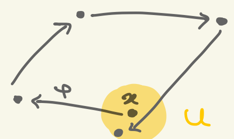
From Gromov-Witten Theory to the closing lemma

joint w. Julian Chaidez.

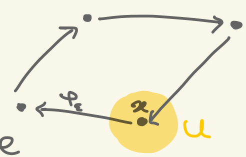
Closing Lemma (dynamical question)	⟵	SFT Spectral gaps (invariants of cobordisms)	⟵	Gromov-Witten inv. (holomorphic curve count)

Q. (Poincaré):



- (M,ω) closed, symplectic
- $\varphi \in Ham(M,\omega)$
- $U \subseteq M$ open.

perturb



Can we "perturb" φ to have a periodic point in U?

Reeb flow analog:

- (Y^{2n-1}, α) contact $(\alpha \wedge (d\alpha)^{n-1} > 0)$
- φ^t Reeb flow $(R \in \ker d\alpha, \alpha(R) = 1)$
- $U \subseteq Y$ open.

Can we perturb α to have a periodic orbit through U?

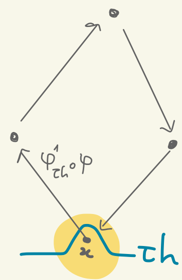
Known.

- Pugh, Pugh-Robinson: Yes, C^1 pert. (60's - 80's)
  "Closing Lemmas"
- Herman: Counter-example, C^∞ pert, dim ≥ 4 (91')
- Irie, ... : Yes, C^∞ pert, dim 2-3. (2015, ...)
  "strong closing lemmas" ← ECH

Def (Irie): $\varphi \in \mathrm{Ham}(M)$ sat. the "strong closing property"

$\forall\ 0 \not\equiv h : [0,1]\times M \to \mathbb{R}_{\geq 0}$

$\exists\ \tau \in [0,1]$ s.t. $\varphi^1_{\tau h} \circ \varphi$ has a periodic pt in $\mathrm{Supp}(h)$.



Conjecture (Irie): Strong closing holds for Reeb flows on ellipsoids, any dim

Proofs:
- Chaidez-Datta-Prasad-T. (CH algebra)
- Xue (KAM normal form)
- Çineli-Seyfaddini (Hamiltonian FH)

Fish-Hofer:

"… the Hamiltonian C^∞ closing lemma is intimately connected to the existence of a sufficiently rich Gromov—Witten theory of the ambient space."

- Hutchings 3D
- Edtmair 2D

Our approach:

closing property $\leftrightsquigarrow$ invariants measuring "change" rather than inv of flow / diffeo.

"change" $\longleftrightarrow$ symplectic cobordism.

Unifying contact & Hamiltonian Settings.

Def. A conformal stable Hamiltonian manifold is (Y, ω, θ) s.t.

- Y^{2n-1} closed
- $\omega \in \Omega^2(Y)$, $\theta \in \Omega^1(Y)$
- $\theta|_{\ker \omega} > 0$, $d\theta = c_Y \cdot \omega$ "conformality const." ≥ 0.

"Reeb vf": $R \in \ker \omega$, $\theta(R) = 1$.

$\rightsquigarrow$ φ^t flow on Y.

Examples:

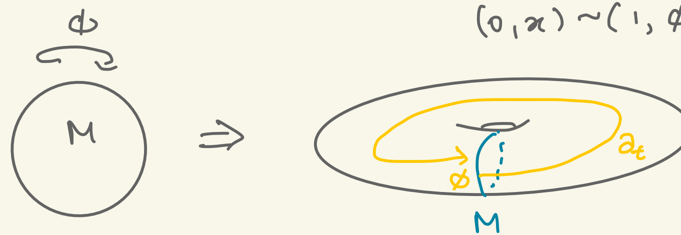
① (Y, α) contact, $\omega = d\alpha$, $\theta = \alpha$

$\Rightarrow c_Y = 1$. $\hat{\omega} = d(e^r \alpha)$

② (M^{2n-2}, ω) closed, $\phi \in \mathrm{Ham}(M, \omega)$

Mapping torus: $M_\phi := [0,1]_t \times M / \sim$

$(0, x) \sim (1, \phi(x))$

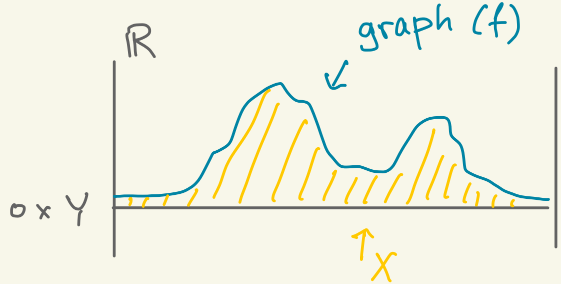


$Y = M_\phi^{2n-1}$, ω lifts to Y, $\theta = dt$

$d\theta = 0 \Rightarrow c_Y = 0$. $\hat{\omega} = d(r\theta) + \omega$.

Made with Goodnotes

Symplectization: $\mathbb{R}_r \times Y$ has a "canonical" symplectic form $\hat{\omega}$



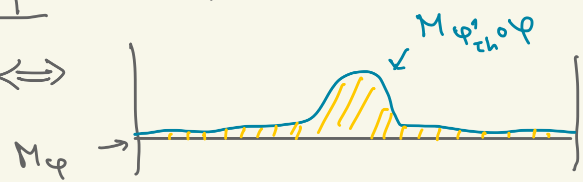
Conformal cobordisms:

Example: Subgraph cobordism:

$$f : Y \to \mathbb{R}_{>0} \rightsquigarrow (X^{2n}, \hat{\omega})$$

Subexample: $\varphi \in Ham(M)$ perturb $\Rightarrow$

$\varphi^1_{th} \circ \varphi \iff$



SFT Spectral gaps:

min-max measurements of areas of hol. curves in cobordisms.

Def. Let $\sigma = (g, m, A)$, $T > 0$

period

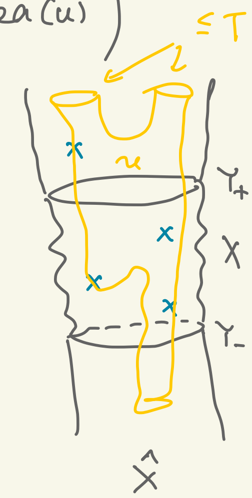
genus

\# pts > 0

$\in H_2(X; \partial X)$

$$g_{\sigma,T}(X) := \sup_{J, P} \left(\min_u \operatorname{area}(u) \right)$$

- J "compatible" a.c.s
- P set of m points
- $u : \Sigma_{\leq g} \to \hat{X}$
  - J hol
  - $u_*[\Sigma] = A$
  - $P \subseteq \operatorname{im}(u)$
  - total period top end $\leq T$



"smallest area required to pass through m points for arbitrary a.c.s"

Remark: period bound is for.

① techical (compactnees)

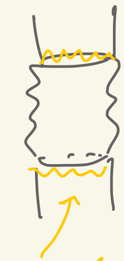
② quatitative closing

Def (Y, ω, θ) conformal

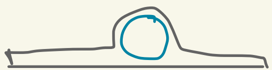
$$g_{\sigma,T}(Y) := \lim_{\varepsilon \to 0} g_{\sigma,T}([-\varepsilon, 0] \times Y)$$

Thm: The gaps satisfy:

① Hofer continuity: $g_{\sigma,T}$ vary continuisly in X wrt "Hofer metric" on cob/mflds.



② "monotonicity":



$$\forall \sigma, T \quad c_G(X) \leq g_{\sigma,T}(X)$$

③ Spectrality:

$$g_{\sigma,T}(X) \in A\,Spec_T(X)$$

"

areas of surfaces in X with bdry on orbit-sets of period $\leq T$

$$\simeq Spec_T(Y_+) - Spec_T(Y_-) + \omega \cdot H_2(X).$$

Application for closing

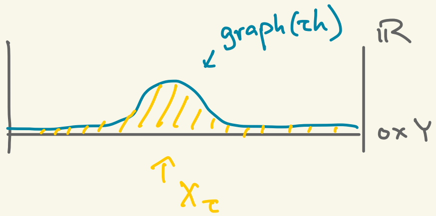
<u>Thm</u>: if $\inf_{\sigma,T} g_{\sigma,T}(Y) = 0$

and $[\omega] \in H^2(Y;\mathbb{Q})$ ← Fails for Herman.

Then Y sat. strong closing.

<u>Sketch</u>:

$0 \not\equiv h : Y \to \mathbb{R}_{\geq 0}$



- $g_{\sigma,T}(X_0) \approx g_{\sigma,T}(Y) \underset{\exists \sigma,T}{\approx} 0$
- $g_{\sigma,T}(X_1) > 0$ depends on h.
- continuity + spectrality + $H^2(Y;\mathbb{Q})$

no orbit $\Rightarrow g_{\sigma,T}(X_\tau) = g_{\sigma,T}(X_0)\ \forall \tau$.

contradiction ▢

Computing gaps: GW invariant.

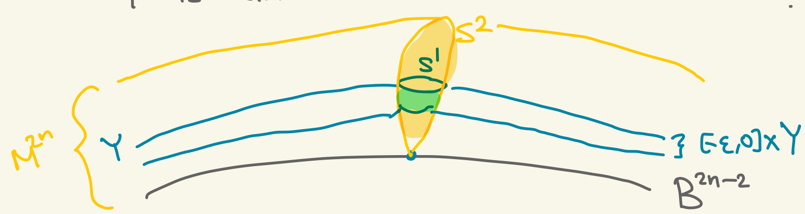
<u>Q</u> for which Y can we show

$$\inf_{\sigma,T} g_{\sigma,T}(Y) = 0 \ ?$$

<u>idea</u>: embed Y^{2n-1} into a closed mfld, use GW inv to produce curves in $[\varepsilon,0] \times Y$.

<u>Example</u>: Suppose the flow on Y generates a free S^1 action (Zoll).

$\Rightarrow Y$ is an S^1 bundle over $B^{2n-2} = Y/S^1$.



Complete Y to an S^2-bundle

(M^{2n}, Ω) over B^{2n-2}.

$$[-\varepsilon, 0] \times Y^{2n-1} \hookrightarrow M^{2n} \quad \text{when } \Omega \cdot [S^2 \times pt] \gtrsim \varepsilon$$

fiber

<u>Lemma:</u> Suppose $GW_{\sigma'}(M; [S^2 \times pt]) \neq 0$.

then

$$g_{\sigma, T_0}([-\varepsilon, 0] \times Y) \leq \Omega \cdot [S^2 \times pt]$$