Bordism of flow modules & exact Lagrangians

Consider a Weinstein manifold \((X, \omega = d\theta)\), & compact exact Lagrangians

\(L, K \subseteq X\).

Classical: If \(L, K\) are "indistinguishable to Floer theory" i.e. \(L = K\) in \(\mathcal{F}(X)\) then \([L] = [K] \in H_n(X; \mathbb{Z}/2)\) or in \(H_n(X; \mathbb{Z})\) if oriented & spin.

Idea of proof:

\[ L \cong K \Rightarrow \exists \alpha \in C^\infty(L, K), \beta \in C^\infty(K, L) \text{ s.t. } \alpha \beta = \text{id}_K, \beta \alpha = \text{id}_L \]

Study the moduli space of holes & strips.

Degenerations

\[ \Theta^K \rightarrow K \] \[ \Theta^L \rightarrow L \]

Disc bubbles are constant by exactness & sweep \(L, K\).
\[
\begin{pmatrix}
\text{CF}(L, K) \\
\oplus \\
\text{CF}(K, L)
\end{pmatrix}
\xrightarrow{\mu_2}
\begin{pmatrix}
\text{CF}(L, L) \\
\oplus \\
\text{CF}(K, K)
\end{pmatrix}
\xrightarrow{\phi_C}
H_*(X)
\]

"length two Hochschild cx"

Note \( TL \otimes C \cong TX \big|_L \) so Pontrjagin numbers of \( L \) are determined by Chern classes 

\[\varphi(L) \otimes [L] = [\lambda] \] 

so \( p_i(L) \to p_i(K) \)

under 

\[ H^*(L; \mathbb{Z}) \cong H^*(K; \mathbb{Z}) \times \mathbb{Z} \]

Motivating Question: 

Can we draw stronger conclusions, e.g. if \( L, K \) are framed, can we deduce constraints on their classes in \( H_*(X) \)?
Natural strategy: (a) Build (Donaldson-) Fukaya category over spectra $E$

(b) Prove $L = \mathcal{F}(X; E)$

\[ \text{But you still have to do this!} \]

New idea: Obstruction theory for lifting quasi-isomorphisms $Z$ to ones on $S$ (the sphere spectrum)

\[ \text{SETTING: Assume } TX \text{ is stably trivial as a } \mathbb{C}^r\text{-bundle. An exact Lag. has a } \]

\[ \text{stable Gauss map } L \to \mathcal{W}_0 \text{; objects of } \mathcal{F}(X; S) \text{ will be graded Lag.}
\]

\[ \text{with a nullhomotopy of } (\star) \text{.} \]

\[ \text{New idea: Stable Lag. Grassmanian} \]

\[ \text{Theorem: (PS)} \]

There is a unital associative category $\mathcal{F}(X; S)$ with objects spectral

\[ \text{Lagr. branes } \& \text{ morphisms } \mathbb{D}_\mathbb{C} \text{ graded modules over } \mathcal{F}(X; S) \; \text{stabilization } \mathcal{F}(X; S) = \mathcal{F}(X) \]
Theorem: (PS)

Setting as above, $L, K = X$ with $H_r$-sphere.

$$H \leq T(x, k) \text{ then } [L] = [K] \in \mathcal{T}_n^{(r)}(x) \setminus \mathcal{T}_{n-1}(x) \cdot R_n^{k-1}(pt)$$

So $[L] = [K] \in \mathcal{T}_n^{k-1}(pt)$ is 2-torsion.

Corollaries

(i) The explicit 8-spheres do not embed in $A_2 = T^2 S^8 \# T^2 S^8$

(ii) If $n \equiv 0, 4, 6, 7 \mod 8$, & $\exists$ elt of order $> 2$ in $\Theta_{2n} / \Theta_{2n+1}$

Then $\pi_0 \text{Sym}_+(X_n, d) \to \text{Aut}_+ \mathcal{T}(X_n, d; 2)$ is not injective

'degree $d \leq c_{n+1}$, for any $d \geq 3$

Viewpoint à la Abouzaid-Blumberg:

Flow theory associates to $L, K$ a flow category $\mathcal{M}_{L,K}$ & we study flow modules.
\[ M^k : \text{objects } L \# K, \text{ grading } 1.1 \text{ in } \mathbb{Z} \]

\[ M_{xy} = M_{xy}, \text{chiral space of floor strips, dimension } |x| - |y| - 1 \]

Composition \[ M_{xy} \times M_{yz} \rightarrow M_{xzt} \text{ inclusion of codim 1 boundary leaf} \]

\[ \text{w/r} \]

**Theorem (Fukaya, Oh, Ohta, Ono; Long)**

\[ M_y \text{ is a smooth manifold with corners. Moreover, when } X \text{ is stable brane, } L, K \text{ are spectral branes, } M_y \text{ admit stable braneys compatibly with breaking.} \]

**Key ingredient:** presumably \[ \omega = u_1 \# u_2 \text{ with length } T, \text{ the actual nearly Floer solv} \]

\[ \omega \text{ written as } \exp(w(S^1)) \text{ for a v.f.d. } S^1 \text{ s.t. } \| S \| \leq C \cdot e^{-T} \text{ for some } s_0. \]

For smoothness, need \[ \| (\partial_x - \partial_T) \cdot S^1 \| \text{ (think part) } \leq C \cdot e^{-T} \]
Definition let \( F, G \) be two categories. A morphism \( \alpha \) from \( F \) to \( G \)

has maps \( \text{Hom}_{F,Y} \) for \( Y \in F, J \in G \) with \( \text{dim} \text{Hom}_{F,Y} = |x| - |y| \)

\[ \text{Hom}_{F,Y} = \bigcup_{x \in F} \text{Hom}_Y \times \text{Hom}_G \cup \bigcup_{y \in G} \text{Hom}_F \times \text{Hom}_Y \]

If \( F = \# [i] \) and object \( X, |X| = i \), then a morphism is a \underline{flow module}.

Notation of: composition: \( F \xrightarrow{w} G \xrightarrow{y} H \)

bordism of morphisms

tracing of morphisms of framed flow categories.

E.g. for composition, \( \tilde{Q}_{nk} := \text{eq} \big( \sqcup \text{Hom}_Y \times \text{Hom}_G \times \text{Hom}_H \times (0,\epsilon)^k \big) \rightarrow \sqcup \text{Hom}_Y \times \text{Hom}_G \times (0,\epsilon)^k \)

and then take \( \text{Inv}^{-1} \) for flow: \( \tilde{Q}_{nk} \rightarrow \text{R} \) proper & re-orient.

\( \to \) category Flow or Flow of (framed) flow categories.
Now $f(x, S)$ has $(F, (L, L'; S)) = [x[i], R_{L'}]$.

So a morphism has span

$$W_{xy} \text{ for } y \in R_{L'} = \text{link of dimension } i - |y|$$

$$\& \quad W_{xy} = W_{xy_1} \times R_{L'}$$

A driven of $R_{j'(p')} \text{ just multiplies all the spaces } W_{xy}$ by a fixed cloud (fixed) $R_{L'}$.

Variation: A $T_{\leq 0}$-morphism from $F \to G$ is the data of the $W_{xy}$ whenever

$$1x1 - 1y1 \in \gamma. \text{ I truncation } [F, G]_{L_{\leq 0}} \to [F, G]_{L_{\leq 0}}. \text{ if } n \geq m.$$  

Say it is a $T_{\leq 0}$-morphism if it comes from truncation.

Example: $T_{\leq 0}$-premorphism defines a linear map $CM\times F \to CM\times G$.
(where  $\mathcal{C}_k$ is the More complex of the flow category)

$\phi : \mathcal{T}_{\leq 0} \to \mathcal{T}$ is a morphism in a chain map.

Upshot: $\mathcal{T}_{\leq k}$ flow exists, $\phi : \mathcal{T}_{\leq 0} \to \mathcal{T}$.

Want to lift a quasi-morph $\mathcal{T}_{\leq i} \to \mathcal{T}_{\leq i+1}$.

Proposition

If $\mathcal{L} : \mathcal{T}_{\leq i} \to \mathcal{G}$ is a $\mathcal{T}_{\leq k}$-morphism, it comes from truncation

or obstruction class $[\mathcal{O}] \in H\mathcal{M}_{i-k-2}(\mathcal{G}, \mathcal{K}^{\mathcal{L}})$ vanishes.

Our setting: $\mathcal{L}, \mathcal{K}$ are $\mathcal{H}_{\mathcal{K}}$-morphs, $\mathcal{L} \simeq \mathcal{K}$ in $\mathcal{T}_{\leq 0} \mathcal{T}$. so $\alpha : \mathcal{L} \to \mathcal{K}$

$\mathcal{L}$ lift to $\mathcal{T}_{\leq 1}$-morphism: $H\mathcal{M}_{n-2}(\mathcal{M}_{n-1}, \mathcal{K}^{\mathcal{L}}) = 0$

$\mathcal{T}_{\leq n-2}$-morphism $H\mathcal{M}_{n-1}(\mathcal{M}_{n-1}, \mathcal{L}) = 0$
Hit an obstruction at final stage to get \( \mathbb{Z}_\omega \)-morphism in \( H\omega (M^e, \mathbb{L}^e) \).

Example:

\[
\begin{align*}
M_{\mathbb{Z}_\omega} \text{ has dim. } (p-1)q + 1 &= n-1 \text{ framed } \sim \theta \in \mathbb{R}^2_{n-1} \\
K &= \mathbb{T}(x,y) \quad \theta = 0 \quad \text{but trying to access } [L] - [K] \in \mathbb{R}^2
\end{align*}
\]

Bass-Sullivan theory:

"Border of nils with singularities" want to allow cone \( (P) \) as a singularity type. Indeed look for bordism of nils-with-lacs s.t. faces \( Px (-) \).

"don't want" to bordism theory \( \mathbb{R}e^P \) s.t. \( \mathbb{R}e^P \xrightarrow{x \neq P} \mathbb{R}e \).

Mild have a closed if \( dm = Px \omega \) & this arrow sends it to \( \omega \).
Next: develop the modules with Bass–Jullien singularities. Amounts to working over a quotient $S/J$ by a class you want to kill. The subdiv is construct their class in quotient $\mathfrak{N}^p(x)$ as given.