Symplectic capacities of domains close to the ball & Banach-Mazur geodesics in the space of contact forms

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Symplectic Zoominar - February 16, 2024

A **normalized symplectic capacity** on $(\mathbb{C}^n, \omega_0)$ is a function

$$c : \mathcal{P}(\mathbb{C}^n) \longrightarrow [0, +\infty]$$

s.t.

- If $\exists\ \varphi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0)$ s.t. $\varphi(X) \subset Y$ then $c(X) \leq c(Y)$
- $c(rX) = r^2 c(X) \quad \forall r > 0$
- $c(B) = c(Z) = \pi$

$$B := \{ z \in \mathbb{C}^n \mid |z| < 1 \}$$

$$Z := \{ z \in \mathbb{C}^n \mid |z_1| < 1 \}$$

Ball capacity

$$c_B(X) := \sup\{\pi r^2 \mid \exists\, \varphi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0) \text{ s.t. } \varphi(rB) \subset X\}$$

Cylindrical capacities

$$c_Z(X) := \inf\{\pi r^2 \mid \exists\, \varphi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0) \text{ s.t. } \varphi(X) \subset rZ\}$$

- Any normalized capacity c satisfies

$$c_B \leq c \leq c_Z$$

Open question: Do all normalized capacities coincide on bounded convex domains?

- Many spectral normalized capacities have be shown to coincide with the systole of X

$$\mathrm{sys}(X) := \left\{ \begin{array}{l} \text{minimal action closed} \\ \text{characteristic on } \partial X \end{array} \right\}$$

when X is a smooth bounded convex domain

Viterbo's conjecture: If c is normalized capacity and X bounded convex domain then

$$c(X)^n \leq n!\, \mathrm{vol}(X) \quad (*)$$

with equality iff X symplectomorphic to Euclidean ball.

- $(*)$ trivially holds for c_B.

Thm 1 (A. - Benedetti - Edtmair) All normalized symplectic capacities coincide on a C^2-neighborhood of B.

If X is C^2-close to B then:

(i) $\exists\ \varphi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0)$ mapping X into cylinder of width $\mathrm{sys}(X)$

(ii) $\exists\ \varphi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0)$ mapping the ball of width $\mathrm{sys}(X)$ into X

Moreover $c(X)^n \leq n!\, \mathrm{vol}(X)$ with equality iff X symplectomorphic to a ball by symplectomorphism of $\mathbb{C}^n$.

C^2- closeness is optimal :

Thm 2 (ABE) There exists sequence X_k of smooth domains C^1- converging to B such that

$$c_B(X_k) < \operatorname{sys}(X_k) \leq c_Z(X_k)$$

Previous results:

- (A.- Bramham - Hryniewicz - Salomão)
Viterbo's conjecture for $c = sys$ and domains which are C^3-close to B in $\mathbb{C}^2$.

- (A. - Benedetti) Viterbo's conjecture for any c and domains which are C^3-close to B in $\mathbb{C}^n$.

- (Edtmair) Equality of all normalized capacities for domains which are C^3-close to B in $\mathbb{C}^2$.

Plan of this talk

1. Sketch the proof of Thm 2 and Thm 1(i) ($c_Z = sys$)

2. Sketch the proof of Thm 1(ii) ($c_B = sys$)

3. Relate part 2 to geodesics in the space of contact forms with Banach-Mazur pseudometric

Compactly supported Hamiltonian diffeomorphisms of $(\mathbb{C}^n, \omega_0)$

$H \in C_c^\infty(\mathbb{T} \times \mathbb{C}^n) \qquad \mathbb{T} := \mathbb{R}/\mathbb{Z}$

$\iota_{X_{H_t}} \omega_0 = dH_t \quad , \quad \phi_H^t$ flow of X_H

$\mathrm{Ham}(\mathbb{C}^n, \omega_0) := \{ \varphi \mid \varphi = \phi_H^1 \text{ for some } H \}$

- Action of $z \in \mathrm{Fix}\,\varphi$:

$$A_\varphi(z) := \int_{t \mapsto \phi_H^t(z)} \lambda_0 + \int_{\mathbb{T}} H_t(\phi_H^t(x))\,dt$$

- Calabi invariant of φ:

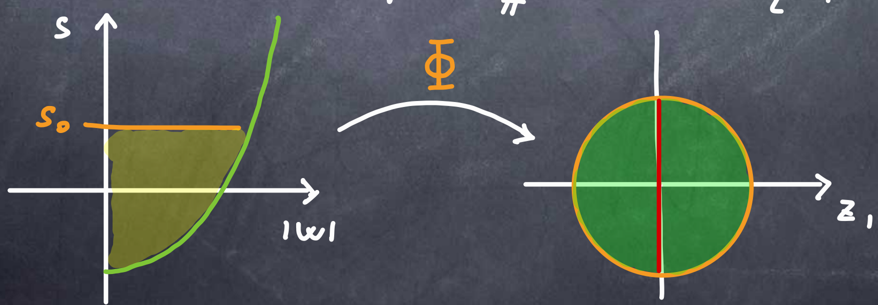
$$\mathrm{CAL}(\varphi) := \int_{\mathbb{T} \times \mathbb{C}^n} H \, dt \wedge \omega_0^n$$

A useful symplectomorphism

$$\Omega := \{ (s,t,w) \in \mathbb{R} \times \mathbb{T} \times \mathbb{C}^{n-1} \mid s > \pi(|w|^2 - 1) \}$$

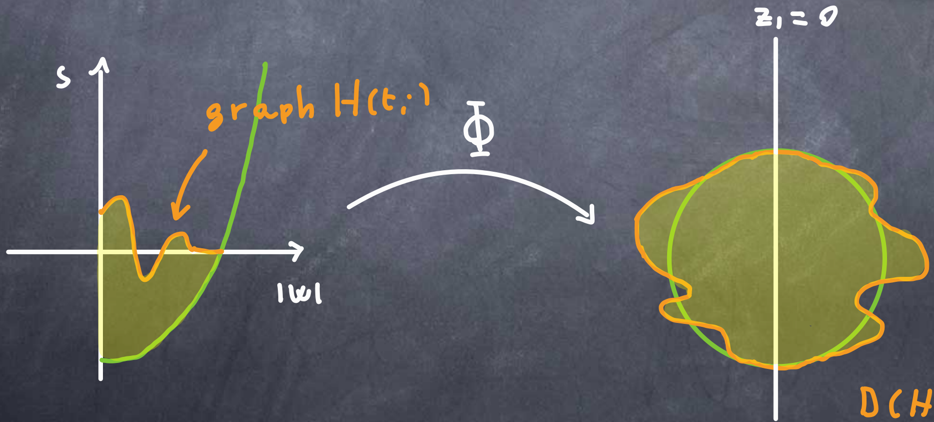
$$\Phi : \Omega \to \mathbb{C}^n \,, \quad (s,t,w) \mapsto e^{2\pi i t}\left(\sqrt{1 + \tfrac{s}{\pi} - |w|^2},\, w\right)$$

- Φ is a symplectomorphism onto $\mathbb{C}^* \times \mathbb{C}^{n-1}$ mapping $\{s < s_0\}$ to the ball of radius $\sqrt{1+\frac{s_0}{\pi}}$ minus $\{z_1 = 0\}$



From Hamiltonian diffeos to domains

- $H \in C^\infty(\mathbb{T} \times \mathbb{C}^{n-1})$ supported in $\mathbb{T} \times B^{n-1}$ and s.t. $H(t,w) > \pi(|w|^2 - 1)$ there
- $D(H) := \Phi(\{(s,t,w) \in \mathbb{R} \times \mathbb{T} \times B^{n-1} \mid s < H(t,w)\}$
  $\cup (B^n \cap \{z_1 = 0\}))$



Prop H as above, $\varphi := \phi_H^1$.

(i) $\mathrm{vol}(D(H)) = \frac{\pi^n}{n!} + \frac{1}{(n-1)!} \, \mathrm{CAL}(\varphi)$

(ii) 1-1 correspondence between periodic points of φ and closed characteristics on $\partial D(H)$ other than those in $\{z_1 = 0\}$.

$w \in \mathrm{Fix}\, \varphi^k \longleftrightarrow$ closed char. γ action

$$\int_\gamma \lambda_0 = k\pi + A_{\varphi^k}(w)$$

(iii) $\{H^\lambda\}_{\lambda \in [0,1]}$ as above s.t. $\phi^1_{H^\lambda} = \varphi \quad \forall \lambda$

$\Rightarrow \quad \exists \, \psi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0)$ s.t.

$$\psi(D(H^0)) = D(H^1)$$

proof of Thm 2

- $H \in C_c^\infty(\mathbb{T} \times B^{n-1})$ s.t. $\mathrm{CAL}(\phi_H^1) < 0$ and all fixed points of ϕ_H^1 have action ≥ 0.

- $H^\lambda(t,w) := \lambda^2 H(t, \frac{w}{\lambda}) \Rightarrow \phi^t_{H^\lambda}(w) = \lambda \phi^t_H(\frac{w}{\lambda})$

  $H^\lambda \to 0$ in C^1 (but not in C^2)

  for λ small H^λ satisfies assumptions of Prop, $\mathrm{vol}(D(H^\lambda)) < \frac{\pi^n}{n!}$,

  $\mathrm{sys}(D(H_\lambda)) = \pi$

$\Rightarrow$ $c_B(D(H_\lambda)) < \mathrm{sys}(D(H_\lambda))$

□

From contact forms to domains

$$\lambda_0 := \frac{1}{2}\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j) \qquad d\lambda_0 = \omega_0$$

$$\alpha_0 := \lambda_0|_{S^{2n-1}} \qquad \xi_{st} := \ker \alpha_0$$

$$\mathcal{F}(S^{2n-1}, \xi_{st}) := \{ f\alpha_0 \mid f > 0 \}$$ space of contact forms on (S^{2n-1}, ξ_{st})

$$\underset{\alpha}{\overset{\Psi}{}} \mapsto D(\alpha) = \{ ru \mid u \in S^{2n-1},\ 0 \leq r < f(u)^{\frac{1}{2}} \}$$

- $D(c\alpha_0)$ = ball of radius $\sqrt{c}$
- $\alpha \leq \beta \implies D(\alpha) \subset D(\beta)$

Prop $\alpha \in \mathcal{F}(S^{2n-1}, \xi_{st})$

(i) $\mathrm{vol}(D(\alpha)) = \frac{1}{n!} \int_{S^{2n-1}} \alpha \wedge d\alpha^{n-1}$

(ii) 1-1 correspondence between closed Reeb orbits of α and closed charact. on $\partial D(\alpha)$. Period = Action

(iii) $\varphi \in \mathrm{Cont}_0(S^{2n-1}, \xi_{st})$ $\varphi^*\beta = \alpha$

$\Rightarrow \exists\, \psi \in \mathrm{Symp}(\mathbb{C}^n, \omega_0)$ s.t. $\psi(D(\alpha)) = D(\beta)$.

Strategy for proving Thm 1 (ii) ($c_B = sys$):

$\forall\, \alpha \in \mathcal{F}(S^{2n-1}, \xi_{st})$ C^2-close to α_0 find

$\varphi \in \mathrm{Cont}_0(S^{2n-1}, \xi_{st})$ s.t. $\varphi^*\alpha \geq \frac{sys\,\alpha}{sys\,\alpha_0}\alpha_0$

$\Rightarrow$ $D(\alpha) \overset{s}{\cong} D(\varphi^*\alpha) \supset D\left(\frac{sys\,\alpha}{sys\,\alpha_0}\alpha_0\right)$

$\Rightarrow$ $c_B(D(\alpha)) \geq sys(D(\alpha))$

Thm 3 (ABE) ξ co-orient. contact structure on closed manifold M, $\alpha_0 \in \mathcal{F}(\xi)$ Zoll.
If $\alpha \in \mathcal{F}(\xi)$ is C^2-close to α_0 then
$\exists\, \varphi \in \mathrm{Cont}_0(M,\xi)$ s.t. $\varphi^*\alpha = f\alpha_0$
with f s.t. $f^{-1}(\min f)$ and $f^{-1}(\max f)$ contain non-empty sets which are invariant under the Reeb flow of α_0.

proof. Bottkol 1980 + [A.-Benedetti] + Moser. $\square$

- Apply to (S^{2n-1}, α_0): $\mathrm{sys}(\alpha) \leq \mathrm{sys}(\alpha_0) \cdot \min f$

$$\Rightarrow \quad \varphi^*\alpha \geqslant \min f\, \alpha_0 \geqslant \frac{\mathrm{sys}(\alpha)}{\mathrm{sys}(\alpha_0)}\, \alpha_0 \qquad \square$$

Banach-Mazur pseudo-metric on $\mathcal{F}(M,\xi)$

[after Ostrover-Polterovich, Rosen-Zhang, Stojisavljević-Zhang, Usher]

$$d(\alpha,\beta) := \inf\left\{ \max f - \min f \;\middle|\; \begin{array}{l} \exists\, \varphi \in \mathrm{Cont}_0(M,\xi) \\ \varphi^*\beta = e^f \alpha \end{array} \right\}$$

- d is a pseudo-metric on $\mathcal{F}(M,\xi)$ invariant under separate action of $\mathbb{R}^+ \times \mathrm{Cont}_0(M,\xi)$

<u>Thm 4 (ABE)</u> If inf in defn of $d(\alpha,\beta)$ is achieved by some f and φ then

(i) $\exists$ probability measure invariant under Reeb flows of both α and $\varphi^*\beta$ supported in $f^{-1}(\min f)$ (resp. $f^{-1}(\max f)$)

(ii) $\exists$ minimizing geodesic from α to β.

Thm 5 (ABE) If $\alpha \in F(M, \xi)$ is C^2-close to Zoll contact form α_0 then inf in the defn of $d(\alpha_0, \alpha)$ is achieved.
In particular, $\exists$ minimizing geodesic from α_0 to α and

$$d(\alpha, \alpha_0) = \log \frac{T_{max}(\alpha)}{T_{min}(\alpha)}$$

where $T_{max}(\alpha)$ and $T_{min}(\alpha)$ are the maximum and minimum period of "short" closed Reeb orbits of α.