On Mañé's critical value for the two component Hunter-Saxton-System

(Levin Maier, Heidelberg University)

Plan:

1) Background on geometric hydrodynamics
2) Toy model: magnetic geodesics on $\mathbb{S}^3$
3) The (M2HS) as magnetic geodesic eq.
4) Mañé's critical value for (2HS).
5) Key lemma: dynamical reduction to $\mathbb{S}^3$

Background:

Arnold 66: (M, g) closed Riemannian mnfd.

sol. to Euler eq: $\partial_t v + \nabla_v v = -\nabla p$

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Khesin & Misiolek, Lenells:

sol. to HSS eq: $u_{tx} = -\frac{1}{2}u_x^2 - u\,u_{xx} - 2c^2$

$$\Big\updownarrow 1:1$$

sol to $\nabla^{\dot H^1}_{\dot\varphi} \dot\varphi = 0$ on $\left(\mathrm{Diff}_0(S^1), \mathcal{G}^{\dot H^1}\right)$

Newton's equation: $\nabla_{\dot\varphi} \dot\varphi = - \nabla U(\varphi)$

Khesin, Misiolek & Modin 21:

TABLE 1. Examples of Newton's equations.

Wasserstein–Otto geometry	Fisher–Rao geometry
Newton's equations on Diff(M)	
• Inviscid Burgers equation (§4.1)	• μ-Camassa–Holm equation (§8.1)
• Classical mechanics (§4.2)	• Optimal information transport (§7)
• Barotropic inviscid fluid (§4.3)	
• Fully compressible fluid (§6.1)	
• Magnetohydrodynamics (§6.2)	
Newton's equations on Dens(M)	
• Hamilton–Jacobi equation (§4.2)	• ∞-dim Neumann problem (§8.2)
• Linear Schrödinger equation (§9.2)	• Klein–Gordon equation (§8.3)
• Nonlinear Schrödinger (§9.2)	• 2-component Hunter–Saxton (§9.5)

<u>Def.</u>: • (M, g) Riemannian manifold, $\sigma \in \Omega^2(M)$: $d\sigma = 0$

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<u>Rmk.</u>: $\nabla_{\dot\gamma}\dot\gamma = s Y_\gamma \dot\gamma$ is a linear deformation of $\nabla_{\dot\gamma}\dot\gamma = 0$ in the velocity $\dot\gamma$.

<u>Example</u>: • $\mathbb{S}^3 = \{ z \in \mathbb{C}^2 \mid \|z\| = 1 \} \subseteq (\mathbb{C}^2, \underbrace{\langle \cdot, \cdot \rangle}_{\text{Herm. product}})$

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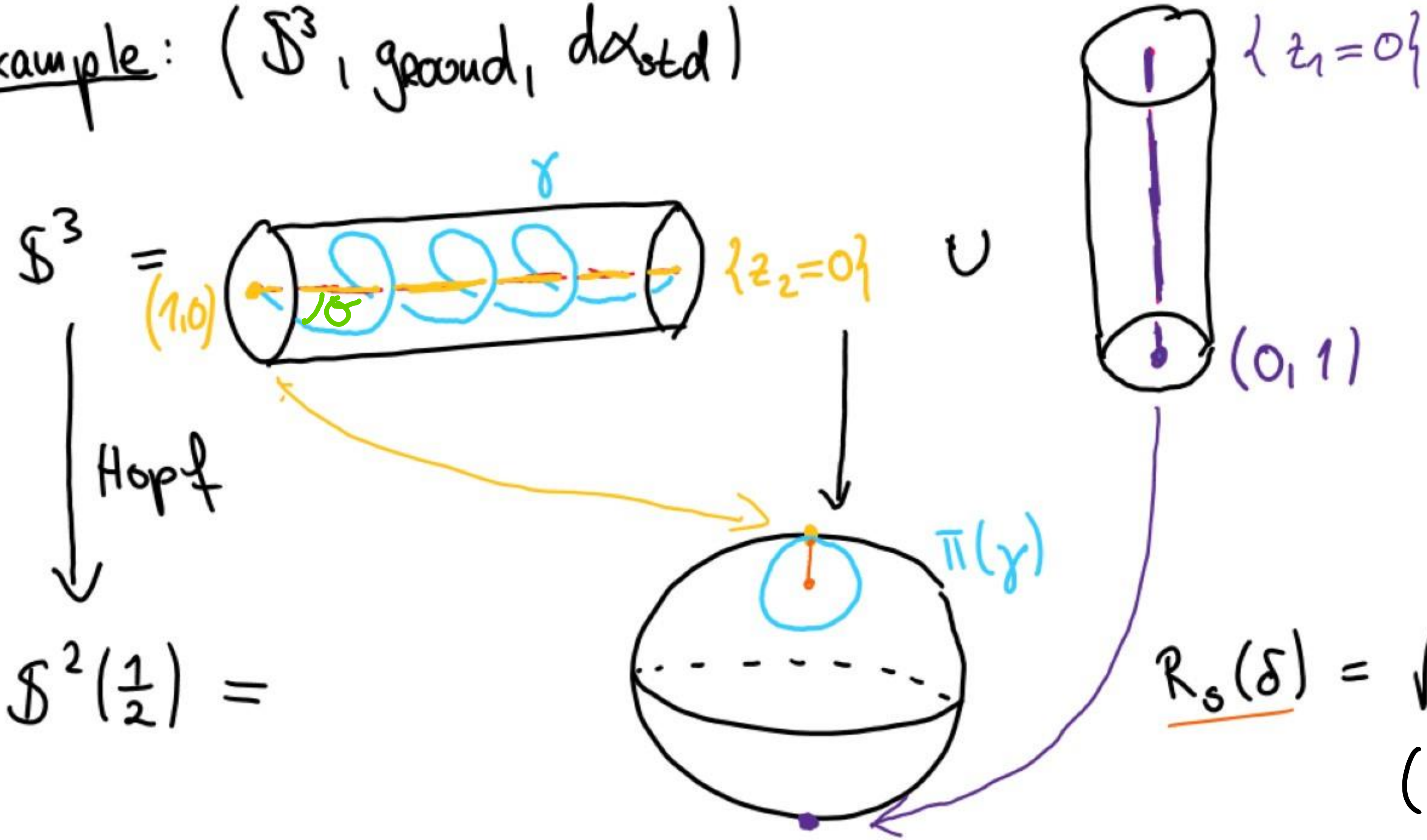
$\Rightarrow \gamma(t) = p_1 e^{i\theta_1 t} + p_2 e^{i\theta_2 t} \quad p_1 \perp_{\mathbb{C}} p_2$ [ABeM24]

Example: $(\mathbb{S}^3, \text{ground}, d\alpha_{std})$



$$R_s(\delta) = \sqrt{\frac{(1-\delta)(1+\delta)}{s^2 + 4(c^2 - s\delta)}}$$

$(\cos\sigma = \delta)$

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Rmk.: • $c = 2$ is Mañé's critical value

• $\Leftarrow$ in 1) follows also from [Con04]

Main Results:

Thm 1 [M24] $\mathcal{G}^m = \mathrm{Diff}_0^{m+1}(\mathbb{S}^1) \ltimes H^m(\mathbb{S}^1, \mathbb{S}^1_{4\pi})$ with $m > \frac{5}{2}$.

Then (φ, τ) is a magnetic geodesic in $(\mathcal{G}^m, \langle \cdot, \cdot \rangle_{\dot{H}^1}, Y^{\dot{H}^1})$ of strength s if and only if $(u = \varphi_t \circ \varphi^{-1}, \rho = \tau_t \circ \varphi^{-1})$ is a solution of

$$\begin{cases} u_{tx} = -\frac{1}{2} u_x^2 - u \cdot u_{xx} + \frac{1}{2}\rho^2 - (s\rho + 2(c^2 - s\delta)) \\ \rho_t = -(\rho u)_x + s\, u_x \end{cases} \qquad (\mathrm{M2HS})$$

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Rmk.:
- (M2HS) is a linear deformation of (2HS).
- $s = 0 \rightarrow (\varphi, \tau)$ geodesic in $\mathcal{G}^m \longleftrightarrow (u, \rho)$ sol. to (2HS) [L13]
- (M2HS) has as (2HS) blow up's.

<u>Thm 2 [M24] :</u> $\mathcal{M}^{\circ}_{AC} = M^{\circ}_{AC} \ltimes L^2(S^1, \mathbb{R})$ where M°_{AC} is defined through:

$$M^{\circ}_{AC} = \{ f \in AC([0,1],[0,1]) \mid f \text{ non decreasing}, f(0)=0 \text{ and } f(1)=1 \}.$$

i) The weak magnetic flow in $\mathcal{M}^{\circ}_{AC}$ exists globally.

ii) If (φ, τ) is a weak magnetic geodesic in $\mathcal{M}^{\circ}_{AC}$ then $(u \circ \varphi = \varphi_t, \rho \circ \varphi = \tau_t)$ is a global weak solution to (M2HS).

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Rmk.: For $s=0$ this was proven by [Wu11].

<u>Thm 3</u> (Mañé's critical value for 2HS)

i) If $s < 2$ there exists for all $(g,h) \in \mathring{\mathcal{M}}_{AC}$ a weak unit speed maignectic geodesic (φ,τ) of strength s in $\mathring{\mathcal{M}}_{AC}$ connecting $(id, 0)$ and (g,h).

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Rmk.: • $c=2$ is Mañé's critical value of (M2HS) weak magnectic flow

"Completion to $\mathcal{M}^\circ_{AC}$"

$$
\begin{tikzcd}
\left(\mathcal{M}^\circ_{AC}, \langle\cdot,\cdot\rangle_{\dot H^1}, \Phi^* d\alpha\right) \arrow[r, bend right, "\Phi"'] \arrow[dd] & \left(\mathcal{U}^\circ, \mathrm{Re}\langle\cdot,\cdot\rangle_{L^2}, d\alpha_{std}\right) \arrow[l, bend right, "\Phi^{-1}"'] \arrow[d, phantom, sloped, "\subseteq"] \\
& \left(S^\infty_{L^2}, \mathrm{Re}\langle\cdot,\cdot\rangle_{L^2}, d\alpha_{std}\right) \\
T_{id}\mathcal{M}^\circ_{AC} & \left(\mathbb{S}^3, \text{ground}, d\alpha_{std}\right) \arrow[u] \arrow[l, dashed]
\end{tikzcd}
$$

weak solutions to
(M2HS)

Further directions

- $c_{HZ}(D, L(p,1), d\lambda) = 2\pi \quad \forall p \in \mathbb{N}$ prime [BM24]
- (M2HS) for closed Riemannian mnfd's (M,g).
- Integrability and well posedness of (M2 HS).
- Conjugate point's on $\mathcal{G}$ w.r.t. magnetic flow.
- Katok examples for PDE's.
- „Dream": ∞-dim c_{HZ} to prove non-squeezing for HS eq.

Example: • $S^{\cancel{3},\infty} = \{ f \in L^2(S^1,\mathbb{C}) \mid \|f\|_{L^2} = 1 \} \subseteq (L^2(S^1,\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2})$

• $g = \mathrm{Re}\langle\cdot,\cdot\rangle|_{TS^\infty}$ round metric

• $(\alpha)_z(\cdot) = \frac{1}{2}\mathrm{Im}\langle z,\cdot\rangle|_{TS^\infty} \Rightarrow d\alpha(\cdot,\cdot) = \mathrm{Im}\langle\cdot,\cdot\rangle|_{TS^\infty}$

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Thm.: γ magnetic geodesic on S^∞ with $\gamma(0) = p$, $\dot\gamma(0) = v$

$\Rightarrow$ γ stays on $S^3(p,v) = S^\infty \cap \langle p,v\rangle_{\mathbb{C}}$

<u>Thm.:</u> [M24]

① $\forall f, g \in \mathcal{S}^\infty$ $\exists$ unit speed magnetic geodesic of strength s connecting f and g in $\mathcal{S}^\infty$ if and only if $s \in [0, 2)$

② $\forall s \geq 2$ $\nexists$ unit speed magnetic geodesic of strength s connecting $f \equiv 1$ and $g \equiv id_{S^1}$ in $\mathcal{S}^\infty$.

<u>Rmk.:</u> ① $c = 2$ is Mañé's critical value for ∞-dim magnetic system.

② can't apply [Con04] anymore.

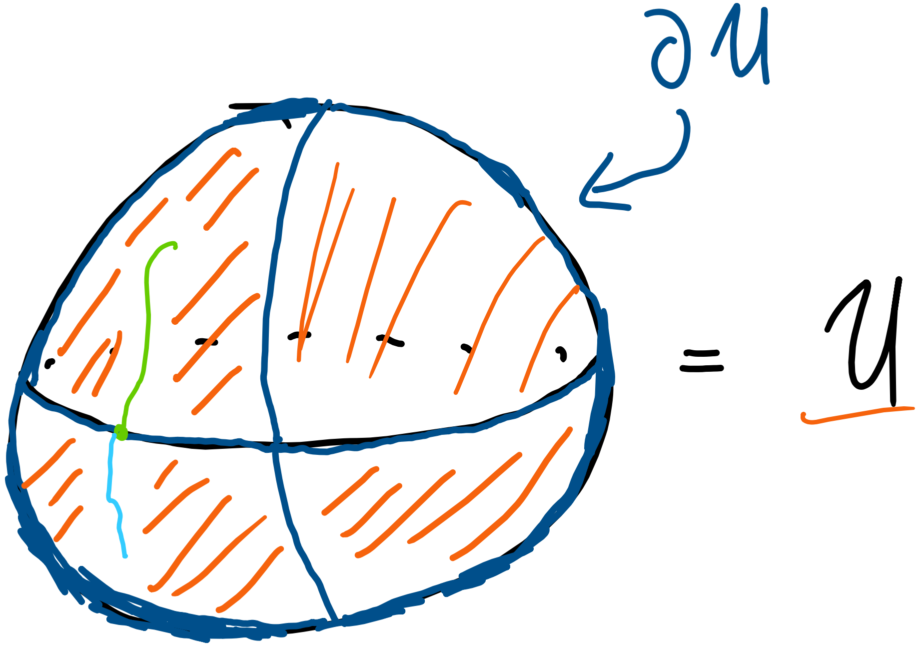
Blow up's and weak solution's:

- $\underline{\Phi(\varphi,\tau)} \cap \partial \mathcal{U} \neq \emptyset$

$\Leftrightarrow$ PDE has blow up

- $\underline{\text{Extend } \Phi(\varphi,\tau) \text{ beyond } \partial \mathcal{U}}$

$\Rightarrow$ leads to global weak solutions.



Weak solutions to (M2HS)

$(u,\rho) : [0,\infty) \times \mathbb{S}^1 \to \mathbb{R}$ is called global weak sol. to (M2HS) with initial data $(\tilde{u},\tilde{\rho}) \in H^1(\mathbb{S}^1,\mathbb{R}) \times L^2(\mathbb{S}^1,\mathbb{R})$ if

1. $u(t,\cdot) \in H^1(\mathbb{S}^1,\mathbb{R}) \quad \forall t \in \mathbb{R}$
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3. $t \mapsto u_x(t,\cdot)$, $t \mapsto \rho(t,\cdot) \in L^\infty([0,\infty), H^0(\mathbb{S}^1,\mathbb{R}))$.
4. $t \mapsto u(t,\cdot) \in AC([0,\infty), H^1(\mathbb{S}^1,\mathbb{R}))$ and for a.e. $t \in \mathbb{R}$

$$\begin{pmatrix} u_t + u\,u_x \\ \rho_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} A^{-1} \partial_x \left(u_x^2 + \rho^2\right) - s \int_0^x \left(\rho - \int_{\mathbb{S}^1} \rho \, dx\right) dx \\ -(\rho u)_x + s\, u_x \end{pmatrix}.$$

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