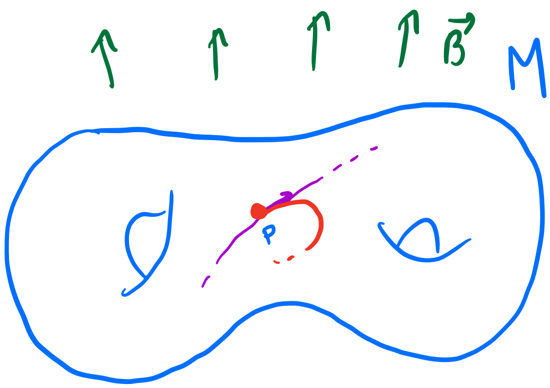
ON THE GEOMETRY OF MAGNETIC FLOWS.

(Valerio Assenza - Impa)

Motivations



Definition A magnetic system is the data of (M, g, σ)

- (M, g) is a Riem. Mfd
- $\sigma \in \Omega^2(M)$, $d\sigma = 0$ MAGNETIC FORM

$\Omega : TM \longrightarrow TM$ (LORENTZ) compatible with g and σ

$$g(\Omega(v), w) = \sigma(v, w) \quad \forall v, w \in TM$$

(Ω is skewsymmetric.)

Equation (Newton).

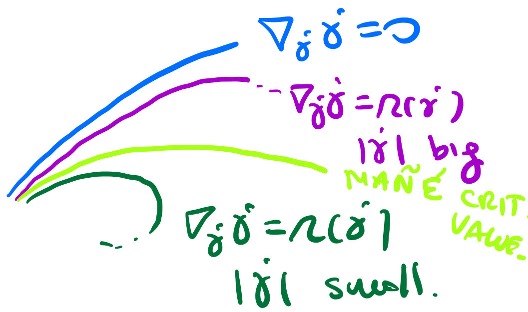
$$\nabla_{\dot\gamma}\dot\gamma = \Omega(\dot\gamma) \quad (\star)$$

$\gamma : \mathbb{R} \longrightarrow M$ solution of $(\star)$ is called MAGNETIC GEODESIC

Rk. The energy $E(\gamma,\dot\gamma) = \frac{1}{2}|\dot\gamma|^2 =$ CONST.

- $\varphi^{(g,\sigma)}$ MAGNETIC FLOWS (by lifting $(\star)$ on TM)
- $\Sigma_s = \{ \frac{1}{2}|v|^2 = s \}$ φ-INVARIANT.
  $s \in (0,+\infty)$

$$\varphi_s^{(g,\sigma)} = \varphi^{(g,\sigma)}\Big|_{\Sigma_s}$$



Q How the shape of M change under the magnetic action?

Idea

Riem. Structure + terms of perturbation due to the magn. wt.

MAGNETIC CURVATURE

$\in (0,+\infty)$, $v \perp w$ $(|v| = |w| = 1)$

$$M^{\Omega}(v,w) = R(w,v)v - \sqrt{}(\nabla_w \Omega)(v) + A^{\Omega}(v,w)$$

RK comes by looking liner. problem (Jacobi equation) | second variation of the ACTION (VAR. SETTING).

- $Sec^{\Omega}(v, \quad) := g(M^{\Omega}(v,w), w)$. Sectional Magnetic curvature.
- $Ric^{\Omega}(v,) := \mathrm{trace}(M^{\Omega}(v,-))$. Ricci magnetic curvature.
- $Scal^{\Omega}(p) = \int_{S_pM} Ric^{\Omega}(v)\,dv$. Scalar magnetic curvature.

M is a closed manif. $\sigma = b \cdot \mathrm{vol}(g)$ $b \in C^{\infty}(M)$

$$Sec^{\Omega} = Ric^{\Omega}$$

- $$K^{b} = K - db(i \cdot v) + b^{2}$$ Gaussian Magnetic Curvature.

POSITIVELY CURVED MAGNETIC SYSTEMS.

- σ is symplectic

  $\exists\, s_0 > 0 \mid \mathrm{Sec}^{\Omega}_s > 0 \quad \forall s \in (0, s_0)$

- σ is nowhere vanishing ($\sigma_p \neq 0 \ \forall p \in M$)

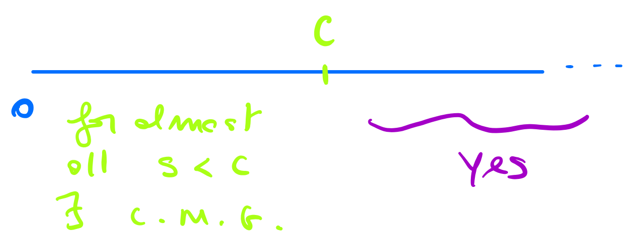
  $\exists\, s_0 > 0 \mid \mathrm{Ric}^{\Omega}_s > 0 \quad \forall s \in (0, s_0)$

SYMPLECTIC MAGNETIC SYSTEMS ON SMALL ENERGY $\sim$ POSITIVELY CURVED MANIFOLDS.

NOWHERE VANISH. MAGNETIC SYSTEMS ON SMALL ENERGY $\sim$ POSITIVELY RICCI-CURVED MANIFOLDS.

APPLICATION: CLOSED MAGNETIC GEODESIC.

Q Given $s \in (0, +\infty)$, Does Σ_s carries a periodic orbit?



Theorem: Let $< c$ such that $\mathrm{Ric}^{\Omega} > 0$. Then Σ carries a contractible closed magnetic geodesic.

Idea

MAGN. BONNET - MYERS

$\mathrm{Ric}^{\Omega}_{s} > 0$ $\Longrightarrow$ Recover P-S for Action.

$\ell(\gamma) < C \cdot \mathrm{index}(\gamma)$

—

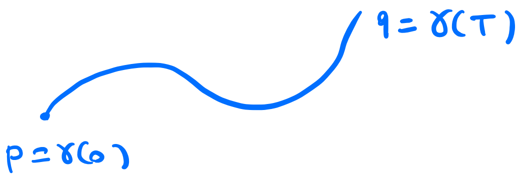
Corollary: σ is nowhere vanishing. Then Σ_s carries a c. c. m. g for EVERY s small.

APPLICATION (2): MAGNETIC FLOWS WITHOUT CONJ POINTS.

γ magnetic geodesic $\quad (\nabla_{\dot\gamma} = \text{" } \cdot \text{ "})$

Jacobi:

$$\ddot J + R(J,\dot\gamma)\dot\gamma - \nabla_J \Omega(\dot\gamma) - \Omega(\dot J) = 0 \quad (**)$$



q is conj to p if
$\exists\, J \not\equiv 0$ solution of $(**)$
s.t.
$J(0) = 0,\ \dot J(0) \perp \dot\gamma$
$J(T) \parallel \dot\gamma$.

Remark: If $\mathrm{Sec}^{\Omega}_{s} < 0 \Rightarrow \varphi^{(g,\sigma)}_{s}$ is without conj. points.

Back in dim 2:

Theorem $\varphi^{(g,b)}$ is without conjugate points then

$$\int_{SM} K^b \leq 0$$

with "=" iff either $M = \mathbb{T}^2$, g is flat and $b=0$

or genus$(M) > 1$, K and b const.

and $= c(g,b)$ Mañé crit.

Last comment:

By Gauss Bonnet:

$$\int_{SM} K^b = 4\pi^2 \left(2 \ (2-2h) + \frac{1}{2} \int_M b^2 \right)$$