New(ish) Invariants of Legendrian Links

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These slides available at math.duke.edu/~ng/zoominar.pdf

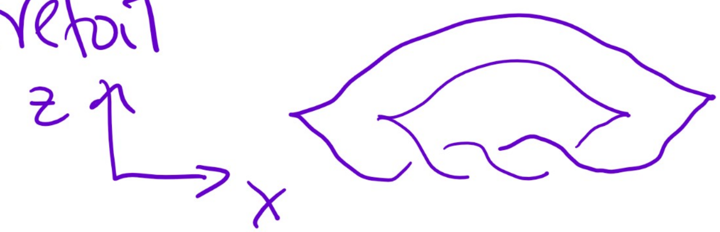
Today's (simple) setting: standard contact $\mathbb{R}^3$:

$\mathbb{R}^3$ equipped with the standard contact structure

$$\xi_{std} = \ker(\underbrace{dz - y\,dx}_{\alpha_{std}}).$$

A link $\Lambda \subset \mathbb{R}^3$ is Legendrian if it's everywhere tangent to ξ_{std}: $\quad \alpha_{std}|_\Lambda \equiv 0.$

ex trefoil



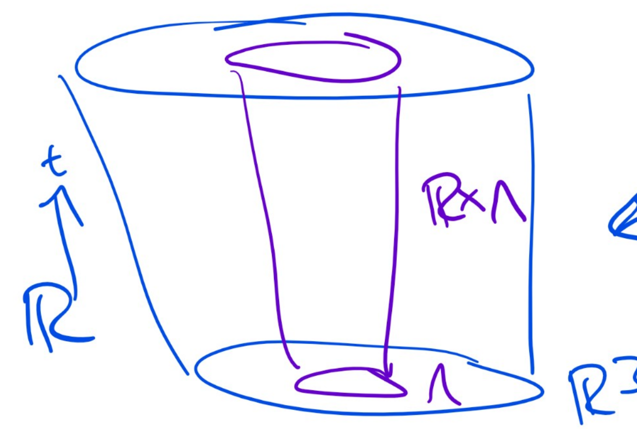
Associated construction:

Symplectization of $\mathbb{R}^3$: $(\mathbb{R}_t \times \mathbb{R}^3, \omega_{std} = d(e^t \alpha_{std}))$

Note: Λ Legendrian in $\mathbb{R}^3$

$\Rightarrow \mathbb{R}\times\Lambda$ Lagrangian in $\mathbb{R}\times\mathbb{R}^3$

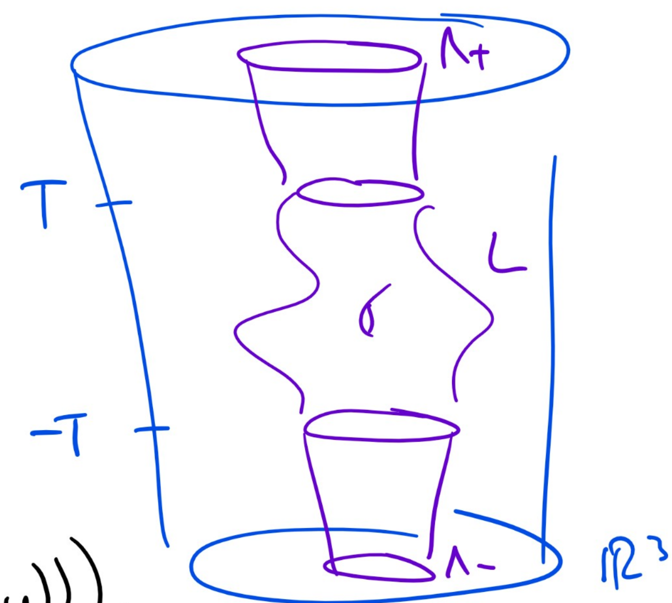
$(\omega_{std}|_{\mathbb{R}\times\Lambda} \equiv 0)$.



Can build a category whose objects are (oriented) Legendrian links in $\mathbb{R}^3$. Morphisms are:

<u>Def</u> Λ_+, Λ_- Legendrian links in $\mathbb{R}^3$. An exact Lagrangian cobordism $\Lambda_- \to \Lambda_+$ is $L \subset \mathbb{R} \times \mathbb{R}^3$ with:

- $L \cap \{t > T\} = (T, \infty) \times \Lambda_+$
  $L \cap \{t < -T\} = (-\infty, -T) \times \Lambda_-$
  for some $T \gg 0$
- $e^t \alpha_{std}|_L$ is exact: $e^t \alpha_{std} = df$
  for some $f : L \to \mathbb{R}$,
  f constant at ends.
  ($\Rightarrow L$ is Lagrangian in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha_{std}))$).

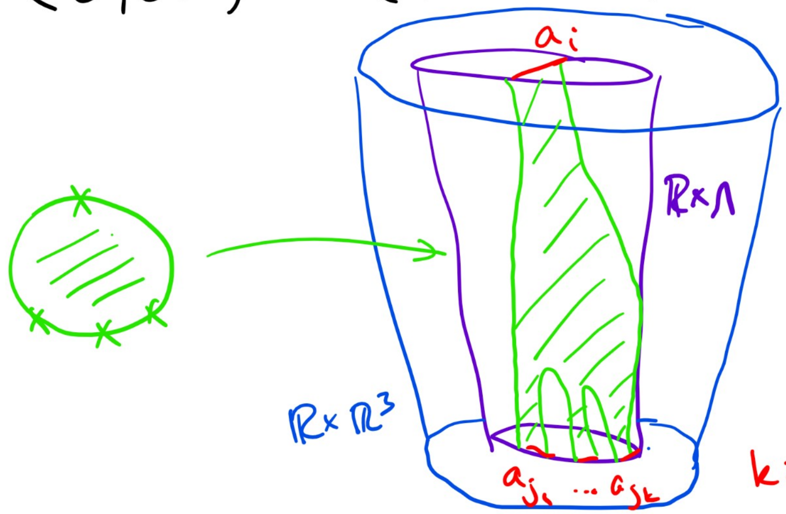


Special case: a filling of a Legendrian link Λ is an exact Lagrangian cobordism $\emptyset \to \Lambda$.

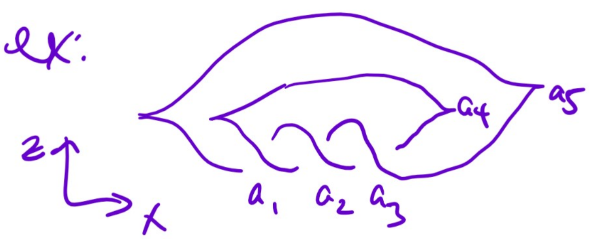
Legendrian contact homology

– Floer-theoretic invt of Legendrian links.

~25 years ago: $\Lambda \subset \mathbb{R}^3$ leg. link $\rightsquigarrow$ Chekanov-Eliashberg differential graded algebra $(\mathcal{A}_\Lambda, \partial)$

$\mathcal{A}_\Lambda = k[a_1, \ldots, a_n]$ where k is a field and $a_1, \ldots, a_n$ are the Reeb chords of Λ (also generators corresponding to base points – we'll ignore)	$\partial : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ counts holomorphic disks $(D^2, \partial D^2) \longrightarrow (\mathbb{R}\times\mathbb{R}^3, \mathbb{R}\times\Lambda)$  1 "positive puncture" $\partial(a_i) = \sum_{\text{rigid moduli spaces}} (*)\, a_{j_1} \cdots a_{j_k}$ $k \geq 0$ "negative punctures"

ex:



$\partial(a_4) = 1 + a_1 + a_3 + a_1 a_2 a_3$

Then $\partial^2 = 0$ and the homology $H_*(\mathcal{A}_\Lambda, \partial) =: LCH_*(\Lambda)$ is the Legendrian contact homology of Λ.

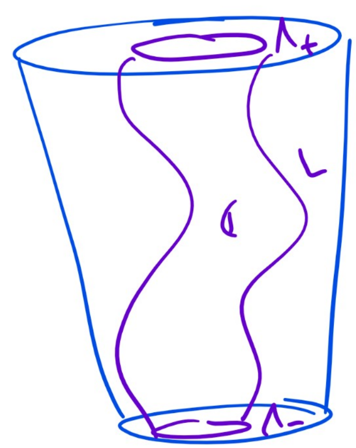
Invariance:

Thm (Chekanov, Eliashberg, Etnyre-N-Sabloff, ...)
If $\Lambda, \Lambda' \subset \mathbb{R}^3$ are Legendrian isotopic then $LCH_*(\Lambda) \cong LCH_*(\Lambda')$.

(stronger: dg algebras are "stable tame isomorphic" $\Rightarrow$ chain homotopy equivalent).

Functoriality:

Thm (Ekholm-Honda-Kálmán 2012, building on Eliashberg-Givental-Hofer)
An exact Lagrangian cobordism $L : \Lambda_- \to \Lambda_+$ induces a dg algebra map $\Phi_L : (\mathcal{A}_{\Lambda_+}, \partial) \to (\mathcal{A}_{\Lambda_-}, \partial)$, invt up to chain homotopy under Hamiltonian isotopy.



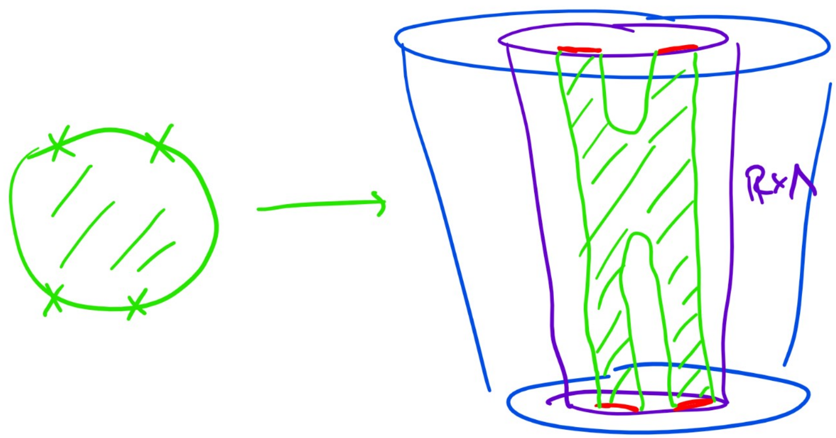
$$\begin{array}{c} \longrightarrow (\mathcal{A}_{\Lambda_+}, \partial) \\ \big\downarrow \Phi_L \quad \text{cobordism map between dg algebras} \\ \longrightarrow (\mathcal{A}_{\Lambda_-}, \partial) \end{array}$$

Rational Symplectic Field Theory

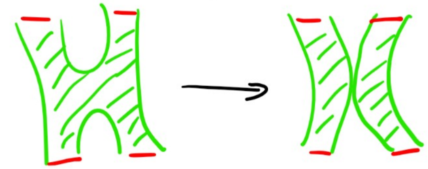
Eliashberg-Givental-Hofer ~2000: Legendrian contact homology is the first level of a multilevel invariant associated to $\Lambda \subset \mathbb{R}^3$.

Next level: rational symplectic field theory:

counts hol. disks $(D^2, \partial D^2) \to (\mathbb{R}\times\mathbb{R}^3, \mathbb{R}\times\Lambda)$ with any # of positive punctures.



Serious issue with boundary degeneration:



→ Ekholm 2006 for multi-component Legendrians Λ

→ N. 2008 for Legendrian knots $\Lambda\subset\mathbb{R}^3$, using ideas from string topology (Cieliebak-Latschev, Cornea-Lalonde).

An L_∞ structure for LCH

Thm (N. ~~2008~~ 2023) Λ = Legendrian link in $\mathbb{R}^3$. The Chekanov-Eliashberg dg algebra $(\mathcal{A}_\Lambda, \partial)$ can be extended to an L_∞ algebra* $(\mathcal{A}_\Lambda, \{l_n\})$ such that $l_1 = \partial$.

L_∞ algebra: (graded) symmetric multilinear maps

$$l_n : (\mathcal{A}_\Lambda)^{\otimes n} \to \mathcal{A}_\Lambda \qquad n \geq 1$$

[think: l_2 = Lie bracket]

satisfying the L_∞ relations:

- $0 = l_1^2(x)$ ← [l_1 = differential]
- $0 = l_1 l_2(x,y) - l_2(l_1 x, y) \pm l_2(x, l_1 y)$ ← [l_1 = derivation w.r.t. l_2]
- $0 = l_2(l_2(x,y),z) \pm l_2(l_2(y,z),x) \pm l_2(l_2(z,x),y) + l_3(l_1 x, y, z) \pm l_3(x, l_1 y, z) \pm l_3(x,y,l_1 z) + l_1 l_3(x,y,z)$ ← [l_2 satisfies Jacobi up to correction terms]
- ....

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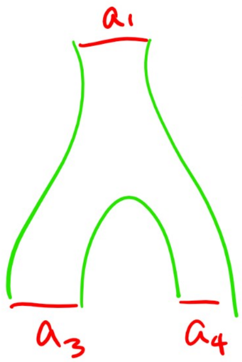
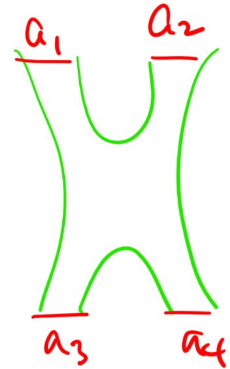
* a.k.a. homotopy Lie algebra; $(\mathcal{A}_\Lambda, \{l_n\})$ is actually a homotopy Poisson algebra since l_n satisfy Leibniz rule with respect to usual (associative) multiplication on $\mathcal{A}_\Lambda$.

The L_∞ operation $l_n : (A_\Lambda)^{\otimes n} \to A_\Lambda$ counts holomorphic disks with n positive punctures, as in rational SFT:



Contributes $a_3 a_4$ to $l_1(a_1) = \partial(a_1)$



Contributes $a_3 a_4$ to $l_2(a_1, a_2)$

Rmks:
- deal with boundary degenerations by adding string-topology correction terms to l_2, as in N. 2008 "Rational SFT"
- link case is completely new, and trickier than knot case.

Thm (N. 2023) The L_∞ algebra $(A_\Lambda, \{l_n\})$ is partially* invariant under Legendrian isotopy of Λ, up to L_∞ equivalence.

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* through l_2

Conjecture ① The L_∞ algebra is fully invariant

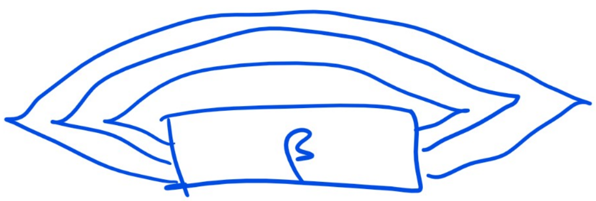
② The L_∞ algebra is functorial under exact Lagrangian cobordisms.

Thm (N. 2023) The L_∞ algebra $(\mathcal{A}_\Lambda, \{l_n\})$ is partially* invariant under Legendrian isotopy of Λ, up to L_∞ equivalence.

* through l_2

Cor l_2 induces a Poisson bracket on $LCH_*(\Lambda) = H_*(\mathcal{A}_\Lambda, l_1)$, and $LCH_*(\Lambda)$ is invariant as a Poisson algebra.

Ex Positive braid $\beta \Rightarrow$ Legendrian link Λ_β = "rainbow closure" of β.



In this case $l_n = 0$ for $n \geq 3$

(the L_∞ algebra is "strict")

$\Rightarrow$ the braid β induces a Poisson structure on

$$k[\underbrace{a_1, \ldots, a_n}_{\text{crossings of } \beta\Delta^2}].$$

For $\beta = \sigma_1^{2m}$, recover the classical Flaschka-Newell bracket (1992). Is the Poisson bracket new in general?

Fillings & Augmentations

Let L be a filling of a Legendrian link Λ.
Then the cobordism map is an augmentation of Λ:
a dg algebra map

$$\varepsilon : (\mathcal{A}_\Lambda, \partial) \longrightarrow (k, 0).$$

$(\varepsilon \circ \partial = 0)$

$$\longrightarrow \quad \begin{array}{c} (\mathcal{A}_\Lambda, \partial) \\ \downarrow \Phi_L \\ (\mathcal{A}_\emptyset, 0) \end{array}$$

Better:

filling L of Λ, with a rank 1 local system on L (group homomorphism $H_1(L) \to k^\times$)	$\xrightarrow{\text{cobordism map}}$	augmentation of Λ

Def The augmentation variety of Λ is

$$\mathrm{Aug}(\Lambda) = \{\text{augmentations } (\mathcal{A}_\Lambda, \partial) \to (k, 0)\} \,/\, \text{homotopy}.$$

Any filling produces a chart $\cong$ algebraic torus on the augmentation variety: $(k^\times)^{b_1(L)} \longrightarrow \mathrm{Aug}(\Lambda)$.

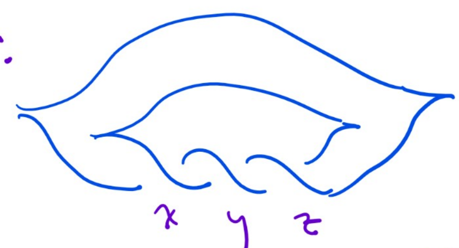
($\hookrightarrow$: Gao-Rutherford 2021)

For a Legendrian Λ, the augmentation variety

$$\mathrm{Aug}(\Lambda) = \{\text{dga maps } (\mathcal{A}_\Lambda, \partial) \to (k, 0)\}/\text{homotopy}$$

- is a Legendrian-isotopy invariant: it's essentially $\mathrm{Spec}\ LCH_0(\Lambda)$
- is an algebraic stand-in for $\{\text{fillings of } \Lambda \text{ (with rank 1 local systems)}\}$.

Ex Λ = trefoil:



$\mathrm{Aug}(\Lambda) = \{1 + x + z + xyz = 0\} \subset k^3$.

Ekholm-Honda-Kálmán: $\exists\ \boxed{5}$ fillings of Λ, each giving a chart

$$(k^\times)^2 \hookrightarrow \mathrm{Aug}(\Lambda) \qquad (\text{ex: } (s_1, s_2) \mapsto (s_1, s_2 - s_1^{-1}, -s_2^{-1} - s_1^{-1}s_2^{-1}))$$

and these 5 charts are <u>distinct</u> ($\Rightarrow$ fillings are distinct) and cover $\mathrm{Aug}(\Lambda)$.

Sabloff form

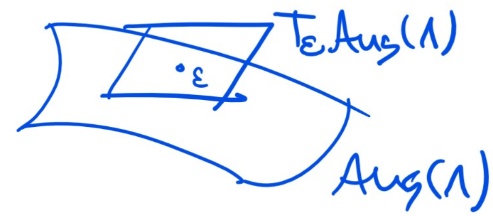
The Poisson bracket on $LCH_*(\Lambda)$ dualizes to:

Thm (Casals-Gao-N-Shen-Weng, in progress)
There is a symplectic form ω_{Sab} on $Aug(\Lambda)$, invariant under Legendrian isotopy.

ex: trefoil

$$\omega_{Sab} = \frac{dx \wedge dz}{xz}.$$

Ekholm-Etnyre-Sabloff 2008 (essentially): for any $\varepsilon \in Aug(\Lambda)$, there is a nondegenerate (skew symmetric) pairing on $T_\varepsilon Aug(\Lambda)$: Sabloff duality.



So ω_{Sab} globalizes Sabloff duality to a closed 2-form on $Aug(\Lambda)$.

Conjecture The Sabloff form is functorial:

$$\rightarrow \quad \begin{array}{c} (\mathcal{A}_{\Lambda_+}, \partial) \\ \downarrow \Phi_L \\ (\mathcal{A}_{\Lambda_-}, \partial) \end{array} \quad \rightarrow \quad \left.\begin{array}{c} Aug(\Lambda_+) \\ \uparrow \Phi_L^* \\ Aug(\Lambda_-) \end{array}\right\}$$

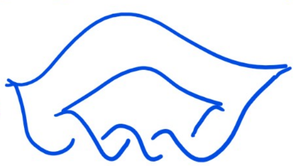
Conj: this map respects ω_{Sab}.

Augmentations and Cluster Theory

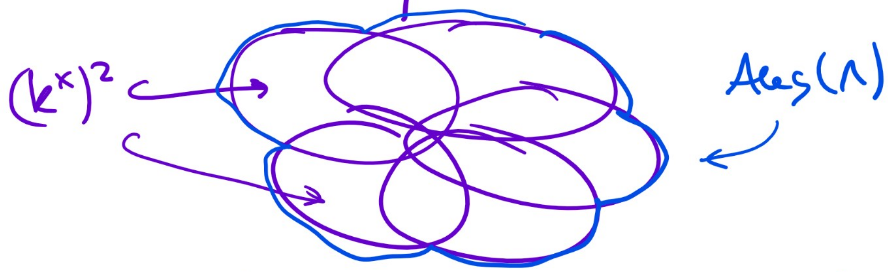
Close connection between symplectic topology and cluster algebras via wall-crossing (Auroux, Fukaya, Kontsevich, Seidel, Soibelman, ...).

For Legendrian links, Shende-Treumann-Williams-Zaslow 2015: cluster structures through sheaf quantization associated to fillings.

__Ex__ (trefoil) Recall: augmentation variety $\mathrm{Aug}(\Lambda) = \{1 + x + z + xyz = 0\}$

$\Lambda =$ 

is covered by 5 charts $\longleftrightarrow$ 5 fillings of Λ.



These charts give $\mathrm{Aug}(\Lambda)$ the structure of a __cluster (A-) variety__ (Fock-Goncharov 2009): covered by __cluster seeds__ (chart + cluster coordinates + quiver) related on overlaps by __cluster mutation__.

Augmentation varieties & Cluster varieties

For many classes of Legendrian links Λ, the augmentation variety $\mathrm{Aug}(\Lambda)$ is a cluster $\mathcal{A}$-variety:

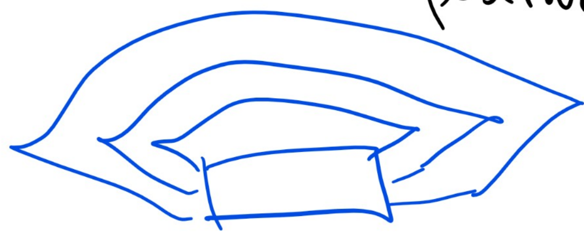
- rainbow closures of positive braids (Gao-Shen-Weng, Casals-Gorsky-Gorsky-Simental, 2020)
- grid plabic graphs (Casals-Weng 2022)
- Legendrian 2-bridge links (Capovilla-Searle - Hughes-Weng 2023)

...

Geometric intuition:

Clusters		Augmentations
cluster seed	$\longleftrightarrow$	exact Lagrangian filling
cluster mutation	$\longleftrightarrow$	Polterovich surgery

Λ = rainbow closure of a positive braid $\Longrightarrow$ $Aug(\Lambda)$ is a cluster $\mathcal{A}$-variety



These cluster varieties carry a natural symplectic form (inherited from each chart).

> **Thm** (Casals-Gao-N-Shen-Weng, in progress)
>
> Λ = rainbow closure of a positive braid
>
> $\Rightarrow$ the cluster symplectic form on $Aug(\Lambda)$ agrees with the Sabloff form ω_{Sab}.

Future work?

Cluster A-varieties come with dual "cluster X-varieties":
for Λ = positive braid rainbow closure, $Aug(\Lambda)$ has both structures.
(Casals-Gorsky-Gorsky-Le-Shen-Simental 2022)

$\Rightarrow$ cluster quantization of $Aug(\Lambda)$.

Thm/Conj (Casals-Gao-N-Shen-Weng-Zaslow, in progress)
Λ = rainbow closure of a positive braid. Then the Chekanov-Eliashberg dg algebra $(\mathcal{A}_\Lambda, \partial)$ can be quantized/q-deformed such that the semiclassical limit recovers ℓ_2 from the L_∞ structure.

$\Rightarrow$ in special cases, recover known quantizations for quantum groups...