Floer-theoretic corrections to the geometry of moduli spaces of Lagrangian tori

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SYZ mirror symmetry

Strominger-Yau-Zaslow (SYZ) construction of mirror to $(X, \omega)$ (Kähler) rel. anticanonical divisor $D \subset X$ ($[D] = c_1(TX)$) (log CY pair)

1. find a Lagrangian torus fibration $\pi : X - D \to B$ (w/ fibers of vanishing Maslov class, generically unobstructed, ...)

   $\to$ uncorrected mirror of $X - D$: \[ \left\{ \left( F_b, \nabla \right)/ \nabla \text{ unitary rk. 1 loc. system}/F \right\} \]

   analytic space, loc. $\sim (\mathbb{C}^*)^n$ (moduli space of Log. tori $\pi + D$ in $X - D$).

2. Floer-theoretic corrections to geometry of $X'$ from holomorphic discs in $(X, F_b)$
   - Maslov index $\mu(u) = 2[D] \cdot [u] = 0$ discs deform analytic structure of $X'$
     (fix issues w/ singular fibers of $\pi$)
   - $\mu = 2$ discs $([D], [u] = 1)$ $\to$ superpotential $W \in \mathcal{O}(X')$
     $(X', W)$ Landau-Ginzburg model.

NEW PHENOMENON

- $\mu < 0$ discs $([D], [u] < 0)$ $\to$ extended deformations - $X'$ no longer analytic space!!
The uncorrected moduli space $X'_0$

$\pi: X \to B$ Laga. form fibration

$X'_0 := \left\{ (F_b, D) \mid F_b = \pi^{-1}(b) \text{ smooth fiber of } \pi \right\}$

$\nabla \in \text{hom}(\pi_1(F_b), U(1)_K) = H^1(F_b, U(1)_K)$

analytic space / $K^*$ (away from sing. fibers):

$\{g_i\}$ basis of $H^1(F_b) \quad \Rightarrow$ local words: $z_i(f_b, D) = T^{\omega(g_i)}. \nabla(g_i) \in K^*$

Here $K = \left\{ \sum a_i T^{\lambda_i} / a_i \in k, \ \lambda_i \in \mathbb{R}, \ \lambda_i \to +\infty \right\}$ Novikov field in formal variable $T$

- Floer-theoretic weights of holomorphic discs in $(X, F_b)$: $F_b \to \bar{\Omega}$

$\beta \to T^{\omega(\beta)} \nabla(\delta\beta) \in K^*$ are monomials in $z_1, ..., z_n$

$\Rightarrow$ Laga. Floer theory has analytic dependence on $(F, D) \in X'_0$.

$\Rightarrow$ Family Floer approach to mirror symmetry (Fukaya, Abuzaid, Tu, Yuan, ...
Holomorphic discs and the Floer-theoretic obstruction

If there are no holom. discs, \( X_0^\nu = \{(F_b, D) \mid F_b = \pi^{-1}(b), D \in \text{hom}(\pi_1(F_b), U(1)_K)\} \) is a moduli space of objects of \( F(X - D) \), with its natural analytic structure.

Holomorphic discs deform the Fukaya category. Floer obstruction (F000) \( m_0 \in CF(L, L) \)
\[
M_1(X, L; J, \beta) = \left\{ u: (D^2, \partial D^2) \to (X, L) \mid \tilde{\partial}_J u = 0 \right\} / \text{Aut}(D^2, \partial D^2) \xrightarrow{\text{ev}_b} L \xrightarrow{\text{ev}_\beta} \text{ev}_\beta: u \mapsto u(1) \in L
\]
\[
m_0^{(F_b, D)} = \sum_{\beta \in \pi_2(X, F_b)} \text{ev}_A^{\beta} [\overline{M}_j(X, F_b; J, \beta)] T^w(\beta) \Delta(\beta) \in C^*(F_b; K).
\]
\( \text{ev}_A^{\beta} \) is a stable map compactification. \( \beta \) (deg = 2 - \mu(\beta)).

Expect: \( \mu = 0 \) discs (eg. disc in \( X - D \)) occur along union of (thickened) codim \( _R 1 \) walls in \( B \)
\( \to \) outside of walls, \( F_b \) is weakly unobstructed: \( \min \mu = 2, \) so \( m_0 = W(F_b, D) \cdot 1 \)

superpotential \( W(F_b, D) = \sum_{\mu(\beta) = 2} \eta_\beta \in \Omega (\nu_0^\nu), \eta_\beta = \deg(\text{ev}_\beta) \in \mathbb{Z} \).
The superpotential \( W(F_0, \theta) = \sum \eta_{\beta} z_{\beta} \) is analytic over domains delimited by the walls in \( B \), but counts of \( \mu = 2 \) discs have wall-crossing discontinuities due to bubbling of \( \mu = 0 \) discs.

\[
\begin{align*}
\eta_{\beta+k} &= 1, \quad \eta_{\beta} = 1 \\
W &= z^\beta (1 + z^k) + \ldots \\
\text{wall} \\
\eta_{\beta+k} &= 0, \quad \eta_{\beta} = 1 \\
W &= z^\beta + \ldots
\end{align*}
\]

But: formulas for \( W \) match under wall-crossing coord. changes \( (\phi : z^{\beta} \mapsto z^{\beta} (1 + z^k + \ldots)) \)

\( \rightarrow \) Corrected mirror \( X^\nu = \text{reglue local pieces of } X^{\nu_0} \text{ via these coord. changes; } W \in O(X^\nu) \text{ globally.} \)

Consistency of wall-crossing transformations (cocycle condition) \( \Rightarrow X^\nu \) well-defined analytic space

Kontsevich-Soibelman algorithm to construct consistent scattering diagram of walls (Gross-Siebert)

\( \ldots \) but consistency fails when there are \( \mu < 0 \) discs !!!
Main example

\[ X = \text{blowup of toric 4-fold } \mathbb{C}^2 \times K_{\mathbb{P}^1}, \text{ along } \{1\} \times L_0 \cup \{1\} \times \mathbb{C} \times L_{\infty} \]

\[ D = \text{proper transform of } U \text{ toric strata} \]

\[ \pi: X \setminus D \rightarrow B \cong \mathbb{R}_+^2 \times \text{int}(\Delta) \]

\[ \text{fibers away from exc. divisors } E_0 \cup E_{\infty} = \text{lifts of product tori in } \mathbb{C}^2 \times K_{\mathbb{P}^1} \]

For \( \mathbb{C}^2 \times K_{\mathbb{P}^1} \), the mirror is \( (K^*_{\mathbb{P}^1})^4 \), \( W = z_1 + z_2 + (1 + q^2 + q z_3 + q^{-1} z_3^{-1}) z_4 \)

Toric \( \mu=2 \) discs along coordinate axes, intersecting one toric divisor

+ \( q^2 z_4 \) term: disc \( S \leq W \neq 0 \text{ in } K_{\mathbb{P}^1} \).
\[ X = \text{Bl}(\mathbb{C}^2 \times K_{P^1}, C \times L_0 \cup 1 \times C \times L_\infty) \]
\[ \pi: X \dashrightarrow \mathbb{B} \cong \mathbb{R}^+ \times \operatorname{int}(\Delta) \]

with singular fibers over

fibers away from exc. divisors \( E_0 \cup E_\infty \) lifts of product tori in \( \mathbb{C}^2 \times K_{P^1} \)

Wall-crossing occurs as radii \( r_1, r_2 \) in \( \mathbb{C}^2 \) through 1: new \( \mu=2 \) disc in \( X \):

- proper transforms of \( \mu \geq 4 \) disc in \( \mathbb{C}^2 \times K_{P^1} \) through \( C \times 1 \times L_0 (r_2 > 1) \) or \( 1 \times C \times L_\infty (r_i > 1) \).

\[ r_1 < 1 \quad r_1 > 1 \]

\[ W_{-+} = z_1 + z_2 (1 + q q' z_4 + q' z_3 z_4) + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

\[ W_{++} = z_1 (1 + q q'' z_4 + q'' z_3 z_4) + z_2 (1 + q q' z_4 + q' z_3 z_4) + q' q'' z_1 z_2 z_4 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

\[ r_2 > 1 \]

\[ W_{--} = z_1 + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

\[ W_{+-} = z_1 (1 + q q'' z_4 + q'' z_3 z_4) + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

\[ W_{--} = z_1 + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

\[ (x_2, x_3) \text{ disc st. } x_2 = 1 @ x_3 = 0 \]

union \[ S_{x_3 = 1} \uparrow \]

\[ (x_2, x_3, x_4) \text{ disc st. } x_2 = 1 @ L_0. \]

\[ (x_3, x_4) \text{ disc st. } (x_3, x_4) = (1, 1) @ x_4 = 0 \]

union \[ S_{x_3 = 1} \uparrow \]

\[ (x_3, x_4, x_5) \text{ disc st. } (x_3, x_4) = (1, 1) @ x_4 = 0 \]
The mirror of $X = Bl(C^2 \times K_{P^1}, C \times L_0 \cup 1 \times C \times L_{\infty})$

Wall crossing at $r_1 = 1$
\[ r_2 \geq 1 \]
\[ w_{-+} = z_1 + z_2 (1 + q q' z_4 + q' z_3 z_4) + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]
\[ \varphi_{-0}: z_2 \mapsto z_2 (1 + q q' z_4 + q' z_3 z_4) \]

$W_{+-} = z_1 (1 + q q'' z_4 + q'' z_3 z_4 + q' q'' z_2 z_4) + z_2 (1 + q q' z_4 + q' z_3 z_4) + q q'' z_1 z_2 z_4 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$

$r_2 < 1$
\[ W_{--} = z_1 + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]
\[ \varphi_{0-}: z_1 \mapsto z_1 (1 + q q' z_4 + q' z_3 z_4) \]

$r_1 < 1$
\[ \varphi_{0+}: z_2 \mapsto z_2 (1 + q q' z_4 + q' z_3 z_4 + q' q'' z_1 z_4) \]

$r_1 > 1$
\[ W_{+-} = z_1 (1 + q q'' z_4 + q'' z_3 z_4) + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4 \]

$\varphi_{+0} \circ \varphi_{0-} \neq \varphi_{0+} \circ \varphi_{-0}$: the wall-crossing diagram is inconsistent!!

This is caused by $\mu = -2$ stable disc at $(r_1, r_2) = (1, 1)$: $z_4$-disc $\cup \\hat{S}_{(x_1, x_2)} = (1, 1)$ (weight $q' q'' z_4$).
$X^\vee$ as a deformed Landau-Ginzburg model.

The mirror constructed above is consistent mod $q'q''$. The extra terms can be viewed as a deformation of (alg. geom of) $(X^\vee, W)$ by a class in

$$H^*(MF(X^\vee, W)) = H^*(X^\vee, (\Lambda^*T_{X^\vee}, dz))$$

determined by $\mu = -2$ discs: leading term $w^{(2)} \in \mathcal{C}^2(X^\vee, \Lambda^2 T_{X^\vee})$ takes value

$$w^{(2)}_{00} = q'q''z_4 \partial \log z_1 \wedge \partial \log z_2$$
on overlap of coord. charts around $(r_1, r_2) = (1, 1)$.

(complete to a cocycle by adding $w^{(1)}_{+0} = q'q''z_1 z_4 \partial \log z_2$

$$w^{(1)}_{0+} = q'q''z_2 z_4 \partial \log z_1$$

$$w^{(1)}_{++} = q'q''z_1 z_2 z_4$$

$$\Rightarrow 2dz(w^{(2)}) = 8w^{(1)}$$

$$2dz(w^{(1)}) = 8w^{(0)}$$

$\rightarrow X^\vee$ deformed Landau-Ginzburg model: $W$ loc. analytic, but coord. charts fail to satisfy cocycle condition by $1dz(w^{(2)})$. 
A family Floer approach to correcting the mirror

\[ \pi : X \to B \text{ Lagrangian fibration, } \chi^{v_0} = \{(F_b, D) \mid F_b = \pi^{-1}(b) \text{ smooth fiber} \} \]

\[ \pi^* \mathcal{O} = \mathcal{O}_{\text{an sheaf on } B^0} \]

\[ B^0 \text{ (smooth part)} \quad \text{uncorrected mirror} \]

completion of \[ K[H_1(F_b)] = K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \text{ (\exists monomials } z^\beta \text{ = disc weights)} \]

Lagrangian Floer theory for discs in \((X, F_b) \forall b \in B^0\), with universal weights \( \mathcal{O}_{\text{an}} \)

\[ \Rightarrow \text{curved Aoo-operations on } C^\bullet(B^0; C^\bullet(F_b) \otimes \mathcal{O}_{\text{an}}). \]

\[ m_0 = \sum_{\beta \in \pi_2(X, F_b)} e_{\chi, \mathcal{J}}[\mathcal{M}(X, F_b; \mathcal{J}, \beta)] z^\beta \in C^\bullet(B^0; C^\bullet(F_b) \otimes \mathcal{O}_{\text{an}}) \]

(deg = 2 - \mu(\beta)).

Expect:

1) \( \exists \alpha^{(i)} \in C^\bullet(B^0; H^i(F_b) \otimes \mathcal{O}_{\text{an}}) \) st. \( m_0 = \alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} + \ldots \)

\( \text{codim } i \text{ walls of } \mu = 2 - 2i \text{ discs} \)

2) \( m_0 \) satisfies master equation \( \delta m_0 + \frac{1}{2} \{m_0, m_0\} = 0 \)

\( \text{deg } = -1 \) bracket
The master equation & mirror geometry

\[ m_0 = \alpha^{(0)} + \alpha^{(1)} + \ldots \in \bigoplus_{i \geq 0} C^i(B^0; H^i(F_b) \hat{\otimes} \mathcal{O}_{an}) \]

\[ \alpha^{(i)} \text{ records codim. } i \text{ walls of } \mu = 2 - 2i \text{ discs} \]

on \( H^*(F_b) \hat{\otimes} \mathbb{K}[H_1(F_b)] : \{ z \alpha, z \alpha' \} = z^{\delta+\delta'} (\alpha \wedge \gamma \alpha' + (-1)^{|\alpha|} \gamma' \alpha \wedge \alpha') \quad (\text{deg. } -1) \)

(\( \cong H_*(ZF_b), \text{ Chas-Sullivan string bracket} \))

\[ \forall \alpha, \alpha' \in H^*(F_b) \quad \delta, \gamma' \in H_*(F_b). \]

\[ \rightsquigarrow \text{ extend (w/ cup-product) to } \{ \ldots \} \text{ on } C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an}). \]

- Using \( H^*(F_b, \mathbb{R}) \cong T_b B, \) get \( H^*(F_b) \hat{\otimes} \mathcal{O}_{an} \rightsquigarrow \pi_*^{\vee} (\wedge^* T_{X_0}) \)

\[ z^{\delta} \gamma_{j_1}^{*} \ldots \gamma_{j_k}^{*} \mapsto z^{\delta} \partial \log z_{j_1} \wedge \ldots \wedge \partial \log z_{j_k} \]

under which \( m_0 = \alpha^{(0)} + \alpha^{(1)} + \ldots \mapsto W = W^{(0)} + W^{(1)} + \ldots \in \bigoplus_{i \geq 0} C^i(X^{1,0}, \wedge^i T_{X_0}) \)

\[ \{ \ldots \} \mapsto \text{Schouten-Nijenhuis bracket} \]

\[ \delta m_0 + \frac{1}{2} \{ m_0, m_0 \} = 0 \Rightarrow \delta W + \frac{1}{2} \{ W, W \} = 0 \]
The geometry of the master equation

\[ W = W^{(0)} + W^{(1)} + \ldots \in \oplus_{i \geq 0} C^i(X^{(0)}, \Lambda^i T_{X^{(0)}}) \quad \text{Schouten-Nijenhuis bracket} \]

\[ \delta W + \frac{1}{2} [W, W] = 0 \iff (S + [W, \cdot])^2 = 0 \iff \begin{cases} (S + [W^{(1)}, \cdot]) W^{(0)} = 0 \\ (S + [W^{(1)}, \cdot])^2 = [2 dW^{(0)}(W^{(2)})] \\ (S + [W^{(1)}, \cdot]) W^{(2)} = 2 dW^{(0)}(W^{(3)}) \end{cases} \]

View \[ S + [W^{(1)}, \cdot] \] as deformation of analytic structure

\[ \begin{cases} \check{\text{Čech}}: \text{deform gluing of } U_i \leftrightarrow U_j \text{ by vector field } W^{(1)}_{ij} \\ \text{Dolbeault}: \text{deform } \overline{\partial} \text{ operator by } W^{(1)} \in L^{0,1} \otimes T^{1,0} \end{cases} \]

(1) \Rightarrow \[ W^{(0)} \] is analytic w.r.t. deformed structure

(2) \Rightarrow deformation actually fails to be an analytic space by \[ 2 dW^{(0)}(W^{(2)}) \in C^2(X^0, T_{X^0}) \]

\[ \begin{cases} \check{\text{Čech}}: \text{failure of coord. changes to satisfy cocycle condition} \\ \text{Dolbeault: failure of integrability of the deformed ex. structure = Nijenhuis tensor} \end{cases} \]

Note: \[ \text{crit}(W^{(0)}) \] is still honestly analytic, since \[ 2 dW^{(0)}(W^{(2)}) = 0 \text{ along critical locus.} \]
A Morse model for Floer operations on $C^*(B^0; H^*(F_b) \otimes \Theta_{an})$

Fix a Morse function $f : X^0 = \pi^{-1}(B^0) \to \mathbb{R}$ (+ Morse-Smale metric).

- triangulation $P$ of $B^0$ ⇒ Morse $f^P$ on $B^0$ with 1 cut pt $b_0$, index $k$ $\forall \sigma \in 2^P \setminus \{\emptyset\}$, $W^{-1}(b_0) = \sigma$.
  - add perfect Morse function on $F_{b_0}$.

$\Rightarrow C \! M^0(f) \cong C^*(B^0; H^*(F_b))$ for open cover of $B^0$ by stars of Vert$(P)$.

Define Floer operations $(m_k)_{k \geq 0}$ on $C \! M^0(f; \Theta_{an})$ by counting (perturbed) $J$-holom. discs $\Sigma$.

- achieve transversality using domain stabilization (cf. Charest-Woodward; divisors vary over $B^0$)
- domain-dependent perturbations
- convergence under assumption that $P$ fine enough ($\Rightarrow \text{val}(z_{b_0}) > 0$ at $b_0$)