \psi(x,v) := \left(\exp_x(-b(j R_{jv,v}) jv),\ P_\gamma v(1)\right)$$

$\mathbb{C}H^1$

$\mathbb{R}^{2,1} \cong \mathfrak{sl}(2,\mathbb{R})$

$\cdot \tilde{\times} \cdot \leftrightarrow [\cdot,\cdot]$



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$$\|$$

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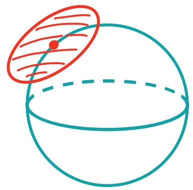
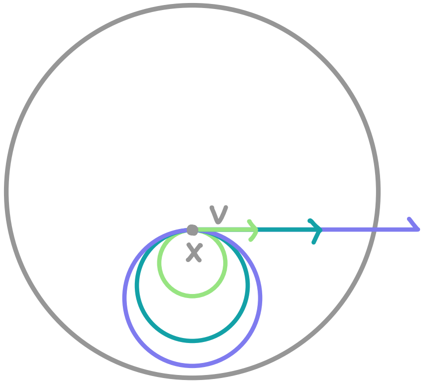
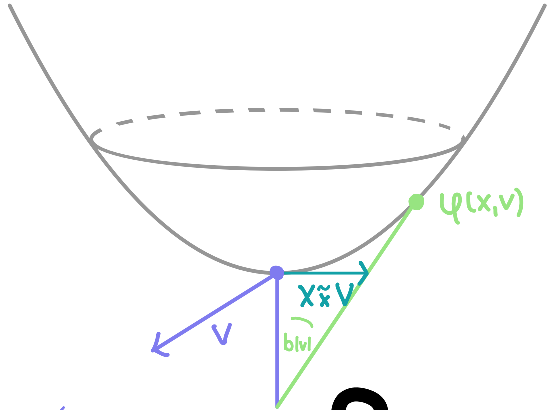
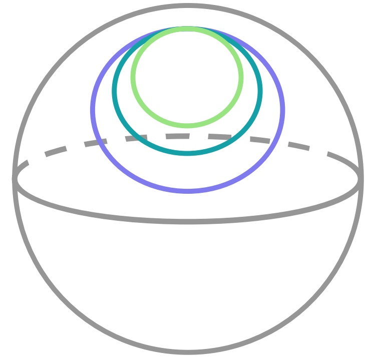
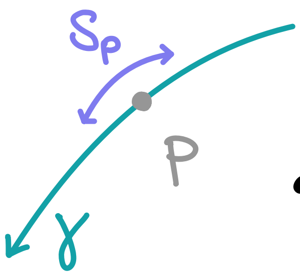
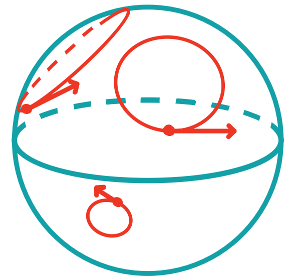




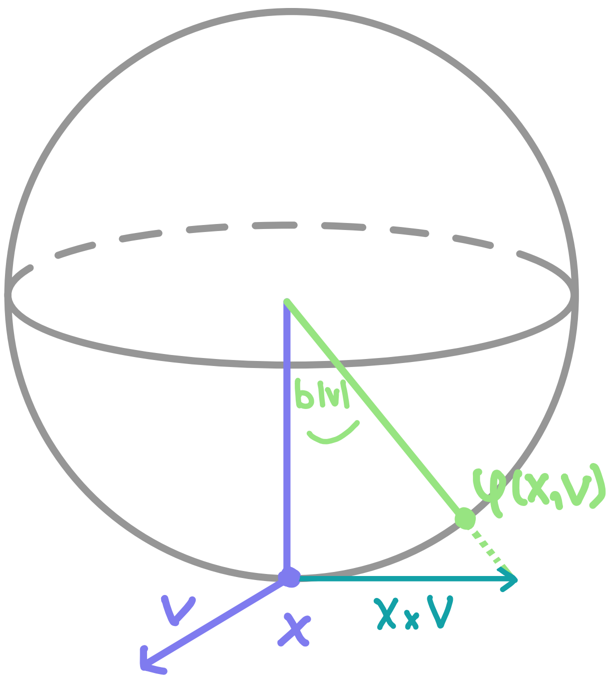


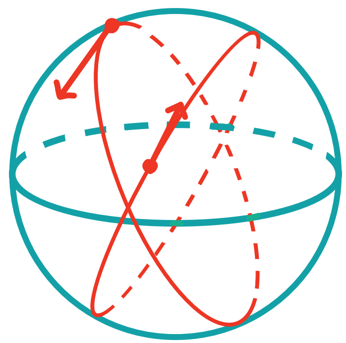


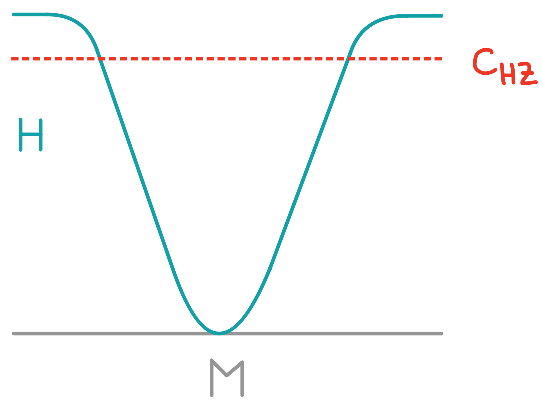




Thank you for listening!!!

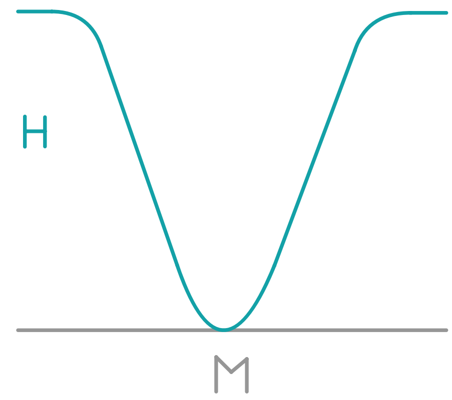






Hofer-Zehnder capacity

$$c_{HZ}(M,\omega) := \sup\{\operatorname{osc} H \mid H \in C^\infty(M,\mathbb{R}) \text{ nice}\}$$



where nice means:

* H constantly attains its maximum near ∂M
* $\nexists$ non-constant periodic orbit with $T \leq 1$

Corollary

Further, denote by d the rank of N, then

compact type

$$\pm 2\pi\left(\sqrt{1 \pm r^2/d} - 1\right) \leq c_{HZ}(D_r N, d\lambda - \pi^*\sigma) \leq \pm 2\pi d\left(\sqrt{1 \pm r^2} - 1\right)$$

non-compact type

for any constant $r > 0$ satisfying $1 \pm r^2 > 0$.