Persistent Legendrian Contact Homology

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based on joint work with

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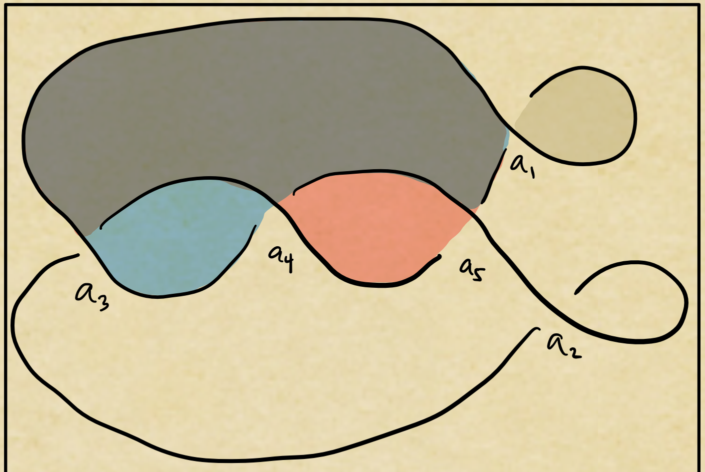
Outline

1. Rapid introduction to Legendrian contact homology
2. Legendrian embeddings and the height filtration.
3. Persistent homology & interleaving distance
4. The continuity of pLCH.
5. Legendrian isotopies

Legendrian contact homology

LCH is an invariant of Legendrian isotopy classes in $(\mathbb{R}^3, \xi_{std})$, computed from the Lagrangian projection $\pi(\Lambda)$ of a generic representative $\Lambda \subset \mathbb{R}^3$ of the isotopy class $\mathcal{L}$.



$|a_1| = |a_2| = 1, \quad |a_3| = |a_4| = |a_5| = 0$

$\partial_\Lambda a_1 = 1 + a_3 + a_5 + a_3 a_4 a_5 \pmod 2$

The Chekanov-Eliashberg dga $(\mathcal{A}_\Lambda, \partial_\mathcal{A})$ of $\Lambda \subset (\mathbb{R}^3, \xi_{std})$ has

- generators : $\pitchfork$ intersection points in $\pi(\Lambda)$
- grading : defined using the rotation of ξ along Λ
- differential: a count of (holomorphic) polygons in $\mathbb{R}^2_{xy}$ with bdry on $\pi(\Lambda)$.

The LCH of $\mathcal{L}$ is then $H_*(\mathcal{A}_\Lambda, \partial_\Lambda)$.

Legendrian contact homology

Like Floer homology, LCH is modeled after Morse homology, with the relevant functional being action (more soon). However, $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is never finite rank, and thus the graded rank of LCH can be infinite.

One fix is linearization. We obtain from any augmentation $\varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ a differential graded vector space $(A_\Lambda, \partial_1^\varepsilon)$, and

$$LCH_*^\varepsilon(\Lambda) := H_*(A_\Lambda, \partial_1^\varepsilon).$$

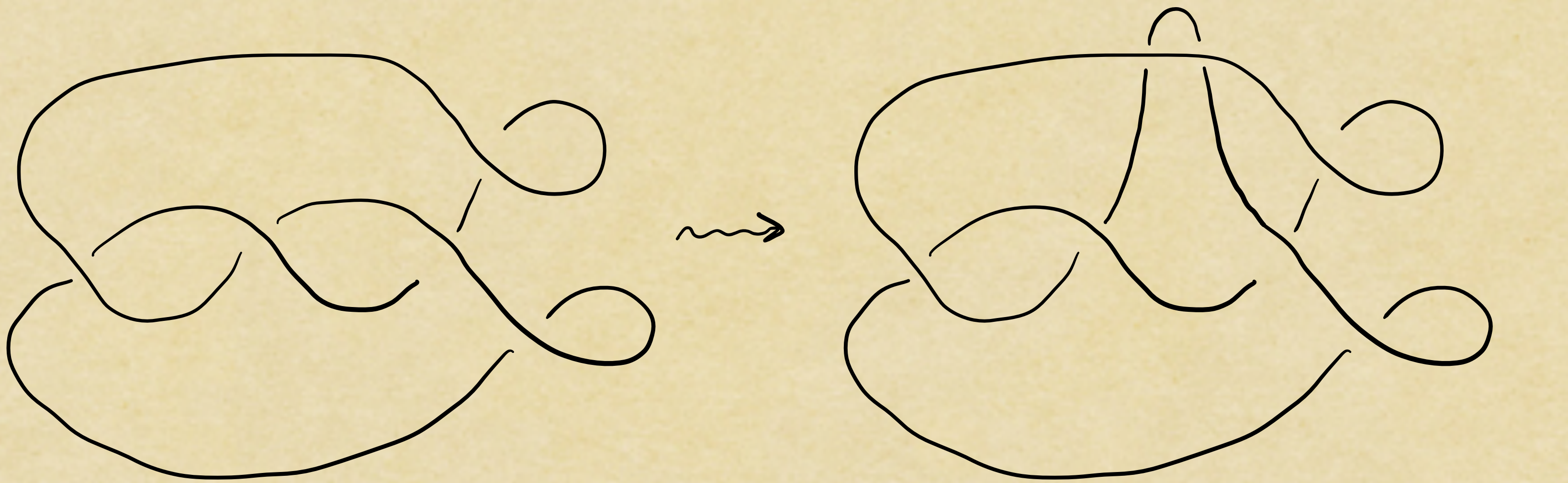
(Caveat: Many Λ fail to admit augmentations.)

__Ex__. The LCH of our standard trefoil admits 5 distinct augmentations, but each has linearization

$$H_k = \begin{cases} (\mathbb{Z}_2)^2, & k = 0 \\ \mathbb{Z}_2, & k = 1 \\ 0, & \text{else} \end{cases}$$

Legendrian embeddings

LCH is invariant under Legendrian isotopies:



The C.E. dga $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is certainly not invariant, and thus we might hope to extract information about Legendrian embeddings from this chain-level data.

(c.f. the Floer chain complex $\underline{CF_*(H,J)}$, whose homology is always $QH^*(M)$.)

filtered by $\mathcal{A}_H$

Legendrian embeddings

For a fixed Legendrian embedding $\Lambda \subset (\mathbb{R}^3, \xi_{std})$, $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is filtered by the action:

$$\mathcal{A} : \left\{ \begin{array}{c} \text{paths in } \mathbb{R}^3 \\ \text{w/ endpoints} \\ \text{on } \Lambda \end{array} \right\} \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto \int_\gamma dz - y\,dx .$$

We can thus define the persistent Legendrian contact homology $F^\bullet LCH_*(\Lambda)$ to be the persistent homology of $(\mathcal{A}^\bullet_\Lambda, \partial^\bullet_\Lambda)$ and analogously define the persistent linearized Legendrian contact homology $F^\bullet LCH^\varepsilon_*(\Lambda)$ for any augmentation ε.

(c.f. Dimitroglu-Rizell — Sullivan)

Persistent homology

An $\mathbb{R}$-filtration on a dga $(\mathcal{A}, \partial)$ is a map

$$h : \mathcal{A} \longrightarrow \mathbb{R}$$

satisfying $h \circ \partial \leq h$. This property ensures that, for any $t \in \mathbb{R}$, $(\mathcal{A}^t, \partial^t)$ is a chain complex, where

$$\mathcal{A}^t = h^{-1}((-\infty, t]) \;\;\&\;\; \partial^t = \partial|_{\mathcal{A}^t}.$$

These complexes are equipped with transfer maps $\psi_s^t : \mathcal{A}^s \longrightarrow \mathcal{A}^t$, $\forall\, s \leq t$, s.t. (1) $\psi_t^t = id_{\mathcal{A}^t}$

(2) $\psi_s^t \circ \psi_r^s = \psi_r^t$, $\forall\, r \leq s \leq t$.

The persistent homology of $(\mathcal{A}, \partial)$ consists of the spaces

$$H_k^t(\mathcal{A}, \partial) := H_k(\mathcal{A}^t, \partial^t)$$

and the induced transfer maps

$$u_s^t : H_*^s(\mathcal{A}, \partial) \longrightarrow H_*^t(\mathcal{A}, \partial).$$

Persistent homology

Ex. (linear version)

$A = \mathbb{Z}_2\langle q_1, q_2, q_3, q_4, q_5\rangle$. $|q_1| = |q_2| = 1$, $|q_3| = |q_4| = |q_5| = 0$.

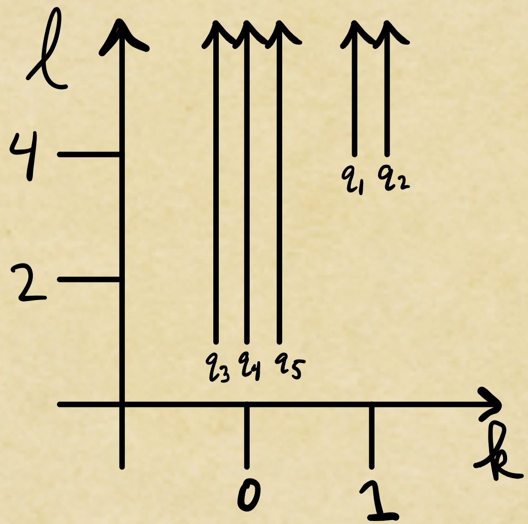
$\partial q_1 = \partial q_2 = q_3 + q_5$, $\partial q_3 = \partial q_4 = \partial q_5 = 0$

Define $h : A \longrightarrow \mathbb{R}$ by $h(q_1) = h(q_2) = 4$

$h(q_3) = h(q_4) = h(q_5) = 1$

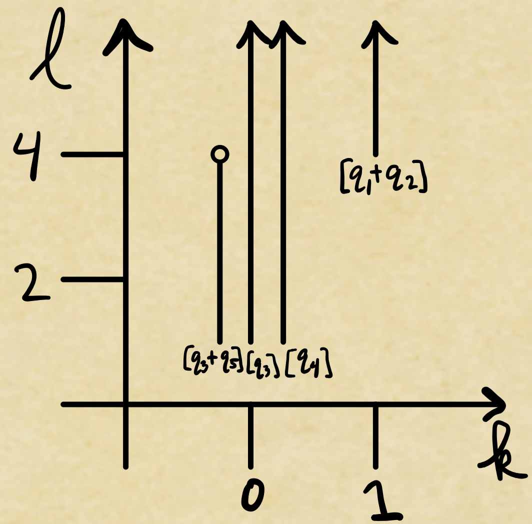
& $h(q + q') = \max\{h(q) + h(q')\}$.

A_k^ℓ



$\rightsquigarrow$

H_k^ℓ



Persistent homology

Persistent homology is an example of a **persistence module**, which is a functor

$$(\mathbb{R}, \leq) \longrightarrow \mathrm{Vect}(\mathbb{R}).$$

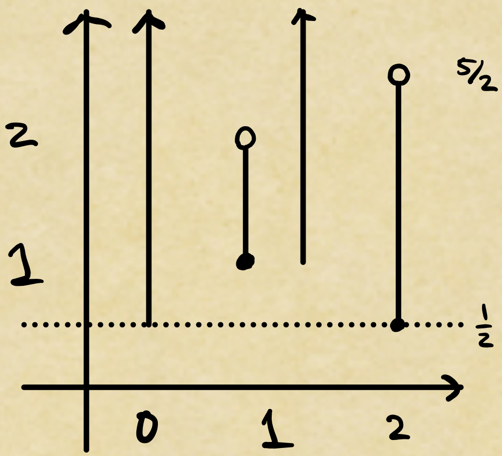
Given persistence modules $U^\bullet$ & $V^\bullet$ with transfer maps u_s^t & v_s^t, a **pMod homomorphism of degree δ** is a collection of maps $\varphi^t : U^t \longrightarrow V^{t+\delta}$ s.t.

$$\begin{tikzcd}
U^s \arrow[r, "u_s^t"] \arrow[d, "\varphi^s"] & U^t \arrow[d, "\varphi^t"] \\
V^{s+\delta} \arrow[r, "v_{s+\delta}^{t+\delta}"] & V^{t+\delta}
\end{tikzcd}$$

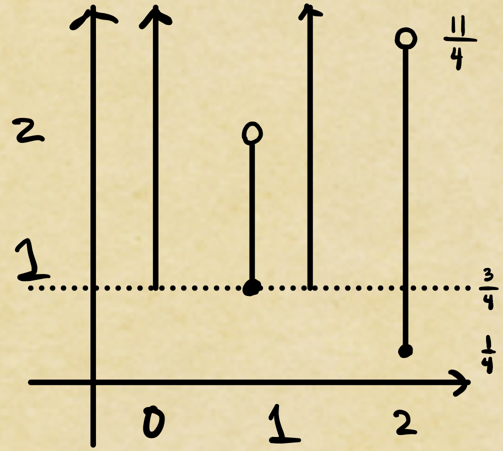
commutes. A **2δ-interleaving** of $U^\bullet$ and $V^\bullet$ is a pair of morphisms $\varphi^t : U^t \to V^{t+\delta}$ & $\psi^t : V^t \to U^{t+\delta}$ s.t. $\psi^{t+\delta} \circ \varphi^t = \mathrm{id}_U^{2\delta}$ & $\varphi^{t+\delta} \circ \psi^t = \mathrm{id}_V^{2\delta}$.

Persistent homology

Ex.



$U^\bullet$



$V^\bullet$

$(2\cdot\frac{1}{4})$-interleaved

The interleaving distance on pMods is defined by

$$d(U^\bullet,V^\bullet) := \inf\{\delta \mid U^\bullet \text{ \& } V^\bullet \text{ are } 2\delta\text{-interleaved}\}.$$

We set $d(U^\bullet,V^\bullet)=\infty$ if no interleaving exists.

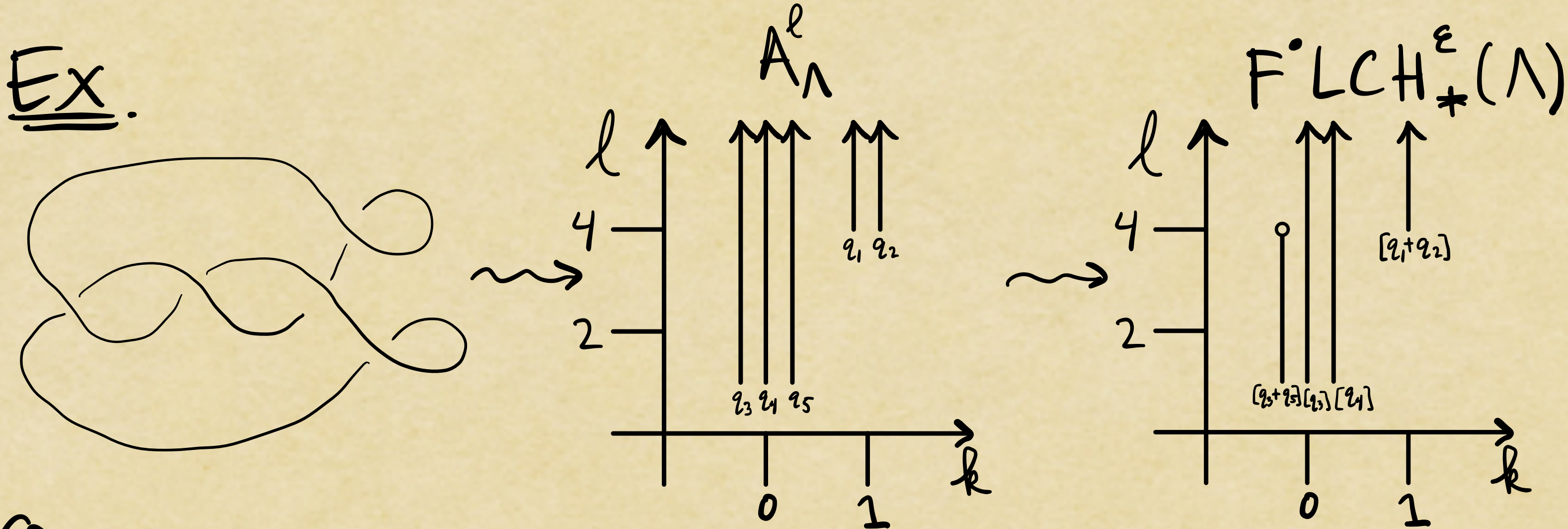
Fact: Interleaving distance is a pseudometric.

$(d(\mathbb{Z}_2[a,b), \mathbb{Z}_2[a,b]) = 0)$

Persistent Legendrian contact homology

The C.E. dga $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is filtered by action, and this filtration is inherited by each linearization $(A_\Lambda, \partial_1^\varepsilon)$. We can now visualize the persistent linearized Legendrian contact homology $F^\bullet LCH_*^\varepsilon(\Lambda)$.

<u>Ex</u>.



<u>Q</u>. Is the map

$$pLCH : \left\{ \begin{array}{c} \text{embedded} \\ \text{Legendrians} \end{array} \right\} \longrightarrow \{ \text{pMods} \}$$

Continuous w.r.t. interleaving distance?

Persistent Legendrian contact homology

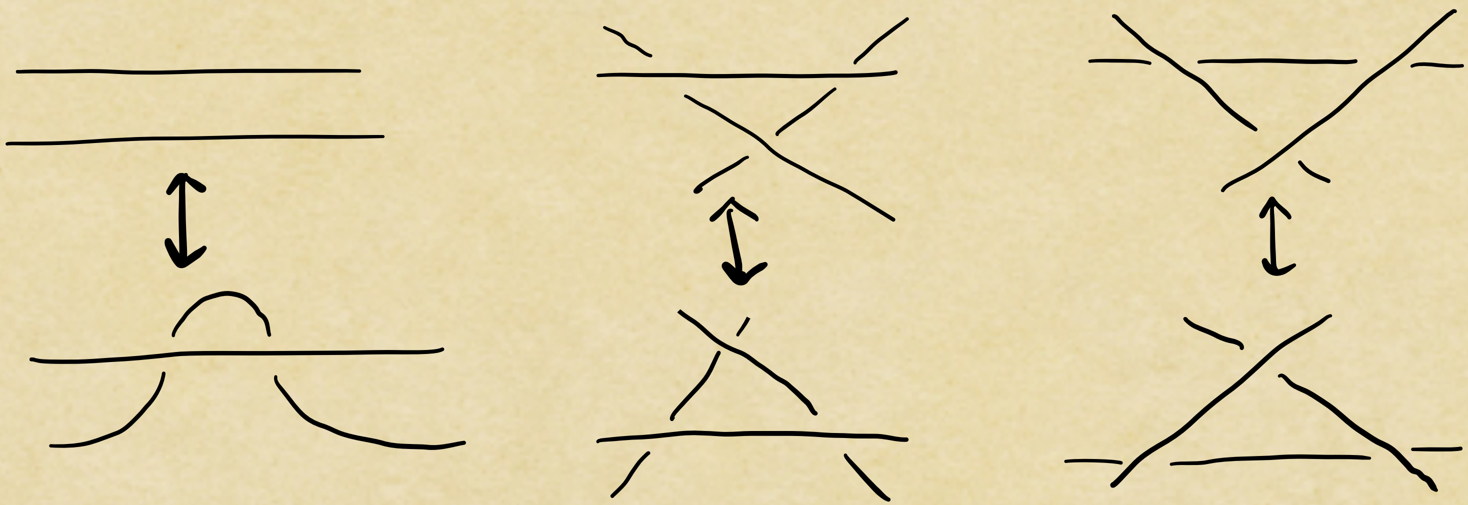
Q. Is the map

$$pLCH : \left\{ \begin{matrix} \text{embedded} \\ \text{Legendrians} \end{matrix} \right\} \longrightarrow \{ \text{pMods} \}$$

Continuous w.r.t. interleaving distance?

Some Legendrian isotopies induce planar isotopies in the Lagrangian projection, and thus affect only the filtration of pLCH.

Others induce Legendrian Reidemeister moves:



Persistent Legendrian contact homology

Q. Is the map

$$pLCH : \left\{ \begin{matrix} \text{embedded} \\ \text{Legendrians} \end{matrix} \right\} \longrightarrow \{ \text{pMods} \}$$

continuous w.r.t. interleaving distance?

Thm. Legendrian Reidemeister moves induce 2δ-interleavings of persistent linearized LCH.

More carefully:

Thm. Let Λ_t, $t \in [0,1]$, be a Legendrian isotopy corresponding to a single Reidemeister move, with $\Pi_{xy}(\Lambda_t)$ a planar isotopy on $[0, t_0)$ & $(t_0, 1]$, for some $t_0 \in (0,1)$. For any $\delta > 0$, $\exists\, \delta' > 0$ s.t.

$$F^{\bullet} LCH_*^{\varepsilon}(\Lambda_{t_0 - \delta'}) \quad \& \quad F^{\bullet} LCH_*^{\varepsilon}(\Lambda_{t_0 + \delta'})$$

are 2δ-interleaved, for any augmentation ε.

Persistent Legendrian contact homology

Q. Is the map

$$\mathrm{pLCH} : \left\{ \begin{matrix} \text{embedded} \\ \text{Legendrians} \end{matrix} \right\} \longrightarrow \{ \text{pMods} \}$$

continuous w.r.t. interleaving distance?

Thm. Legendrian Reidemeister moves induce 2δ-interleavings of persistent linearized LCH.

Cor. Persistent linearized Legendrian contact homology is continuous with respect to interleaving distance.

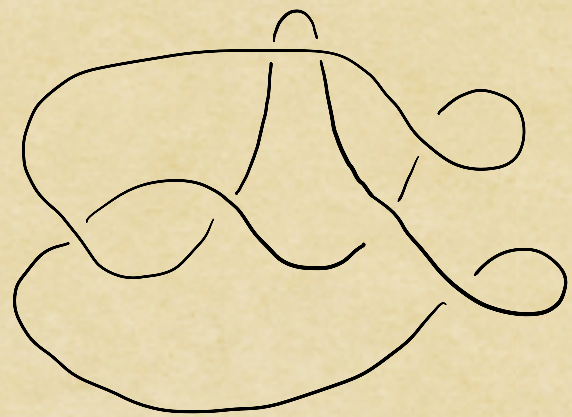
pLCH for isotopy classes

Finally, a few comments on an unrealistic dream.

Though pLCH is a continuous invariant of Legendrian embeddings, Legendrian isotopies can result in long paths in the space of persistence modules.

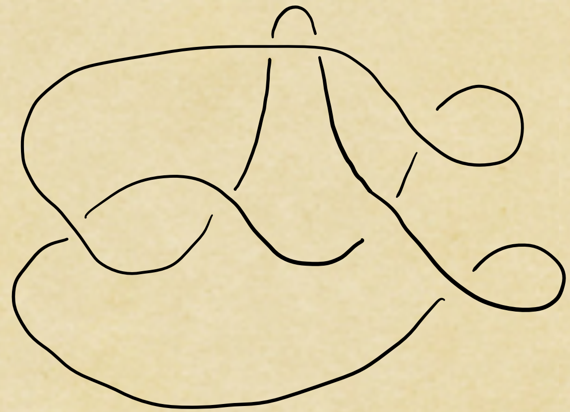
Given a Legendrian isotopy class $\mathcal{L}$, we could ask which persistence modules are realized as $F^{\bullet}LCH_{*}^{\varepsilon}(\Lambda)$, for some $\Lambda \in \mathcal{L}$ and hope that this collection of pMods is an invariant of $\mathcal{L}$.

Because LCH is typically computed from a Lagrangian projection diagram, a first step might be associating a preferred Λ to such a diagram.



pLCH for isotopy classes

A Lagrangian projection diagram determines a linear program whose nonzero solutions correspond to valid height assignments.



By declaring certain parameters for the minimum length of a Reeb chord and minimum area of a region in the diagram, we can lift to a preferred embedding.

Informal conjecture: The pLCH of this lift gives us a means of realizing the Lagrangian crossing number of $\mathcal{L}$, by systematically eliminating very short bars.

Thanks!