Bordism of flow modules & exact Lagrangians

Joint work with Noah Porcelli

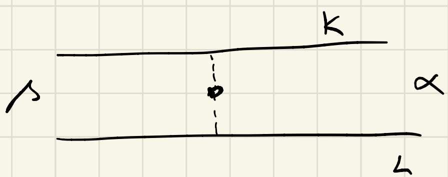
Consider a Weinstein manifold $(X, \omega = d\theta)$, & compact exact Lagrangians

$L, K \subseteq X$.

Classical: If L, K are "indistinguishable to Floer theory" i.e. $L \simeq K$ in $\mathcal{F}(X)$ then $[L] = [K] \in H_n(X; \mathbb{Z}/2)$ or in $H_n(X; \mathbb{Z})$ if oriented & spin

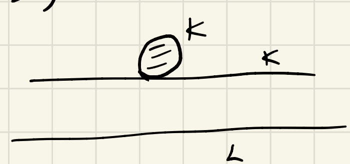
Idea of proof:

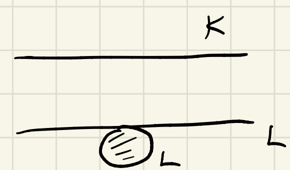
$$L \simeq_{\mathcal{F}} K \Rightarrow \exists\, \alpha \in CF^0(L,K),\ \beta \in CF^0(K,L) \text{ s.t. } \alpha\beta = id_K,\ \beta\alpha = id_L$$



Study the moduli space of hol^c strips.

Degenerations





Disc bubbles are constant by exactness ⇝ sweep L, K

$$\begin{pmatrix} CF(L,K) & & CF(L,L) \\ \oplus & \longrightarrow & \oplus \\ CF(K,L) & & CF(K,K) \end{pmatrix} \xrightarrow{\mathcal{OC}} H_*(X)$$

$$\underbrace{(\alpha,\beta) \xmapsto{\mu_2} (id_L, -id_K)}_{\text{"length two Hochschild cx"}} \longmapsto [L]-[K]$$

Note $TL \otimes \mathbb{C} \cong TX|_L$ so pontrjagin numbers of L are determined by Chern classes of X & $[L]$ if L spin & we work over $\mathbb{Z}$ so $p_i(L) \longmapsto p_i(K)$

under $H^*(L;\mathbb{Z}) \xrightarrow{\cong} H^*(K;\mathbb{Z})$ $x \mapsto \alpha x \beta$

(cf the oriented bordism class of L in $\Omega_*^{or}(pt)$ determined by $p_I(L) \in \mathbb{Z}$, $w_J(L) \in \mathbb{Z}/2$
pontrjagin & Stiefel-Whitney nos.

<u>Motivating Question</u>: Can we draw stronger conclusions, e.g. if L, K are framed, can we deduce constraints on their classes in $\Omega_n^{fr}(X)$?

Natural strategy: (a) Build (Donaldson-) Fukaya category over spectra $\mathbb{E}$

(b) Prove $L \simeq_{\mathcal{F}(X;\mathbb{E})} K$

well known goal

But you still have to do this!

New idea: obstruction theory for lifting quasi-iso's over $\mathbb{Z}$ to ones over $\mathbb{S}$ (the sphere spectrum)

SETTING: Assume TX is stably trivial as a $\mathbb{C}$-v.bundle. An exact Lagr. has a stable Gauss map $L \to U/O$; objects of $\mathcal{F}(X;\mathbb{S})$ will be graded Lagrangians with a nullhomotopy of this.

stable Lag Grassmannian

(a)

Theorem: (PS) There is a unital associative category $\mathcal{F}(X;\mathbb{S})$ with objects spectral Lagrangian branes & morphism gps graded modules over $\Omega_*^{fr}(pt) = \pi_*^{st} = \pi_*(\mathbb{S})$.

If $L \simeq_{\mathcal{F}(X,\mathbb{S})} K$ then $[L] = [K] \in \Omega_n^{fr}(X)$.

(b)

Theorem: (PS)

Setting as above; $L, K \subset X$ exact Lagr $\mathbb{Z}H_*$-spheres.

If $L \simeq_{\mathcal{F}(X,\mathbb{Z})} K$ then $[L] = [K] \in \Omega_n^{fr}(X) / \Omega_{n-1}^{fr}(X) \cdot \underbrace{\Omega_1^{fr}(pt)}_{\cong \mathbb{Z}/2 \text{ gen'd by } \eta \in \pi_1^{st}.}$

So $[L] = [K] \in \Omega_n^{fr}(pt)$ is 2-torsion.

Corollaries

(i) The exotic 8-spheres do not embed in $A_2 = T^*S^8 \#_{pt} T^*S^8$

(ii) If $n \in \{0, 4, 6, 7\} \bmod 8$ & $\exists$ elt of order > 2 in Θ_{n+1} / bP_{n+2}

then $\pi_0 \, \mathrm{Symp}(X_{n,d}) \longrightarrow \mathrm{Auteq}\, \mathcal{F}(X_{n,d}; \mathbb{Z})$ is NOT injective

degree $d \subseteq \mathbb{C}^{n+1}$, for any $d \geq 3$

Viewpoint à la Abouzaid-Blumberg:

Floer theory associates to L, K a flow category $\mathcal{M}^{LK}$ & we study flow modules.

$\mathcal{M}^{LK}$: objects $L \pitchfork K$, grading $|\cdot|$ in $\mathbb{Z}$

$\mathrm{Mor}_{xy} = \mathcal{M}_{xy}$ compactified space of Floer strips, dimension $|x|-|y|-1$

composition $\mathcal{M}_{xy} \times \mathcal{M}_{yz} \hookrightarrow \mathcal{M}_{xz}$ inclusion of codim 1 boundary facet

__using__

__Theorem__ (Fukaya, Oh, Ohta, Ono ; Large)

$\mathcal{M}_{xy}$ is a smooth manifold with corners. Moreover, when X is stably framed & L, K are spectral branes, $\mathcal{M}_{xy}$ admit stable framings compatibly with breaking.

Key ingredient : pregluing $w = u_1 \#_T u_2$ with length T, the actual nearby floer sol$^{\underline{n}}$ is written as $\exp_w(\xi_{u,T})$ for a v.field $\xi_{u,T}$ s.t. $\|\xi\| \leq C \cdot e^{-\delta \cdot T}$ some $\delta > 0$.

For smoothness, need $\|(D_u)^i (\partial/\partial T)^j \xi_{u,T}\|_{[\text{thick part}]} \leq C \cdot e^{-\delta' T}$

Definition Let $\mathcal{F}, \mathcal{G}$ be flow categories. A morphism $\mathcal{W}$ from $\mathcal{F}$ to $\mathcal{G}$

has mflds $\mathcal{W}_{xy}$ $x \in \mathcal{F}, y \in \mathcal{G}$ st $\dim \mathcal{W}_{xy} = |x| - |y|$

$$\partial \mathcal{W}_{xy} = \bigcup_{x' \in \mathcal{F}} \mathcal{F}_{xx'} \times \mathcal{W}_{x'y} \cup \bigcup_{y' \in \mathcal{G}} \mathcal{W}_{xy'} \times \mathcal{G}_{y'y}$$

If $\mathcal{F} = *[i]$ one object $*$, $|*| = i$, then a morphism is a <u>flow module</u>.

<u>Notion of</u>: composition: $\underset{x}{\mathcal{F}} \xrightarrow{\mathcal{W}} \underset{y}{\mathcal{G}} \xrightarrow{\mathcal{V}} \underset{z}{\mathcal{H}}$

bordism of morphisms

framing of morphisms of framed flow categories.

e.g. for composition, (collar co-ordinates) $\tilde{Q}_{xz} := \mathrm{Coeq}\left(\coprod_{y,y' \in \mathcal{G}} \mathcal{W}_{xy} \times \mathcal{G}_{yy'} \times \mathcal{V}_{y'z} \times [0,\epsilon)^2 \rightrightarrows \coprod_{u \in \mathcal{G}} \mathcal{W}_{xu} \times \mathcal{V}_{uz} \times [0,1) \right)$

& then take $f_{xz}^{-1}(1)$ for $f_{xz} : \tilde{Q}_{xz} \to \mathbb{R}$ proper & -ve on cpt locus.

$\rightsquigarrow$ <u>Category</u> Flow or Flow^{fr} of (framed) flow categories.

Now $\mathcal{F}(X, \mathbb{S})$ has $CF_i(L, K; \mathbb{S}) = [*[i], \mathcal{M}^{LK}]_{\mathrm{Flow}^{fr}}$

Flow cat with one object $*$ in degree i.

So a morphism has spaces

$$W_{*y} \text{ for } y \in \mathcal{M}^{LK} = L \pitchfork K \text{ of dimension } i - |y|$$

$$\& \quad \partial W_{*y} = W_{*y'} \times \mathcal{M}^{LK}_{y',y}$$

Action of $\Omega^{fr}_j(pt)$ just multiplies all the spaces W_{*y} by a fixed closed (framed) mfld.

<u>Variation</u>: A $\tau_{\leq n}$-premorphism from $\mathcal{F}$ to $\mathcal{G}$ is the data of the W_{xy} whenever $|x| - |y| \leq n$. $\exists$ truncations $[\mathcal{F}, \mathcal{G}]_{\tau_{\leq n}} \to [\mathcal{F}, \mathcal{G}]_{\tau_{\leq m}}$ if $n \geq m$.

Say it is a $\tau_{\leq n}$-morphism if it comes from truncation.

<u>Example</u>: $\tau_{\leq 0}$-premorphism defines a linear map $CM_* \mathcal{F} \to CM_* \mathcal{G}$

$$x \mapsto \sum_{\substack{y \\ |y|=|x|}} W_{xy} \cdot y$$

(where CM_* is the Morse complex of the flow category)

& a $\tau_{\leq 0}$-morphism is a chain map.

Upshot: $\tau_{\leq k}$ Flow exists, & $\tau_{\leq 0} \mathcal{F}(X; \mathbb{S}) = \mathcal{F}(X; \mathbb{Z})$.

<u>Want</u> to lift a quasi-iso from $\tau_{\leq i} \mathcal{F}(X;\mathbb{S})$ to $\tau_{\leq (i+1)} \mathcal{F}(X;\mathbb{S})$.

<u>Proposition</u>

If $W: * [i] \to G$ is a $\tau_{\leq k}$-morphism, it comes from truncation

$\Leftrightarrow$ an obstruction class $[O] \in HM_{i-k-2}(G, \Omega^{fr}_{k+1})$ vanishes.

<u>Our setting</u>: L, K are $\mathbb{Z}H_*$-spheres, $L \simeq K$ in $\tau_{\leq 0} \mathcal{F}(X;\mathbb{S})$. so $\alpha: L \to K$, $\beta: K \to L$ — degree n

Lift to $\tau_{\leq 1}$-morphism: $HM_{n-2}(M^{LK}, \Omega^{fr}_1) = 0$

$\vdots$

$\tau_{\leq n-2}$-morphism $HM_1(M^{LK}, \Omega^{fr}_{n-2}) = 0$

$\rightsquigarrow$ Hit an obstruction at final stage to get $\tau_{\leq n}$-morphism

in $HMo(M^{\leq k}, \Omega^{fr}_{n-1})$

Example :

M_{pq} has dim. $|p| - |q| - 1 = n-1$ framed $\rightsquigarrow$ $\theta \in \Omega^{fr}_{n-1}$

$K \underset{\mathcal{F}(X;D)}{\simeq} L \iff \theta = 0$ BUT trying to access

$[L] - [K] \in \Omega^{fr}_n$

Baas-Sullivan theory :

"Bordism of mflds with singularities" Want to allow Cone (P)

as a singularity type. Instead look for cobordisms of mflds-with-faces s.t. faces $P \times (\_\_)$

"don't count" $\rightsquigarrow$ bordism theory Ω^P_* s.t.

$$\Omega_* \xrightarrow{\times P} \Omega_* \xrightarrow{\text{inclusion}} \Omega^P_* \dashrightarrow \Omega_*$$

Mfld here is closed if $\partial M = P \times N$ & this arrow sends it to N

Next: develop flow modules with Bass-Sullivan singularities.

Amounts to working over a quotient $\mathbb{S}/_{\theta}$ by a class you want to kill

Deduce $[L] \simeq [K] \in \mho(X; \mathbb{S}_{(n-1)-BS-sings})$.

This suffices to constrain their class in quotient of $\Omega_n^{fr}(X)$ as given. $\square$