Strongly invertible knots, Khovanov homotopy, and localization

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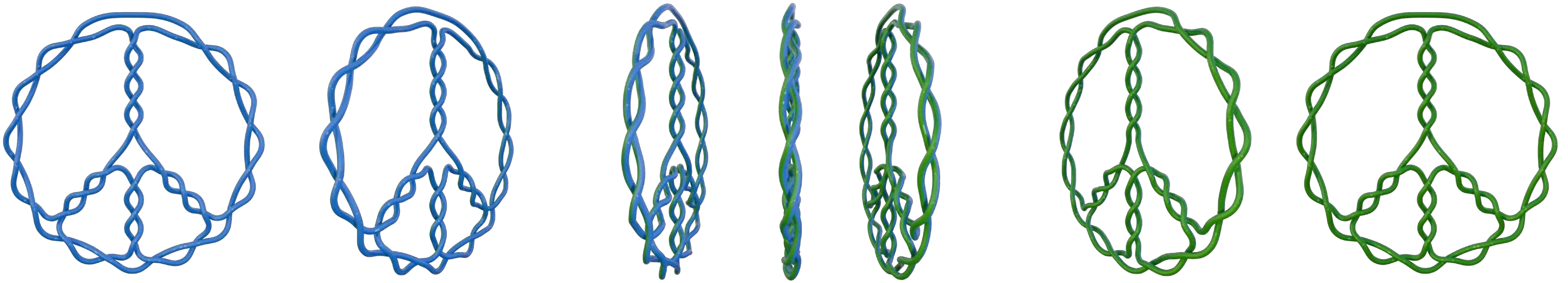

Joint with Sucharit Sarkar

See the future!

- Version of slides with text pre-written: pages.uoregon.edu/lipshitz/zoominar.pdf
- Blank version, for taking your own notes: pages.uoregon.edu/lipshitz/zoominar-blank.pdf

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Views expressed in this talk are his, not those of the National Science Foundation.

The Plan

1. The main theorem and excuses for talking about it here.

2. A brief review of Khovanov homology.

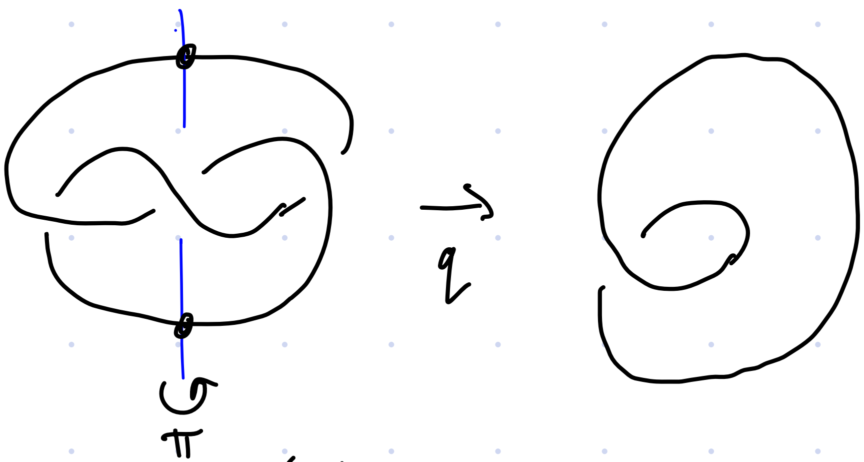
3. A little application.

4. Sketch of the proof of the main theorem.

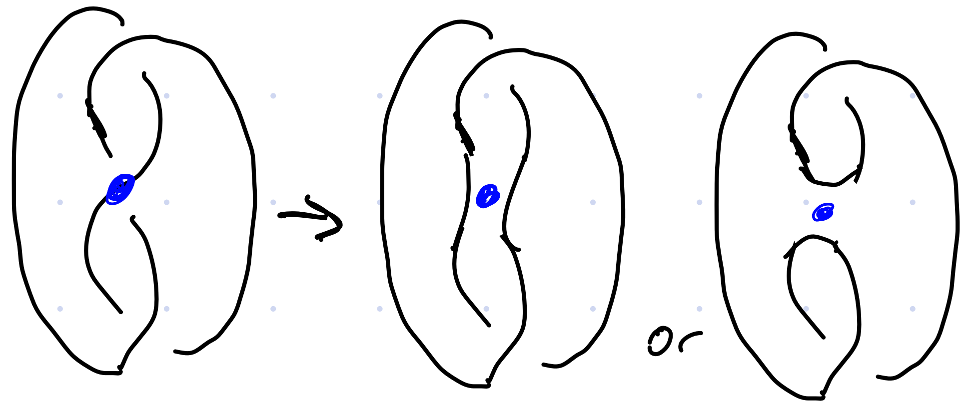
5. Related work.

The Main Theorem

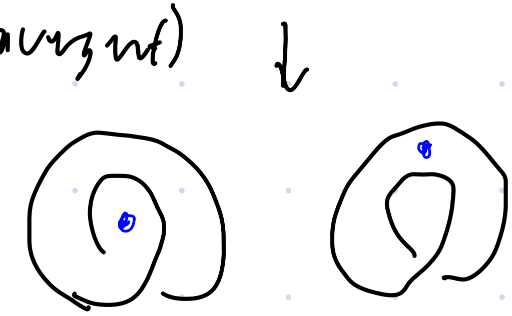
<u>Theorem</u>. (L-Sarkar) Given a strongly invertible knot $K \subset S^3$ with quotient $\overline{K}$, there is a spectral sequence $Kh(K;\mathbb{F}_2)\otimes\mathbb{F}_2[\![\theta,\theta^{-1}]\!] \Rightarrow AKh(\overline{K};\mathbb{F}_2)\otimes\mathbb{F}_2[\![\theta,\theta^{-1}]\!]$.



(Transvergent)



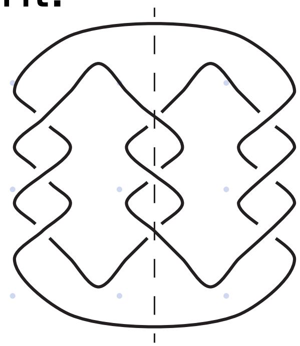
(Intravergent)

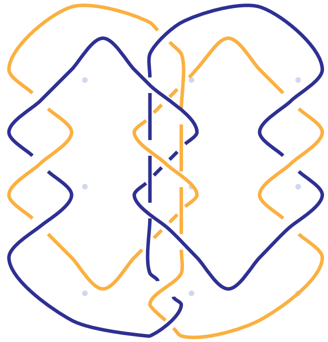


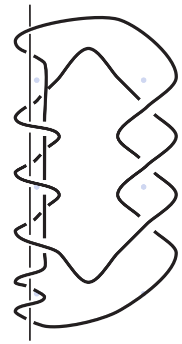
Toy application: 9_{46} has no equivariant slice disk.

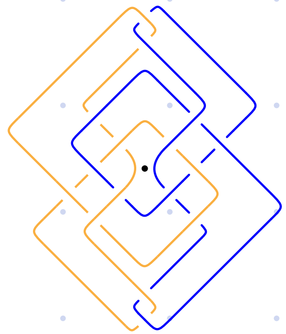
Some More Pictures

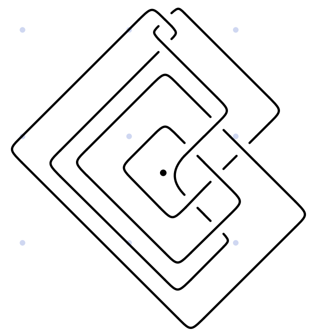
Transvergent:



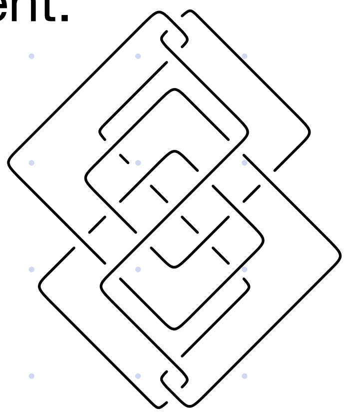


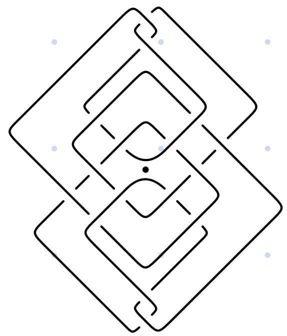


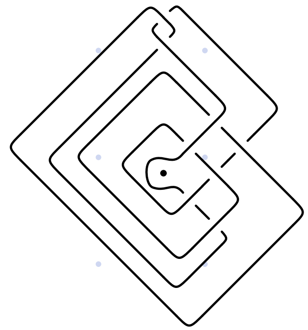




Intravergent:







Khovanov homology

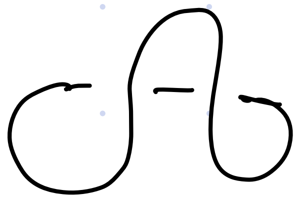
$K \subset S^3 \rightsquigarrow$ Bigraded abelian group $Kh_{i,j}(K)$

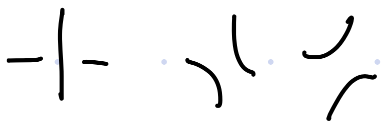
Homology of chain cx $\partial : CKh_{i,j}(K) \to CKh_{i-1,j}(K)$

3_1:

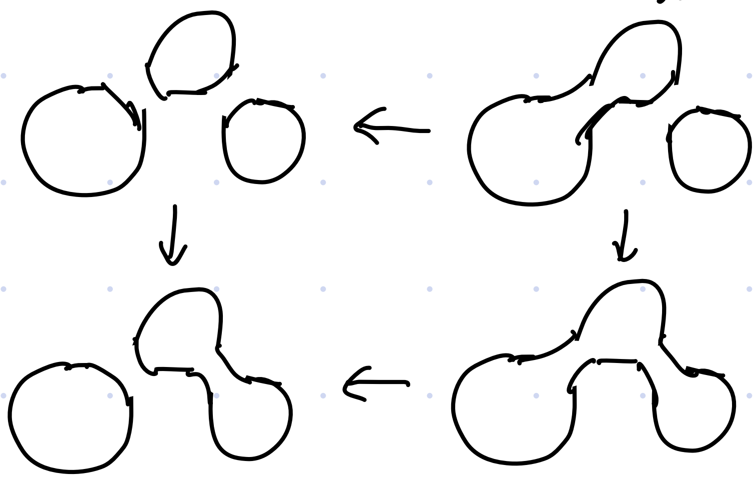
$j \backslash i$	0	1	2	3
9				$\mathbb{Z}$
7			$\mathbb{Z}/2$	
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

K





0 1



00 10

$V = \mathbb{Z}[x]/(x^2)$

$\Delta : V \to V \otimes V$

$\Delta(1) = 1 \otimes X + X \otimes 1$

$\Delta(X) = X \otimes X$

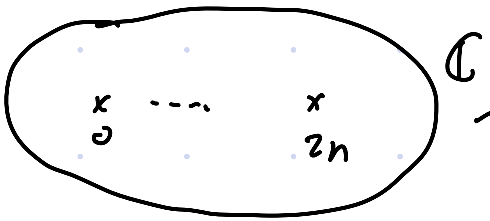
$$\begin{array}{ccc} V\otimes V\otimes V & \xleftarrow{\Delta\otimes Id} & V\otimes V \\ {\scriptstyle Id\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ V\otimes V & \xleftarrow[\Delta]{} & V \end{array}$$

$$CKh: \quad V\otimes V \xleftarrow{\begin{bmatrix} \Delta & Id\otimes m \end{bmatrix}} V \oplus V\otimes V\otimes V \xleftarrow{\begin{bmatrix} -m \\ \Delta\otimes Id \end{bmatrix}} V\otimes V$$

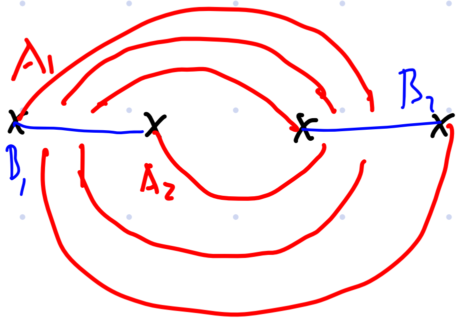
Khovanov homology

$K \subset S^3 \rightsquigarrow$ Bigraded abelian group $Kh_{ij}(K)$

Conjectural symplectic interpretation [Seidel-Smith, Manolescu, Abouzaid-Smith]:



$$\rightsquigarrow p(z) = (z-0)\cdots(z-2n) \rightsquigarrow \begin{array}{c} S = \{u^2+v^2+p(z)=0\} \subset \mathbb{C}^3 \\ {\scriptstyle z}\downarrow \\ \mathbb{C} \end{array}$$



$\rightsquigarrow$ Thimbles $\alpha_1, \alpha_2, \beta_1, \beta_2$

$\rightsquigarrow$ Lagrangians $\alpha_1 \times \alpha_2,\ \beta_1 \times \beta_2 \subset \mathrm{Hilb}^2(S \to \mathbb{C})$

$\rightsquigarrow Kh_{symp}(L) = HF(\alpha_1 \times \alpha_2, \beta_1 \times \beta_2)$

Thm. [Abouzaid-Smith] $Kh_{symp}(L;\mathbb{Q}) \cong Kh(L;\mathbb{Q})$

Thm. [Seidel-Smith] For K 2-periodic, $Kh_{symp}(K;\mathbb{F}_2) \otimes \mathbb{F}_2[\![\theta,\theta^{-1}]\!] \Rightarrow Kh_{symp}(\bar{K};\mathbb{F}_2) \otimes \mathbb{F}_2[\![\theta,\theta^{-1}]\!]$

An application

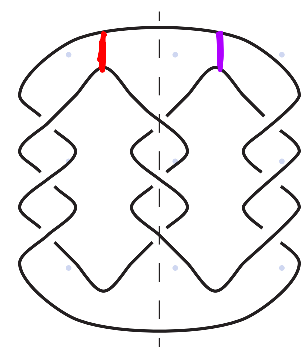
<u>Theorem</u>. [L-S] For K strongly invertible, $Kh(K;\mathbb{F}_2) \Rightarrow AKh(\overline{K};\mathbb{F}_2)$

- First differential is $Id + \tau_*$
- Gradings of d_r differentials are predictable.

So, sometimes can determine τ_* from $Kh(K)$ and $AKh(\overline{K})$.
Given cobordism $F: L_0 \to L_1$ in $[0,1]\times S^3$ get $F_*: Kh(L_0) \to Kh(L_1)$.

- If $F: \emptyset \to L$ is a disk, F_* nontrivial (from "Lee deformation").

For 9_{46}, this gives $\tau_* \circ F_* \neq F_*$ for any slice disk F. So, 9_{46} has no equivariant slice disk.



cf. Sundberg-Swann
Dai-Mallick-Stoffregen

Sketch of the Proof

Given a $\mathbb{Z}/2$ CW complex X, $H_*(X;\mathbb{F}_2)\otimes\mathbb{F}_2[\![\theta,\theta^{-1}]\!] \Rightarrow H_*(X^{fix})\otimes\mathbb{F}_2[\![\theta,\theta^{-1}]\!]$.

$$\begin{tikzcd}
\cdots & C_2(X) \arrow[l] \arrow[d, "\partial"] & C_2(X) \arrow[l] \arrow[d, "\partial"] & C_2(X) \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
\cdots & C_1(X) \arrow[l] \arrow[d, "\partial"] & C_1(X) \arrow[l, "Id+\tau"'] \arrow[d, "\partial"] & C_1(X) \arrow[l, "Id+\tau"'] \arrow[d, "\partial"] & \cdots \arrow[l, "Id+\tau"'] \\
\cdots & C_0(X) \arrow[l] & C_0(X) \arrow[l] & C_0(X) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

cf. Floer homotopy theory

<u>Thm.</u> [L-S] Given K, there is a CW complex $X(K)$ with

$$H_{*+N}(X(K)) \cong Kh_*(K).$$

- For K strongly invertible, can arrange $X(K)$ to have $\mathbb{Z}/2$-action.
- Main work identifying fixed set.
- Technique due to Stoffregen-Zhang ← 2-periodic case
  (cf. Borodzik-Politarczyk-Silvero)

Related Work

Analogues for p-periodic knots:

- $Kh_{symp}(K) \Rightarrow Kh_{symp}(\overline{K})$ ($p = 2$) [Seidel-Smith '06]
- $Kh(K) \Rightarrow AKh(\overline{K})$ (any p) [Stoffregen-Zhang '18, Borodzik-Politarczyk-Silvero '18]
- $\widehat{HFK}(K) \Rightarrow \widehat{HFK}(\overline{K})$ ($p = 2$) [Hendricks '12] uses Seidel-Smith

Miscellaneous:

- $HF(\phi^2) \Rightarrow HF(\phi)$ [Hendricks '14, Seidel '14]
- $HF(L_0, L_1) \Rightarrow HF(L_0^{fix}, L_1^{fix})$, under hypotheses [Seidel-Smith '06, Large '19]
- Symplectic Khovanov spectral exist [Large, Smith, …?]

Analogues for strongly invertible knots:

- $\widehat{HF}(\Sigma(K)) \Rightarrow \widehat{HF}(\Sigma(\overline{K}))$ [L-Sarkar '22]. Uses [Hendricks-Lidman-L], which uses [Large].
- Construction of AKh_{symp}. [Mak-Smith '19]
- New construction of AKh_{symp} and spectral sequence $Kh_{symp}(K) \Rightarrow AKh_{symp}(\overline{K})$, plus symplectic version of Stoffregen-Zhang. [Hendricks-Mak-Raghunath, IP]
  student of Hendricks
- $\widehat{HFK}(T) \Rightarrow \widehat{HFK}(\overline{K})$ [Parikh, IP]
  student of Hendricks

Questions:

quotients knots in lens spaces

- What about for symmetries that aren't manifest in diagrams (e.g., free periods)?
- Do these kinds of ideas have more purely symplectic applications?