Symplectic squeezing of domains in $T^*\mathbb{T}^n$

Qi Feng
(based on joint work with Jun Zhang)

University of Science and Technology of China

March, 2024

Background

Background

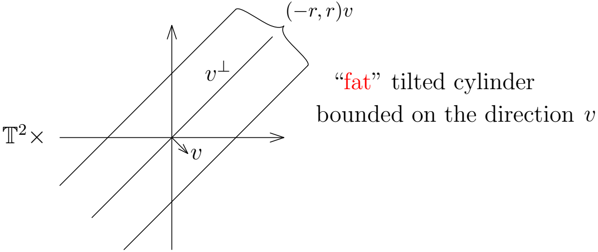
Denote the standard simplex in $\mathbb{R}^n_{\geq 0}$ by $\Delta^n(r) := \{x_1 + \cdots + x_n \leq r\}$. Introduce two domains in $T^*\mathbb{T}^n$:

$$P^{2n}(r) := \mathbb{T}^n \times \Delta^n(r)$$

and

$$Y^{2n}(r, v) := \mathbb{T}^n \times \left((-r, r)v \times v^{\perp}\right).$$

Here, v is a unit vector in $\mathbb{R}^n$ and $v^{\perp}$ denotes the **hyperplane** in fiber $\mathbb{R}^n$ that is perpendicular to v. For instance, when $n = 2$, $v^{\perp}$ is simply a line perpendicular to v.



† There are $(n-1)$-many unbounded directions in the fiber of $Y^{2n}(r, v)$.

Rigidity

- We have some rigidity on vertical cylinder $Y^{2n}(r, v)$.
- We say ϕ a $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial symplectic embedding if ϕ is a symplectic embedding with $\phi_*(\alpha) = \alpha$ for any $\alpha \in \widetilde{\pi}_1(\mathbb{T}^n) := [S^1, \mathbb{T}^n]$.

Theorem (Sikarov, 1989)

There is a $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial symplectic embedding from $P^{2n}(s)$ to $Y^{2n}(\frac{r}{2}, (\mathbf{1}, \mathbf{0}, \cdots, \mathbf{0}))$ if and only if $s \leq r$.

Theorem (Maley-Mastrangeli-Traynor, 2000)

There is a symplectic embedding $\psi \colon P^{2n}(s) \to Y^{2n}(\frac{r}{2}, (\mathbf{1}, \mathbf{0}, \cdots, \mathbf{0}))$ with

$$\psi_* \colon H_1(P^{2n}(s)) \xrightarrow{\simeq} H_1\left(Y^{2n}\left(\frac{r}{2}, (1, 0, \cdots, 0)\right)\right)$$

if and only if $s \leq r$.

BPS capacity

In general, Biran, Polterovich and Salamon define the capacity for a pair (W, A) and a nontrivial free homotopy class $\alpha \in \widetilde{\pi}_1(\mathbb{T}^n)$:

$$c_{\mathrm{BPS}}(W, A; \alpha) := \inf \left\{ c > 0 \,\middle|\, \text{If } \sup_{S^1 \times A} H \leq -c, \text{then } \mathscr{P}_\alpha(H) \neq \emptyset \right\}$$

where $\mathscr{P}_\alpha(H)$ denotes the set of Hamiltonian orbits in the homotopy class α. Note that BPS capacity is invariant under $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial symplectomorphisms.

Theorem (Gong-Xue, 2020)

*Let v be a unit vector in $\mathbb{R}^n$. If v is **rational**, i.e. v is a scalar multiple of an integer vector $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Then*

$$c_{\mathrm{BPS}}(Y^{2n}(r, v), \mathbb{T}^n; \pm\alpha) = r\|\alpha\|.$$

Therefore, there exists a obstruction to a $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial symplectic embedding from $P^{2n}(s)$ to $Y^{2n}(r, v)$ for rational v.

Calculation of BPS capacity

Theorem (Gong-Xue, 2020)

*If v is **irrational**, i.e. v is not a scalar multiple of any integer vector. Then for all $\alpha \in H_1(\mathbb{T}^n, \mathbb{Z}) \setminus \{0\}$ and all $r > 0$, we have*

$$c_{\mathrm{BPS}}(Y^{2n}(r, v), \mathbb{T}^n; \alpha) = \infty.$$

• There is **no** obvious obstruction to $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial symplectic embedding from $P^{2n}(s)$ to $Y^{2n}(r, v)$ for irrational v.

• If we drop the requirement that $\widetilde{\pi}_1$-trivial, we can indeed construct symplectic embedding. See our main theorem.

Question

What is the precise value of $c_{\mathrm{BPS}}(P^{2n}, \mathbb{T}^n; \alpha)$?

Main Theorem

Result

Theorem (F.-Zhang, 2024)

Let v be an ***irrational*** *unit vector in $\mathbb{R}^{n\geq 2}$, then there exist a symplectic embedding from $P^{2n}(r)$ to $Y^{2n}(1,v)$ for any $r>0$.*

Corollary

If v is an irrational unit vector, then any bounded domain of $T^*\mathbb{T}^n$ can symplectically embed into $Y^{2n}(1,v)$.

† The symplectic embedding in our Theorem is not $\widetilde{\pi}_1(\mathbb{T}^n)$-trivial.

- The special case $n=2$ resolves an open problem:

Problem (Gong-Xue, 2020)

Let v be an irrational vector but not an eigenvector of any $A\in \mathrm{SL}_2(\mathbb{Z})$. Is it true that for all $r>0$, there exists a symplectic map from $P^4(r)$ to $Y^4(1,v)$?

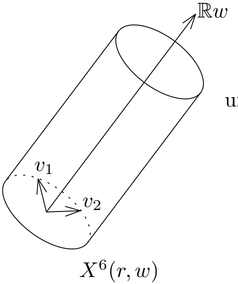
When the dimension $2n \geq 6$ (so $n \geq 3$), instead of the **"fat cylinder"**

$$Y^{2n}(r,v) := \mathbb{T}^n \times (-r,r)v \times v^{\perp}$$

one can consider the following **"thin cylinder"**,

$$X^{2n}(r,w) := \mathbb{T}^n \times D^{n-1}_{\text{perp}}(r) \times \mathbb{R}w$$

where $D^{n-1}_{\text{perp}}(r)$ is a disk of radius r in $\mathbb{R}^n$, $(n-1)$-dimensional, and perpendicular to the line $\mathbb{R}w$ pointing in the direction of w.



"thin" tilted cylinder unbounded on the direction w

$v_1 \perp w, v_2 \perp w$

$\Rightarrow X^6(r,w) \subset Y^6(r,v_1)$

$X^6(r,w) \subset Y^6(r,v_2)$

New question

- If there is an rational vector α perpendicular to w, then

$$X^{2n}(r,w) \subset Y^{2n}(r,\alpha) \Rightarrow c_{\mathrm{BPS}}(X^{2n}(r,w),\mathbb{T}^n;\alpha) \leq r\|\alpha\|.$$

In an opposite direction, motivated by our main Theorem, here is another intriguing question.

Question (Xue, 2024)

Suppose $D^{n-1}_{\mathrm{perp}}(r)$ contains no rational vectors at all, is it possible that there exists a symplectic embedding from $P^{2n}(r)$ to $X^{2n}(1,w)$ for any $r > 0$?

Remark

The condition that $D^{n-1}_{\mathrm{perp}}(r)$ contains no rational vectors at all is equivalent to that $w = (w_1, \cdots, w_n)$ is $\mathbb{Q}$-independent.

Classical non-squeezing in $\mathbb{R}^{2n}$

• Gromov proves that if there exists a symplectic embeddding from $B^{2n}(r)$ to $B^2(1)\times\mathbb{R}^{2n-2}$, then $r\leq 1$. Here $B^2(1)$ is in the coordinate of x_1 and y_1. The bounded direction is important.

Question (Hofer, 1990)

Is there $r>0$, such that the cylinder $B^2(r)\times\mathbb{R}^{2n-2}$ symplectically embeds into $B^{2n-2}(1)\times\mathbb{R}^2$?

Theorem (Pelayo-Ngoc, 2015)

If $n\geq 2$, the cylinder $B^2(r)\times\mathbb{R}^{2n-2}$ can be symplectically embedded into $B^{2n-2}(1)\times\mathbb{R}^2$ for all $r\leq\frac{1}{\sqrt{2^{n-1}+2^{n-2}-2}}$.

• The domains we discuss before are bounded on the fiber, the **Lagrangian** subspace of $T^*\mathbb{T}^n$. It is distinct to $B^2(r)\times\mathbb{R}^{2n-2}$, which is bounded on the **symplectic** subspace of $\mathbb{R}^n$.

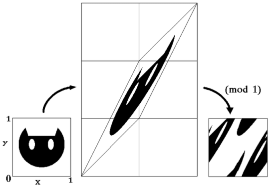
Proof

- Arnold's cat map
- Dirichlet's approximation theorem
- Bézout's identity

Arnold's cat map

- We now consider a symplectomorphism in the form of $\Phi_A = (A^{-1}, A)$ on $T^*\mathbb{T}^2$, where A is the famous **Arnold's cat map**,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$



- There will be two eigenvectors v_1 and v_2 such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. We have the equation for eigenvalues λ_1 and λ_2:

$$\lambda_1 + \lambda_2 = 3, \lambda_1 \lambda_2 = 1 \Rightarrow 0 < \lambda_2 < 1 < \lambda_1.$$

- The iterations of A, stretchs any domain in $\mathbb{R}^2$ along one direction v_1 while shrinks the other direction v_2.

General case

• By Arnold's cat map, we can symplectically embed any bounded domain into $Y^4(1, v_2)$:

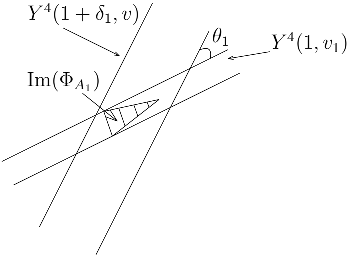
$$\Phi_{A^n} \colon P^4(r_n) \to Y^4(1, v_2), \quad r_n \sim \frac{1}{\lambda_2^n}$$

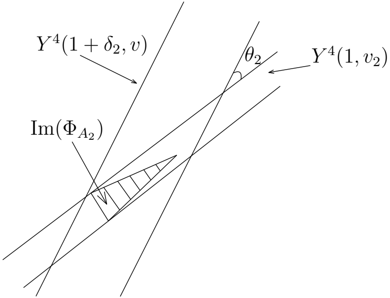
• Note that embeddings produced in this way can only be induced by a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. Its eigenvalues have to be algebraic numbers solving $x^2 - \mathrm{tr}(A)x + 1 = 0$.

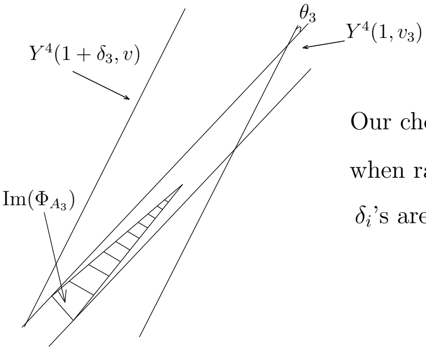
• To prove more general case that v is irrational but not eigenvector of any $A \in \mathrm{SL}_2(\mathbb{Z})$, we need take an appropriate sequence $\{A_i \in \mathrm{SL}_2(\mathbb{Z})\}_{i \in \mathbb{N}}$ such that

$$\Phi_{A_i} \colon P^4(r_i) \hookrightarrow Y^4(1, v_i) \quad \text{with rational } v_i \textbf{ approximates } v$$

$$\text{but} \quad \mathrm{Im}(\Phi_{A_i}) \subset Y^4(1 + \delta_i, v) \text{ such that } \delta_i \text{ is bounded.}$$







Our choice of A_i guarantees that

when rational v_i approximate irrational v

δ_i's are uniformly bounded

- Suppose v is a scalar multiple of $(\kappa, 1)$ for some irrational number κ. Due to **Dirichlet's approximation theorem**, there exists a sequence of coprime pairs $\{(p_i, q_i)\}_{i\in\mathbb{N}}$ satisfying

$$\left|\frac{p_i}{q_i} - \kappa\right| < \frac{1}{q_i^2} \quad \text{and} \quad \lim_{i\to+\infty} q_i = +\infty.$$

Then the wanted unit vector v_i is a rescale of vector (p_i, q_i).

- Consider $\{A_i\}_{i\in\mathbb{N}}$ such that

$$(p_i, q_i)\cdot A_i = (1,0) \quad \text{and} \quad r_i = \sqrt{p_i^2 + q_i^2} \qquad (*)$$

then we verify with the length of projection along v_i direction

$$\max_{x,y\in\Delta^2(r_i)} v_i \cdot A_i(x-y) = \max_{x,y\in\Delta^2(r_i)} \frac{(p_i, q_i)\cdot A_i(x-y)}{\sqrt{p_i^2 + q_i^2}} \leq 1.$$

Approximation (cont.)

This means Φ_{A_i} can embed $P^4(r_i)$ into $Y^4(1, v_i)$. To obtain A_i satisfying $(*)$, we will use **Bézout's identity**.

• For coprime p_i, q_i, there exist $a_i, c_i \in \mathbb{Z}$ such that $a_i p_i + c_i q_i = 1$ and $|a_i| \leq |q_i|, |c_i| \leq |p_i|$. Take

$$A_i = \begin{pmatrix} a_i & -q_i \\ c_i & p_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

• In fact, $\Phi_{A_i}(P^4(r_i))$ is in $Y^4(1+\delta_i, v)$, where δ_i is determined by the rectangle spanned by v_i and w_i (perpendicular to v_i).

$$\begin{aligned} \Phi_{A_i}\left(P^4\left(r_i\right)\right) &\subset \mathbb{T}^2 \times ((-1,1)v_i \times (-\ell_i, \ell_i) w_i) \\ &\subset Y^4(1, v_i) \cap Y^4(1+\delta_i, v) \end{aligned}$$

If $\{\delta_i\}_{i\in\mathbb{N}}$ is bounded, then we finish the proof.

Proof of $n = 2$

Denote θ_i by the angle between v_i and v. We have $2(1+\delta_i) = 2\ell_i \sin\theta_i + 2\cos\theta_i$. Then $\delta_i := \ell_i \sin\theta_i + \cos\theta_i - 1 \le \ell_i \theta_i$. Here,

$$\begin{aligned}\ell_i &:= \max_{x \in \Delta^2\left(\sqrt{p_i^2+q_i^2}\right)} |A_i x| \\ &= \sqrt{p_i^2+q_i^2} \cdot \max\left\{\sqrt{a_i^2+c_i^2}, \sqrt{p_i^2+q_i^2}\right\} \le p_i^2+q_i^2.\end{aligned}$$

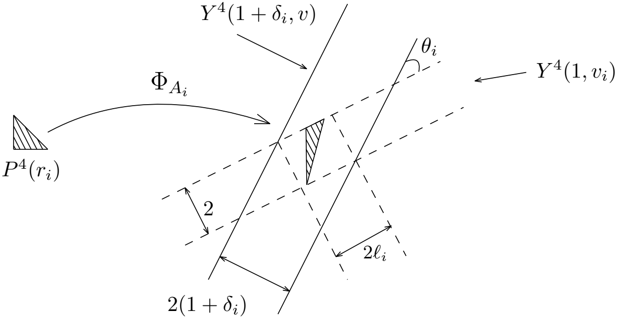


Figure: Symplectic embedding ψ_{A_i} from $P^4(r_i)$ to $Y^4(1+\delta_i, v)$.

Proof of $n = 2$

Then it is easy to verify that

$$\Phi_{A_i}\left(P^4(r_i)\right) \subset Y^4(1, v_i) \cap Y^4(1+\delta_i, v)$$

up to a shift in the fiber (which is also a symplectomorphism of $T^*\mathbb{T}^2$).
Recall that

$$\left|\frac{p_i}{q_i} - \kappa\right| < \frac{1}{q_i^2} \quad \Rightarrow \quad (\theta_i \leq) \tan\theta_i \leq \frac{1}{q_i^2}$$

Therefore,

$$\delta_i = \ell_i \sin\theta_i + \cos\theta_i - 1 \leq \ell_i \theta_i$$
$$\leq \ell_i \tan\theta_i \leq \frac{p_i^2 + q_i^2}{q_i^2}$$

which is bounded as disired.

Remarks on proof

• If v is rational, then v is a scalar multiple of $(p/q, 1)$ for some coprime $p, q \in \mathbb{Z}$. We can still find a sequence of pairs $\{(p_i, q_i)\}_{i\in\mathbb{N}}$ to approximate $(p/q, 1)$. But to satisfy the condition

$$\lim_{i\to+\infty} q_i = +\infty,$$

we only have

$$\left|\frac{p_i}{q_i} - \frac{p}{q}\right| < \frac{1}{q_i}$$

So our approach fails for rational v.

• In higher dimensional, we can still find unit vectors v_i, which is a rescale of coprime $(p_{i,1}, \cdots, p_{i,n})$, to approximate v by higher dimensional vision Dirichlet's approximation theorem. Denote θ_i as the angle between v_i and v, we have

$$\theta_i \sim \frac{1}{p_{i,1}^{1+1/n}}.$$

Key step in higher dimension

Proposition

For any coprime $(p_1, \cdots, p_n)$*, there exist* $A = (a_{ij})_{1 \le i,j \le n} \in \mathrm{SL}_n(\mathbb{Z})$ *with* $(p_1, \cdots, p_n)A = (1, 0, \cdots, 0)$ *and* $|a_{ij}| \le C(n)\sqrt{\sum_{k=1}^n p_k^2}$*, where* $C(n)$ *is a constant only depending on* n*.*

Assume this proposition, denote $r_i = \sqrt{\sum_{j=1}^n p_{i,j}^2}$, then

$$\ell_i = \max_{x \in \Delta^n(r_i)} |A_i x| \le 2 \sum_{j,k=1}^n r_i |(A_i)_{j,k}| \le 2C(n) n^2 r_i^2.$$

We can also verify that

$$\Phi_{A_i}\left(P^{2n}(r_i)\right) \subset Y^{2n}(1, v_i) \cap Y^{2n}(1 + \delta_i, v)$$

where $\delta_i = \ell_i \sin\theta_i + \cos\theta_i - 1 \sim \ell_i \theta_i$. So we have

$$\frac{r_i}{1+\delta_i} \gtrsim \frac{1}{\frac{1}{r_i} + \frac{\ell_i \theta_i}{r_i}} = \frac{1}{\frac{1}{r_i} + \frac{r_i}{r_i^{1+1/n}}} \to +\infty.$$

Discussion

New question

Question (Xue, 2024)

Suppose $w = (w_1, \cdots, w_n)$ is $\mathbb{Q}$-independent, is it possible that there exists a symplectic embedding from $P^{2n}(r)$ to $X^{2n}(1, w)$ for any $r > 0$?

• We can consider a higher-dimensional **analogue of Arnold's cat map**. Here, for instance,

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z})$$

where its three eigenvalues are $\lambda_1 \approx 0.243, \lambda \approx 0.573, \lambda_3 \approx 7.184$. Denote v_3 as the unit eigenvector of λ_3. For any $r > 0$, sufficiently high iterations of A maps $P^6(r)$ into $X^6(1, v_3)$. The components of v_3 are $\mathbb{Q}$-independent.

Difficulties

- Unfortunately, our method in proving main Theorem is **not** applicable to answer Xue's Question.

- For simplicity, let us illustrate the difficulty when $n = 3$. Denote v_1 and v_2 the unit vectors perpendicular to w. The directions labelled by v_1 and v_2 are bounded. We approximate v_1 and v_2 by the sequences $\{v_1^i\}_{i\in\mathbb{N}}$ and $\{v_2^i\}_{i\in\mathbb{N}}$ obtained from Dirichlet's approximation theorem.

- The first step is to find $A_i \in \mathrm{SL}_3(\mathbb{Z})$ that is able to control both directions v_1^i and v_2^i, for instance,

$$A_i \colon v_1^i \mapsto (1,0,0) \text{ and } v_2^i \mapsto (0,1,0).$$

But we cannot construct such $A_i \in \mathrm{SL}_3(\mathbb{Z})$. To make progresses in this direction, one probably needs stronger results from Bézout's identity.

THANK YOU!