Commutative control data for smoothly locally trivial stratified spaces

Yoav Zimhony

Tel Aviv University

March 29, 2024

Outline

Background: Hamiltonian G-spaces

Let

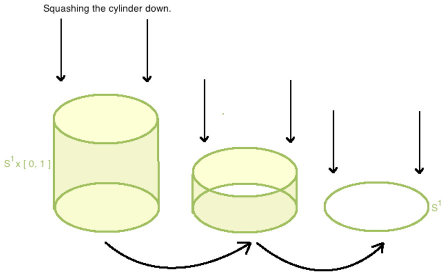
- G — compact Lie group
- $\mathfrak{g}, \mathfrak{g}^*$ — the Lie algebra and its dual
- (M, ω) — symplectic manifold
- $G \circlearrowright (M, \omega)$ symplectically
- $\mu : M \to \mathfrak{g}^*$ — momentum map

Background: Deformation retractions

Definition

Let $C \subset A$ be a subset of a topological space. A **deformation retraction** of A to C is a homotopy $H : A \times [0, 1] \to A$ with:

1. $H(\cdot, 0) = \mathrm{Id}_A$
2. $H(A, 1) = C$
    1. (**Strong**) $H(\cdot, t)|_C = \mathrm{Id}_C$ for all $t \in [0, 1]$
    2. (**Weak**) $H(C, t) \subset C$ for all $t \in [0, 1]$



Strong deformation retraction of the cylinder to S^1

Motivating question

Fix a G-invariant inner product on $\mathfrak{g}^*$, with norm $\|\cdot\|$.
Two interesting subsets:

- zero level set $\mu^{-1}(0)$
- critical set $\mathrm{Crit}\|\mu\|^2$ of the norm-squared momentum map

Question

For each of these subsets, is there a G-invariant neighbourhood of it and a G-equivariant **smooth weak** *deformation retraction from this neighbourhood onto the subset?*

Motivating question: why not strong?

- For submanifolds — strong deformation retractions follow from the tubular neighbourhood theorem
- when they are not manifolds — no smooth strong retractions
- Example: $S^1 \circlearrowright \mathbb{C}^2$ with

$$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2)$$
$$\mu(z_1, z_2) = \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)$$

here

$$\mathrm{Crit}\|\mu\|^2 = \mu^{-1}(0) = \{|z_1|^2 = |z_2|^2\}$$

Motivating question: literature

Question

For each of these subsets, is there a G-invariant neighbourhood of it and a G-equivariant **smooth weak** *deformation retraction from this neighbourhood onto the subset?*

- conjectured by Harada and Karshon (2012)
  - Implies their localization theorem for Duistermaat–Heckmann distributions applies to $\mathrm{Crit}\|\mu\|^2$ without additional assumptions
- **without smoothness** — exists in the literature
  - Lerman (2005) and Woodward (2005), attributed to Duistermaat. Specifically for these subsets
  - Goresky (1976), Pflaum and Wilkin (2019). Using the *stratified structure* of these subsets

Stratified spaces

Definition

Let M be a manifold and $C \subset M$ a closed subset. A **stratification** of C is a decomposition

$$C = \bigsqcup_{X \in \mathcal{S}} X$$

into **strata** X such that:

1. Each X is a sub-manifold
2. $\mathcal{S}$ is locally finite
3. (Frontier) The closure of each stratum X is a union of strata.

A subset $C \subset M$ with a stratification $\mathcal{S}$ is a **stratified subset** of M.

Remark

One often adds additional assumptions which control how strata behave next to each other. Most common is "Whitney (B)" regularity.

Stratified spaces: examples

- embedded sub-manifolds
- the union of the axes in $\mathbb{R}^2$
- real algebraic varieties in $\mathbb{R}^n$
- the orbit type stratification of a manifold with an action of a compact Lie group
- **both $\mu^{-1}(0)$ and Crit$\|\mu\|^2$ can be stratified by orbit types**

Proof strategy

1. (local) prove these subsets, stratified by orbit types, satisfy a strong stratified local regularity condition
2. (local-to-global) using this condition, construct a collection of tubular neighbourhoods of strata that commute in a suitable sense
3. use this collection to obtain a smooth weak deformation retractions

Smooth local triviality with conical fibers

Definition

Let

- $C = \bigsqcup_{X \in S} X$ stratified subset of M
- $X \in \mathcal{S}$ stratum of $\dim X = k$
- $p \in X$

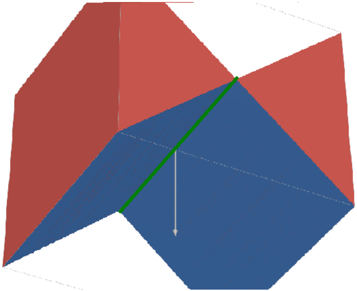
C is **smoothly locally trivial with conical fibers around** p if there exist:

- A neighbourhood U_p of p in M
- A stratified subset $(\tilde{C}_p, \tilde{\mathcal{S}}_p)$ of $\mathbb{R}^{n-k}$ with *conical strata*
- A diffeomorphism

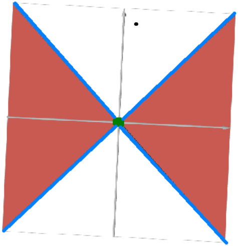
$$\Psi_p : U_p \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$
$$\Psi_p(C \cap U_p) = \mathbb{R}^k \times \tilde{C}_p$$

identifying nearby strata $Y \in \mathcal{S}$ with $\mathbb{R}^k \times \tilde{Y}$ for $\tilde{Y} \in \tilde{\mathcal{S}}_p$

Smooth local triviality with conical fibers: visually

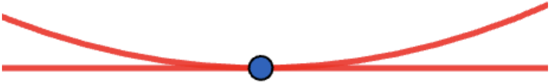


$$= \mathbb{R}^1 \times$$

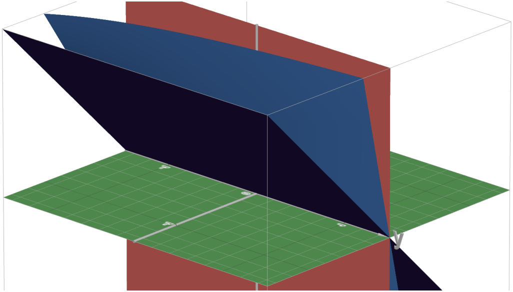


Smooth local triviality with conical fibers: non-examples

- For the zero set of $y(y - x^2)$ around 0 the fibers are not conical



- (Whitney) The zero set of $xy(y - x)(y - (3 + t)x)$ is not smoothly locally trivial.



Smooth local triviality with conical fibers: examples

Theorem (Z. 2023)

the following stratified subsets are smoothly locally trivial with conical fibers.

1. *The orbit type stratification of a manifold with an action of a compact Lie group*
2. $\mu^{-1}(0)$ *and* $Crit\|\mu\|^2$*, stratified by orbit types*

the proofs use, respectively:

1. Koszul's slice theorem
2. the local normal form by Guillemin–Sternberg (1984) and Marle (1985)

Tubular neighbourhood

Definition

Let $X \subset M$ be a sub-manifold. The **normal bundle** of X in M is the quotient vector bundle

$$\nu_X = TM|_X / TX.$$

over X.

Theorem (Tubular neighbourhood)

Let $X \subset M$ be a sub-manifold. There exists an embedding

$$\Psi_X : \nu_X \to M$$

that identifies the zero section with X, and whose differential along X induces the identity map on ν_X.

Collection of tubular neighbourhoods

A **collection of tubular neighbourhoods** for $C = \bigsqcup_{X \in S} X$ is

$$\{(T_X, m_X^t, \rho_X)\}_{X \in \mathcal{S}}$$

where

- $T_X = \Psi_X(\nu_X) \subset M$: image of a tubular neighbourhood embedding
- $m_X^t : T_X \to T_X$: obtained from scalar multilication by $t \geqslant 0$
- $\rho_X : T_X \to \mathbb{R}_{\geqslant 0}$: smooth distance-squared function, i.e.

$$\rho_X \circ m_X^t = t^2 \cdot \rho_X$$
$$\rho_X^{-1}(0) = X$$

Commutative tangential control data

Theorem (Z. 2023)

Let

$$C = \bigsqcup_{X \in S} X \subset M$$

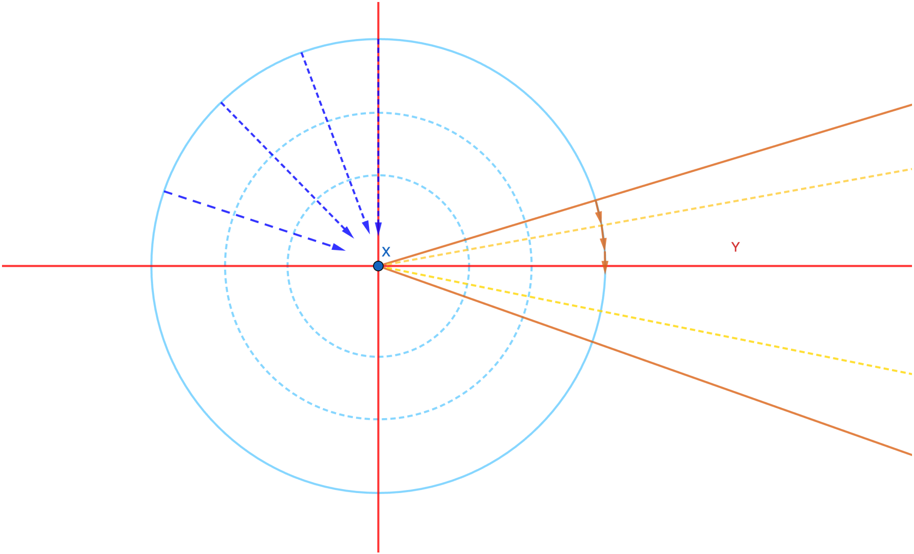
be a stratified subset that is smoothly locally trivial with conical fibers. There exists a collection

$$\{(\boldsymbol{T_X} \subset M, \quad \boldsymbol{m_X^t} : T_X \to T_X, \quad \boldsymbol{\rho_X} : T_X \to \mathbb{R}_{\geqslant 0})\}_{X \in \mathcal{S}}$$

of tubular neighbourhoods satisfying, in the domains of definition:

1. $\rho_X \circ m_Y^s = \rho_X$ *for* $X \neq Y$
2. $m_X^0 \circ m_Y^s = m_X^0$ *for* $X \subsetneq \overline{Y}$
3. *(Tangential)* $m_X^t : T_X \to T_X$ *preserves all other strata for* $t > 0$
4. *(Commutative) The maps* m_X^t, m_Y^s *commute*

Commutative tangential control data for the axes in $\mathbb{R}^2$



Euler-like vector fields

- Euler vector field in $\mathbb{R}^n$:

$$\mathcal{E} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

  - flow: multiplication by $\exp t$
- in vector bundles — Euler vector field on each fiber
- Euler-**like** vector fields along sub-manifolds $X \subset M$
  - a vector field on M, vanishing on X
  - linearization $=$ Euler vector field in the normal bundle
  - introduced by Bursztyn, Lima and Meinrenken (2019)

Properties of Euler-like vector fields

- Euler-like vector field $\mathcal{E}_X$ along $X \implies (T_X, m_X^t)$
  - flow becomes $m_X^{\exp t} : T_X \to T_X$
- being Euler-like along X is
  - a local property
  - preserved under smooth convex combinations

$\implies$ can **patch local tubular neighbourhoods to a global one**.

Also allows the following translations:

(T_X, m_X^t)	$\mathcal{E}_X$
m_X^t preserves a sub-manifold Y	tangent to Y
$f \circ m_X^t = f$	$\mathcal{L}_{\mathcal{E}_X} f = 0$
m_X^t, m_Y^s commute	$[\mathcal{E}_X, \mathcal{E}_Y] = 0$

Obtaining smooth weak deformation retractions

Theorem (Z. 2023)

Let $(C, \mathcal{S})$ be a stratified subset with commutative tangential control data

$$\{(T_X, \rho_X, m_X^t)\}_{X \in \mathcal{S}}.$$

Then there exists a smooth weak deformation retraction from the open set

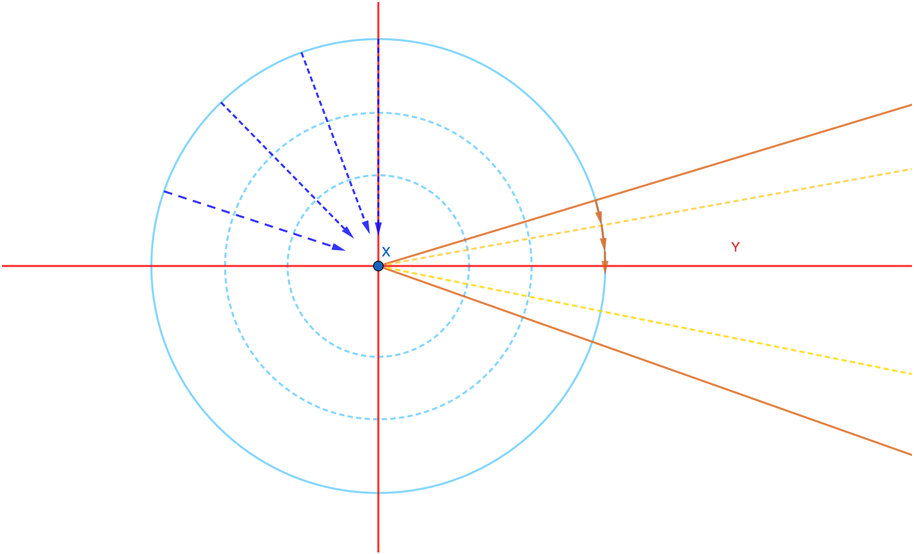
$$\bigcup_{X \in \mathcal{S}} \{\rho_X < 1\}$$

to C, which is the restriction of a smooth homotopy $F : M \times [0,1] \to M$.

Construction: descending recursion on the dimension of strata.

- use m_X^t for the retraction
  - **problem**: can't use $m_X^t : T_X \to T_X$ as is — not smooth on M
- "smoothen" it using control data of lower strata

Construction example



Questions?

Thank you!