Equivariant Lagrangian Correspondence and A Conjecture of Teleman

Yan-Lung Leon Li

(The Chinese University of Hong Kong)

Symplectic Zoominar

12 April, 2024

Joint work with Siu-Cheong Lau and Naichung Conan Leung

(based on arXiv: 2312.13926)

Hamiltonian G-spaces and Symplectic quotients

Setup (Y, ω) : Hamiltonian G-space

$\Leftrightarrow$ $G \overset{\text{Ham}}{\curvearrowright} (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$: moment map $\left(\overset{\Delta}{\Leftrightarrow} \begin{array}{l} \cdot\ G\text{-equivariance} : \mu(g \cdot y) = Ad_g^* \mu(y) \\ \cdot\ \text{Hamilton equation} : \iota_{v^\#} \omega = d\mu(v) \end{array} \right)$

compact　　　Symplectic

Assume $G \curvearrowright \mu^{-1}(0)$: free $(\Rightarrow 0 : \text{regular value of } \mu \Rightarrow \mu^{-1}(0) : \text{smooth})$

$$\mu^{-1}(0) \xrightarrow{\pi} X := \mu^{-1}(0)/G : \text{smooth}$$

Thm [Marsden-Weinstein, Meyer] $\exists!\ \overline{\omega} \in \Omega^2(X)$ such that $\pi^* \overline{\omega} = \omega|_{\mu^{-1}(0)}$.

Symplectic

Def $(X, \overline{\omega})$ Symplectic quotient/reduction

Theme Compare (G-equivariant) geometry and topology of Y and X.

G-invariant Lagrangians and Lagrangian quotient

Setup $\quad G \curvearrowright (Y,\omega) \xrightarrow{\mu} \mathfrak{g}^*$

$$\begin{tikzcd} G \arrow[r, phantom, "\curvearrowright"] \arrow[dr, bend right] & (Y,\omega) \arrow[d, phantom, "\cup\!|"] \\ & L \end{tikzcd}$$

L : G-invariant Lagrangian (+ closed, connected)

Fact $\quad L \subseteq \mu^{-1}(c)$, $\exists!\, c \in Z(\mathfrak{g}^*)$ (WLOG: assume $c=0$)

central

$\rightsquigarrow$

$$\begin{tikzcd} G \arrow[r, phantom, "\curvearrowright"] \arrow[rr, bend left, "\text{free}"] & L \arrow[d] \arrow[r, phantom, "\subseteq"] & \mu^{-1}(0) \arrow[d] \subseteq (Y,\omega) \\ & \bar{L} := L/G \arrow[r, phantom, "\subseteq"] & (X,\bar{\omega}) \end{tikzcd}$$

Lagrangian quotient

Goal. Define G-equivariant Lagrangian Floer Theory of $L \subseteq (Y,\omega)$.

. Compare with Lagrangian Floer Theory of $\bar{L} \subseteq (X,\bar{\omega})$.

Motivating Conjecture [Teleman ICM 2014, abelian case $G=T$]

Conj LG Mirror of Y with LG potential $W_Y : \check{Y} \xrightarrow[\text{hol.}]{} \mathbb{C}$

There exists $\overbrace{(\check{Y}, W_Y)} \xrightarrow{F} \check{T}_{\mathbb{C}}$: holomorphic fibration ["Mirror to $T \overset{\text{Ham}}{\curvearrowright} (Y,\omega)$"]

such that

$$\begin{array}{ccc}
(\check{Y}, W_Y) & \xrightarrow{F} & \check{T}_{\mathbb{C}} \\
\cup & & \cup \\
(F^{-1}(1), W_Y|_{F^{-1}(1)}) & \longrightarrow & 1 \\
\cong \quad \| & & \\
\underbrace{(\check{X}, W_X)} & &
\end{array}$$

LG Mirror of X with LG potential W_X

Rmk

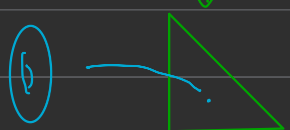
- Closed-string versions: [Pomerleano–Teleman: QH/SH, Iritani–Sanda: QDM]
- Above conjecture arises from 3d Mirror Symmetry.

Example: Toric Varieties

① $\mathbb{CP}^2$

Setup

$$\begin{tikzcd}
S^1 \arrow[r, phantom, "\overset{\text{diag}}{\curvearrowright}"] & (\mathbb{C}^3, \omega_0) \arrow[rr, "\mu = \frac{1}{2}\left(\sum_{i=1}^{3} |z_i|^2\right)"] & & \mathbb{R} \\
T^3 = L \arrow[ur, "\text{Lag}"] \arrow[d] \arrow[r, phantom, "\subseteq"] & S^5(c) \arrow[u, phantom, "\cup"] \arrow[rr] \arrow[d, "\pi"] & & c>0 \arrow[u, phantom, "\cup"] \\
T^2 \cong \overline{L} \arrow[r, phantom, "\subseteq"] & (\mathbb{CP}^2, c\,\omega_{FS}) \arrow[d, "\mu"] & & \\
& \Delta_{\mathbb{P}^2} & &
\end{tikzcd}$$



Lag. Torus fiber moment polytope

More gen.

$G = T^k \hookrightarrow T^m \overset{\text{diag}}{\curvearrowright} Y = \mathbb{C}^m$

X : cpt Toric Fano

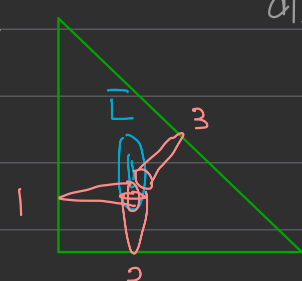
Givental–Hori–Vafa Mirror Construction

$$\begin{tikzcd}
(\mathcal{MC}_{wk}(L), W_L) \arrow[d, phantom, "\cong"] & \\
\left((\check{\mathbb{C}}^\times)^3, W_{\mathbb{C}^3} = Z_1 + Z_2 + Z_3\right) \arrow[r, "Z_1 Z_2 Z_3"] & \check{\mathbb{C}}^\times \\
\{Z_1 Z_2 Z_3 = e^c\} \arrow[u, phantom, "\cup"] \arrow[r] & e^c \arrow[u, phantom, "\cup"] \\
\left((\check{\mathbb{C}}^\times)^2, W_{\mathbb{P}^2} = \overset{1}{Z_1} + \overset{2}{Z_2} + \overset{3}{\frac{e^c}{Z_1 Z_2}}\right) \arrow[u, phantom, "\cong"] & \\
(\mathcal{MC}_{wk}(\overline{L}), W_{\overline{L}}) \arrow[u, phantom, "\cong\ \text{[Cho-Oh]}"] &
\end{tikzcd}$$

; weak Maurer-Cartan space

$\uparrow$

disk potential of $\overline{L}$



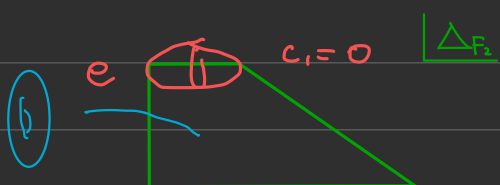
② [Auroux, Fukaya-Oh-Ohta-Ono, Chan-Lau-Leung-Tseng]

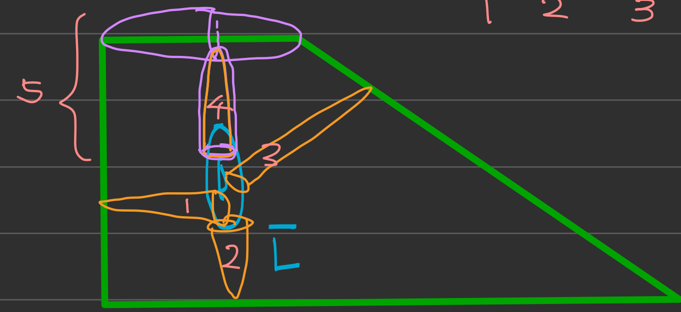
$$G = T^2 \xhookrightarrow{\begin{pmatrix}1&0\\2&1\\1&0\\0&1\end{pmatrix}} T^4 \curvearrowright Y = \mathbb{C}^4$$

$$\check{X} = \Big((\check{\mathbb{C}}^\times)^2,\ W_{LF}(z_1,z_2) = \overbrace{\underset{1}{z_1} + \underset{2}{z_2} + \underset{3}{\frac{q_1}{z_1 z_2^2}} + \underset{4}{\frac{q_2}{z_2}}}^{W_{GHV}} + \underset{5}{\frac{q_2\cdot q^e}{z_2}}\Big)$$

$T^2 \simeq \bar{L} \subseteq X = F_2$: semi-Fano

2nd Hirzebruch surface





Intepretation

$$W_{LF} = W_{GHV} + \frac{q_2\cdot q^e}{z_2}$$

$$\Updownarrow$$

$$W_{\bar{L}} = W_L\big|_{F^{-1}(1)} + (\cdots)$$ ← "obstruction term from $\mu^{-1}(0)$"

Main Theorem

Setup

$$\begin{tikzcd}
 & G \curvearrowright (Y,\omega) \arrow[r, "\mu"] & \mathfrak{g}^* \quad : \text{proper} \\
G \circlearrowright L \arrow[ur, "\text{Lag}"] \arrow[d] \arrow[r, phantom, "\subseteq"] & \mu^{-1}(0) \arrow[u, phantom, "\cup\!\!\mid" , sloped] \arrow[r] \arrow[d, "\pi"'] & 0 \arrow[u, phantom, "\cup\!\!\mid", sloped] \\
\overline{L} \arrow[r, phantom, "\overset{\text{Lag}}{\subseteq}"] & (X, \overline{\omega}) &
\end{tikzcd}$$

$$\begin{array}{c} G \\ \circlearrowright \end{array}$$

$$L^{\pi} := \mathrm{Graph}(\pi) \subseteq \left(Y^{-} \times X,\ (-\omega) \oplus \overline{\omega}\right)$$

Moment Level Correspondence

as a G-equivariant Lagrangian correspondence

Thm [Lau-Leung-L.]

① $\exists\ (CF_G(L), \{m_k^G\}, e_G)$ Equivariant Floer complex

(as unital A_∞ alg. $/\Lambda_0$)

$\parallel$

as mod: $H_G^\cdot(L)$

$\rightsquigarrow$

$$\mathcal{MC}^G_{wk}(L) := \{ b \in CF^{odd}_{G,+}(L) \mid \sum_{k\geq 0} m_k^G(b^{\otimes k}) = W_L^G(b)\cdot e_G \} \xrightarrow{W_L^G} \Lambda_0$$

Equivariant weak Maurer-Cartan space

Equivariant disk potential

Similarly: $\exists\ (CF_G(L^\pi), \{m_k^G\}, e_G)$

$\rightsquigarrow$

$$\mathcal{MC}^G_{wk}(L^\pi) \xrightarrow{W^G_{L^\pi}} \Lambda_0$$

② Fix $b^{\pi} \in \mathcal{MC}^{G}_{wk}(L^{\pi})$,

$$\begin{array}{ccc} \mathcal{MC}^{G}_{wk}(L) \ni b & & W^{G}_{L}(b) \\ \exists\ \psi_{b^{\pi}} \wr\!| \text{ bij.} \quad \updownarrow & \text{Such that} & \| \\ \mathcal{MC}_{wk}(\bar{L}) \ni \bar{b} & & W_{\bar{L}}(\bar{b}) - \overbrace{W^{G}_{L^{\pi}}(b^{\pi})}^{(\cdots)} \end{array}$$

Cor [localised Teleman's Conjecture for $G = T^k$: abelian]

$$\exists \left(\left(\widetilde{\mathcal{MC}}^{G}_{wk}(L), \widetilde{W}^{G}_{L}\right) \xrightarrow{F_L} (\Lambda^{\times})^{k} \right) \leftsquigarrow \text{[Kim-Lau-Zheng: Equiv. Morse model]}$$

$$\begin{array}{ccc} \cup\!| & & \cup \\ \left(\mathcal{MC}^{G}_{wk}(L), W^{G}_{L}\right) & \longrightarrow & 1 \\ \psi_{b^{\pi}} \wr\!| & & \\ \left(\mathcal{MC}_{wk}(\bar{L}), W_{\bar{L}} - W^{G}_{L^{\pi}}(b^{\pi})\right) & & \end{array}$$

Rmk More generally for $b^{\pi} \in \widetilde{\mathcal{MC}}^{G}_{wk}(L^{\pi})$ w/ $F_{L^{\pi}}(b^{\pi}) \neq 1$

Example : Toric Variety

Setup $G = T^k \curvearrowright \mathbb{C}^m \rightsquigarrow (X, \overline{\omega})$: cpt Toric

$$\begin{array}{ccccc} T^k & \curvearrowright & \mathbb{C}^m & \rightsquigarrow & X \\ & & \cup & & \cup \\ T^m & = & L & \longrightarrow & \overline{L} \end{array}$$

$\overline{L}$: Lag Torus fiber

Thm If X : (semi-) Fano $\Rightarrow$ L^π : equiv. (weakly-) unob.

Ex ① $S^1 \curvearrowright \mathbb{C}^3 \rightsquigarrow (\mathbb{P}^2, \omega_A)$

$$\begin{array}{ccc} \mathbb{C}^3 & & \mathbb{P}^2 \\ \cup & & \cup \\ \prod_{i=1}^{3} S^1_{A_i} & \longrightarrow & \overline{L} \cong T^2 \end{array}$$

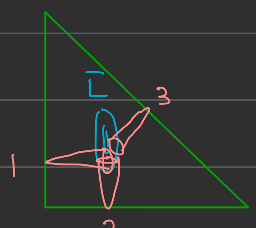
· $\widetilde{W}^G_L = \overbrace{T^{A_1} Z_1 + T^{A_2} Z_2 + T^{A_3} Z_3}^{W_L} + \lambda \log \overbrace{Z_1 Z_2 Z_3}^{F_L}$

· $\widetilde{W}^G_{L^\pi} \equiv 0$

$\rightsquigarrow$

$$W_{\overline{L}} = \left(\underset{1}{T^{A_1} Z_1} + \underset{2}{T^{A_2} Z_2} + \underset{3}{T^{A_3} Z_3}\right)\Big|_{\{Z_1 Z_2 Z_3 = 1\}}$$

$$= T^{A_1} Z_1 + T^{A_2} Z_2 + T^{A_3} \frac{1}{Z_1 Z_2}$$



② $X = F_2 \Rightarrow \widetilde{W}^G_{L^\pi} = -\lambda_2 \log(1 + q^e)$ captures $W_{LF} - W_{GHV}$.

Key Ingredient: Equivariant Lagrangian Correspondence

$G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

$\circlearrowleft \nearrow$ Lag $\quad \cup \quad \cup$

- $L \subseteq \mu^{-1}(0) \to 0 \rightsquigarrow$

$\downarrow \qquad \pi \downarrow$

- $\underline{L} \overset{\text{Lag}}{\subseteq} (X, \overline{\omega})$

- $L^{\pi} \overset{\text{Lag}}{\subseteq} \overline{Y} \times X$

$$\begin{tikzcd} L \subseteq (Y, \omega) \arrow[d, mapsto] \arrow[r, phantom, ""] & (Y,\omega) \arrow[d, "L^{\pi}"] & G \arrow[l, curve] \\ \underline{L} = L^{\pi} \circ L \subseteq (X, \overline{\omega}) & & \end{tikzcd}$$

clean comp.

$\Downarrow$

$L_G \subseteq (Y_G, \omega_G)$ Symplectic Borel space [Kim-Lau-Zheng; Cazassus]

$$\begin{tikzcd} L_G \subseteq (Y_G, \omega_G) \arrow[d, mapsto] \arrow[r, phantom, ""] & (Y_G,\omega_G) \arrow[d, "L^{\pi}_G"] \\ \underline{L} = L^{\pi}_G \circ L_G \subseteq (X, \overline{\omega}) & \end{tikzcd}$$

Equivariant Correspondence Tri-modules

$CF_{can}(\bar{L}) \quad CF_G(L) \quad CF_G(L^\pi)$: A_∞ alg.

Thm [Lau-Leung-L.] $\exists\, (CF_G(\bar{L}; L, L^\pi), \{n^G_{k'',k',k}\})$: A_∞ tri-mod.

Equivariant Correspondence Tri-module

as a G-equivariant extension of Fukaya's Correspondence Tri-module

with $\mathbb{1}_G \in CF^0_G(\bar{L}; L, L^\pi)$

"equivariant bi-cyclic element" $\Rightarrow$ Thm ②

Thank You!