Invariant Sets and Hyperbolic Periodic Orbits of Reeb Flows

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Motivation: Hyperbolic Periodic Orbits \(\implies\) Interesting Dynamics

**Phenomenon:** In some instances, the presence of one or several *hyperbolic* or even *locally maximal* periodic orbits forces a Hamiltonian system to have interesting dynamics.

**Some examples** (for Hamiltonian diffeomorphisms):

- **Homoclinic intersections:** A hyperbolic periodic orbit with transverse homoclinic intersections \(\Rightarrow\) a horseshoe, positive entropy, etc. Note: This is a \(C^1\)-generic condition (Hayashi '97, Xia '96).

- **Spectral norm:** Sufficiently many hyperbolic periodic orbits of \(\varphi \Rightarrow\) a lower bound on the spectral norm \(\gamma(\varphi^k) > \epsilon > 0\), \(\forall k \in \mathbb{N}\); Çineli–G.–Gürel, arXiv:2207.03613 and arXiv:2310.00470. Note: This is a \(C^\infty\)-generic condition.
Motivation

- **Multiplicity**: A hyperbolic fixed point of \( \varphi: \mathbb{C}P^n \to \mathbb{C}P^n \) \( |\text{Per}(\varphi)| = \infty \), G.–Gürel ’14.

  Closely related: Franks Theorem (Franks ’92, ’96): \( |\text{Per}(\varphi)| = 2 \) or \( \infty \) for \( \varphi: S^2 \to S^2 \). Generalizations to \( \mathbb{C}P^n \) – the Hofer–Zehnder conjecture: “\( |\text{Per}(\varphi)| > n + 1 \Rightarrow |\text{Per}(\varphi)| = \infty \)” (Shelukhin 22’).

- **Invariant sets**: Moreover, a *locally maximal* fixed point of \( \varphi: \mathbb{C}P^n \to \mathbb{C}P^n \) \( |\text{Per}(\varphi)| = \infty \), G.–Gürel ’18.
  Def: Locally maximal = isolated as an invariant set; e.g., hyperbolic fixed point is locally maximal.

  **Corollary**: for a *Hamiltonian pseudo-rotation (PR)* of \( \mathbb{C}P^n \) no fixed point is locally maximal.
  Def: \( \varphi: \mathbb{C}P^n \to \mathbb{C}P^n \) is a PR if \( |\text{Per}(\varphi)| = n + 1 \).


**Goal**: Analogs of the last two results to Reeb flows on \( S^{2n-1} \).
Main results: Setting

**Mainly interested in:** The contact sphere \((S^{2n-1}, \alpha)\); \(\text{ker} \, \alpha = \) the standard contact structure; \(\varphi^t = \) the Reeb flow of \(\alpha\). Think of \((S^{2n-1}, \alpha)\) as the boundary of a star-shaped domain \(W \subset \mathbb{R}^{2n}\).

**Closed Reeb orbits:** \(\mathcal{P} = \mathcal{P}(\alpha)\) is the collection of closed Reeb orbits; \(\mathcal{\hat{P}}\) is the set of simple closed Reeb orbits.

**Dynamical Convexity (DC):** \(\mu(x) \geq n + 1\) for all \(x \in \mathcal{P}\), where \(\mu\) is the lower semi-continuous extension of the Conley–Zehnder index (Hofer–Wysocki-Zehnder ’98). Often weaker requirements of this type suffice. Ubiquitous in proofs in higher dimensions.

**Remark:** Convexity \(\Rightarrow\) DC; but a DC hypersurface in \(\mathbb{R}^{2n}\) need not be symplectomorphic to a convex hypersurface (Chaidez–Edtmair ’22; Cristofaro-Gardiner–Hind ’23; Dardennes–Gutt–Ramos–Zhang ’23).

**Many counterparts of the proof work in a more general setting:** \(M = \partial W^{2n}\) where \((W, \alpha)\) is a Liouville domain, etc.
Main results: Multiplicity

**Notation:** \( \hat{\mu}(x) := \lim_{k \to \infty} \mu_-(x^k)/k \) is the mean index of \( x \); \( 2\nu(x) \) is the algebraic multiplicity of the eigenvalue 1 of the Poincaré return map of \( x \).


Assume that \((S^{2n-1}, \alpha)\) has a hyperbolic (simple) closed Reeb orbit \( z \) with \( \hat{\mu}(z) > 0 \) and

\[
\mu(x) \geq \max \{ 3, 2 + \nu(x) \} \quad (DC \text{ type condition}) \tag{1}
\]

for all \( x \in \mathcal{P}(\alpha) \) with \( \hat{\mu}(x) > 0 \). Then the Reeb flow of \( \alpha \) has infinitely many simple periodic orbits: \( |\mathcal{P}(\alpha)| = \infty \).

**Remark:** DC \( \Rightarrow \) (1). As a consequence: DC + a hyperbolic orbit \( \Rightarrow \) \( |\mathcal{P}(\alpha)| = \infty \). **Note:** No non-degeneracy conditions.
Main results: Invariant sets

**Theorem B** (ÇGGM, arXiv:2401.01421)

Assume that \((S^{2n-1}, \alpha)\) is DC, non-degenerate and its Reeb flow has only finitely many simple closed orbits (aka Reeb PR). Then no closed orbit is locally maximal, i.e., isolated as an invariant set.

**Remark:** Hyperbolic closed orbits are locally maximal. Hence, Theorem B is almost Theorem A up to non-degeneracy and a stronger DC type condition.

**Remark:** Reeb PR’s can have interesting dynamics: \(\exists\ C^\infty\)-small ergodic PR perturbations of irrational ellipsoids (Katok ’73; Albers–Geiges–Zehmisch ’22).
Main results: Bonus – Reeb barcode entropy

More general setting:

- A Liouville domain \((W, \alpha)\); Reeb flow \(\varphi^t\) on \(\partial W\).
- The filtered symplectic homology (non-equivariant, ungraded) persistence module \(SH(W) := \{SH^s(W) \mid s \in \mathbb{R}\}\).
- \(b_\epsilon(s) = |\{ \text{bars} > \epsilon \ \text{beginning} < s\}|\).
- The \(\epsilon\)-barcode entropy and barcode entropy of \((W, \alpha)\)

\[
\bar{h}_\epsilon(W) := \limsup_{s \to \infty} \frac{\log^+ b_\epsilon(s)}{s} \quad \text{and} \quad \bar{h}(W) := \lim_{\epsilon \to 0^+} \bar{h}_\epsilon(W) \in [0, \infty],
\]

where \(\log^+ = \max\{\log, 0\}\).
Main results: Bonus – Reeb barcode entropy

**Theorem:** $\bar{h}(\alpha) \leq h_{\text{top}}(\varphi)$ (Fender–Lee–Sohn '23). In particular, $\bar{h}(\alpha) < \infty$.

**Theorem C (ÇGGM, arXiv:2401.01421)**

Let $K \subset \partial W$ be a compact hyperbolic invariant set of $\varphi^t$. Then

$$h_{\text{top}}(\varphi|_K) \leq \bar{h}(W).$$

Combining these two theorems with the results of Lian–Young '12 or Lima–Sarig '19 extending Katok '80 to flows, we have

**Corollary (ÇGGM, arXiv:2401.01421)**

Assume that $\dim \partial W = 3$. Then $\bar{h}(W) = h_{\text{top}}(\varphi)$. 
Discussion and context: Reeb flows in 3D

**Disclaimer:** Theorems A and B are mainly of interest when \( \text{dim} > 3 \).

**Multiplicity in 3D has been extensively studied and well understood.** The 2-or-\( \infty \) conjecture has been proved for most of Reeb flows in 3D: Hofer–Wysocki–Zehnder ’98, Cristofaro-Gardiner, Hutchings, Ramos, Pomerleano, Hryniewicz, Liu ’16–’23, Colin–Dehormoy–Rechtman ’23. Nothing as precise as that is true when \( \text{dim} > 3 \). The (expected) orbit bounds depend very much on the underlying contact manifold and much less is known even for \( S^{2n-1} \geq 5 \).

**Invariants sets in 3D:** Theorem B in 3D \( \iff \) the Franks–Le Calvez–Yoccoz theorem (2D); for the latter theorem is in fact local.

**Related result in a similar spirit:** In 3D, the union of proper closed invariant sets is dense (Cristofaro-Gardiner–Prasad 24’). This does not follow from the Franks–Le Calvez–Yoccoz theorem and the proof also implies Theorem B in 3D.
Discussion and context: Multiplicity for $S^{2n-1}_{\ge 5}$

The question originates in classical mechanics and calculus of variations (Lyapunov, Moser, Rabinowitz, Weinstein, Ekeland, ...).

**Conjecture**: For a Reeb flow on the standard contact $S^{2n-1}$ either $|\mathcal{P}| = n$ and all orbits are elliptic or $|\mathcal{P}| = \infty$ and at least one of the orbits is degenerate or not elliptic. (Along the lines of the Reeb HZ Conjecture aka the Reeb Franks “Theorem”.)

**Comment**: A long shot given how little is known! Theorem A is one of the first steps in the “or” direction.

**Unknown**: If the Reeb flow on the standard contact $S^{2n-1}_{\ge 5}$ must have $> 1$ simple closed Reeb orbits or $> 2$ in the non-degenerate case, without a DC type index condition! (Nondegeneracy $\Rightarrow |\mathcal{P}| \ge 2$; Gürel ’15; Abreu–Gutt–Kong–Macarini ’19, ... .)
Discussion and context: Multiplicity for $S^{2n-1}\geq 5$

**Lower bounds on $|\mathcal{P}|$ with index requirements — Extensively studied:**

- DC type conditions + non-degeneracy $\Rightarrow |\mathcal{P}| \geq n$.
- DC type conditions without non-degeneracy $\Rightarrow |\mathcal{P}| \geq \sim n/2$; improvements in lower dimensions... .

**Credits:** Breakthrough: Long–Zhu ’02. Then in various combinations: Long, Liu, Wang, Hu ’02–’24; Gutt–Kang ’16; Abreu, Macarini, Gürel, G. ’16–’19; ... .

**Related work:** Some upper bounds for “perfect” flows on the sphere and other manifolds; multiplicity results for other manifolds, the contact Conley conjecture, ... .
Discussion and context: Invariant sets in $\dim > 3$

**Theorem B** is the first result of this type. Nothing else seems to be known. No general conceptual picture.

**Somewhat related work:** No hypersurfaces in $\mathbb{R}^4$ with minimal characteristic flow (Fish–Hofer ’23) + refinements (Prasad ’24); Invariant probability measures (Prasad ’21); No hypersurfaces in $\mathbb{R}^{2n}$ with uniquely ergodic characteristic flow (G.–Niche ’15).
Discussion and context: Barcode entropy

Some related results and constructions:

Barcode entropy:
- Barcode entropy for Hamiltonian diffeomorphisms: ČGG ’21–’23
- Barcode entropy for geodesic flows: GGM ’23
- Barcode entropy for Reeb flows: Fender–Lee–Sohn ’23, Fernandes ’24
- Relation of categorical entropy to $h_{\text{top}}$: Bae–Lee ’22
- Lower semicontinuity of Lagrangian volume: ČGG ’22
- Triangulated persistence categories: Biran–Cornea–Zhang ’22, ’23

$b_\epsilon$: In some other settings, $b_\epsilon$ carries useful geometrical info:
About proofs: Background

Three main ingredients:
- Boundary depth upper bound
- Crossing energy lower bound – The key new ingredient (Çineli)
- Index recurrence (IR)

Need to work with specific Hamiltonians rather than symplectic homology and things get a bit technical.
About proofs: Background

**Convenient choice:** *Semi-admissible* Hamiltonians.

![Figure 1: A semi-admissible Hamiltonian](image)

**Fact:** \( \mathcal{S}H^\tau(W) \cong H^f(\tau)(H) \) where \( f(\tau) \approx \tau \) when \( \tau \ll \text{slope}(H) \).
About proofs: Boundary depth upper bound

**Notation:**  \( \text{SH}^\infty(W) \) is the total symplectic homology, i.e., the action range is \([0, \infty)\); e.g., \( \text{SH}^\infty(W) = 0 \) when \( W \) is displaceable (Viterbo ’99, Cieliebak–Frauenfelder–Oancea ’10, Sugimoto ’16, ...); \( \beta_{\text{max}}(W) \) is Usher’s boundary depth, i.e., the maximal bar in \( \text{SH}(W) \).

**Theorem (Irie, Shon–G. ’18):** \( \text{SH}^\infty(W) = 0 \Rightarrow \beta_{\text{max}} < \infty \).

**Remark:** Upper bound = non-equivariant \( SH \)-capacity. In fact, we need a more precise result:

**Theorem (ÇGGM ’23):** Assume that \( \text{SH}(W) = 0 \). Fix \( a > 0 \) and let \( H \) be a semi-admissible Hamiltonian with \( \text{slope}(H) > a \). Then there exists a constant \( C > 0 \) depending only on \( H \) such that for every sufficiently large \( k \in \mathbb{N} \) and any \( \tau < ka \) the inclusion/quotient map

\[
\text{HF}^\tau(kH) \to \text{HF}^{\tau+C}(kH)
\]

is zero. Hence, every bar \( I \) ending \( < ka \) has \( |I| < C \). (Note: \( \text{HF}^\infty(kH) \neq 0 \).)
About proofs: Crossing energy

Ingredients:
- $z$ is a locally maximal (e.g., hyperbolic) closed Reeb orbit of period $T$.
- $H$ is semi-admissible with $\text{slope}(H) > T$.
- $\tilde{z}$ is the corresponding orbit (never locally maximal) of $H$.
- Iterated orbits – $z^k$ and $\tilde{z}^k$. Note: $\tilde{z}^k$ is a one-periodic orbit of $kH$.
- An admissible almost complex structure.

Theorem (Crossing Energy, ÇGGM 2309.04576): Under a minor additional requirement on $H$, there exists $\sigma > 0$ such that $E(u) \geq \sigma$ for any $k \in \mathbb{N}$ and any Floer cylinder $u : \mathbb{R} \times S^1 \to \mathcal{W}$ of $kH$ asymptotic, at either end, to $\tilde{z}^k$.

Remark: A similar result for periodic orbits $z$ in a locally maximal hyperbolic set of the Reeb flow (ÇGGM, arXiv:2401.01421) ⇒ applications to barcode entropy (Theorem C).
About proofs: Crossing energy

Key point of the proof (Çineli): $u$ cannot get too close to $W$ in $\hat{W}$!

Fig 2: Key point: $u$ stays away from $W$.

Remark: This is a new result and it does not follow from any previously known fact about the behavior of Floer cylinders in $\hat{W}$. 
About proofs: Index recurrence

**Setting and notation:** $r$ non-degenerate elements $\Phi_1, \ldots, \Phi_r$ in $\widetilde{Sp}(2n)$ with positive mean indices $\hat{\mu} (\Phi_i) > 0$. Set $\mu_i(k) := \mu(\Phi_i^k)$ for $k \neq 0$ ...

**Index Recurrence Theorem – Non-degenerate Version; GG ’20):** For every $N > 0$ (large) and every $\epsilon > 0$ (small), there exist $r$ integer sequences $k_{ij} \to \infty$ as $j \to \infty$ and $i = 1, \ldots, r$, and an integer sequence $d_j \to \infty$ such that for every $1 \leq |\ell| \leq N$

(i) $|\hat{\mu}_i(k_{ij}) - d_j| < \epsilon$ and

(ii) $\mu_i(k_{ij}) = d_j + \mu_i(\ell)$.

**Explanation:** Arbitrary long segments $[\mu_i(-N), \ldots, \mu_i(N)]$ (with $\mu_i(0)$ deleted) repeat themselves infinitely many times in the sequences $\mu_i(k)$ up to a common index shift $d$; in the derivative sequence $\mu_i(k) - \mu_i(k - 1)$ every interval repeats itself infinitely many times. Hence, recurrence! An **IR event:** $\{d_j, k_{1j}, \ldots, k_{rj}\}$.

**Closely related:** The common jump theorem; Long–Zhu ’02, ... .
About proofs: Index recurrence

We need a very particular case of the IRT.

**Corollary:** Assume that all $\Phi_i$ are dynamically convex: $\mu(\Phi_i) \geq n + 1$. Then there exist $r$ integer sequences $k_{ij} \to \infty$ as $j \to \infty$ and $i = 1, \ldots, r$, and an integer sequence $d_j \to \infty$ such that

(i) $|\mu_i(k_{ij}) - d_j| \leq n - 1$ and $|\hat{\mu}_i(k_{ij}) - d_j| \leq \epsilon$, and

(ii) $|\mu_i(k_{ij}) - \mu_i(k)| \geq n + 1$ when $k \neq k_{ij}$.

![Diagram](image)

**Fig 3:** Indices: an IR event in a DC setting.
About proofs: Outline

**Simplifying assumptions:**
- Non-degeneracy and DC.
- Working with $\text{SH}(W)$ including crossing energy rather than $\text{HF}(H)$.
- Focus on Theorem A.

**Theorem (a weaker version of Theorem A):** A non-degenerate dynamically convex Reeb flow on $S^{2n-1}$ with a hyperbolic closed Reeb orbit has infinitely many closed Reeb orbits.
Generators of the complex $\operatorname{CSH}(W)$ where $W$ is a star shaped domain filling of $(S^{2n-1}, \alpha)$: Two generators $\check{y}$ and $\hat{y}$ with $|\check{y}| = \mu(y)$ and $|\hat{y}| = \mu(y) + 1$ for every $y \in \mathcal{P}$ and one generator of degree $n$ for the interior of $W$.

By contradiction, assume that $\mathcal{P}$ is finite: $\mathcal{P} = \{x_0 = z, x_1, \ldots, x_r\}$ with actions $a_0, \ldots, a_r$; $z$ is hyperbolic. Can assume $a_0/\hat{\mu}(z) = 1$.

Consider an IR event: $\{d, k_0, \ldots, k_r\}$ suppressing $j$ (large!) in the notation. Note: $d = \check{\mu}(z) = \mu(z)$.

**Key observation:** $\check{z}^{k_0}$ is a non-exact cycle in $\operatorname{CSH}(W) \Rightarrow \operatorname{SH}(W) \neq 0 \Rightarrow$ contradiction.
Two groups of orbits:

- Group I: \(a_i/\hat{\mu}(x_i) = a_0/\hat{\mu}(z) = 1\); action close to \(d\).
- Group II: \(a_i/\hat{\mu}(x_i) \neq a_0/\hat{\mu}(z)\); action far from \(d\).

No differential connecting to \(\check{z}\):

- Iterates of Group I within an IR event: action difference is too small (Energy Crossing).
- Iterates of Group II within an IR event: action difference is too large (Upper bound on the boundary depth).
- Other iterates: index difference \(> 1\).
About proofs: Outline

Visualizing an IR event on the action/index plane:

Fig 4: IR event on the index/action plane.
Thanks!