TORIC AND SEMITORIC SYMPLETIC GEOMETRY: PROGRESS AND CHALLENGES

CRM-Montréal,Princeton/IAS,Tel Aviv & Paris Symplectic Zoominar

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The goal of this talk is to give an **overview, for non-specialists**, of finite dimensional integrable systems on symplectic manifolds, and then focus on two special types: toric and semitoric. I will mention quantum applications and open problems. There are three parts:

- **Integrable systems (15 minutes);**
- **Toric systems (30 minutes);**
- **Semitoric systems (15 minutes).**

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PART 1: INTEGRABLE SYSTEMS

(approx. 15 minutes)

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[1 of 71] Famous dynamical systems: INTEGRABLE

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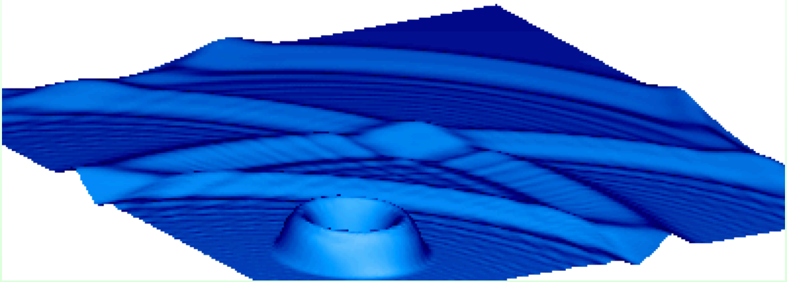

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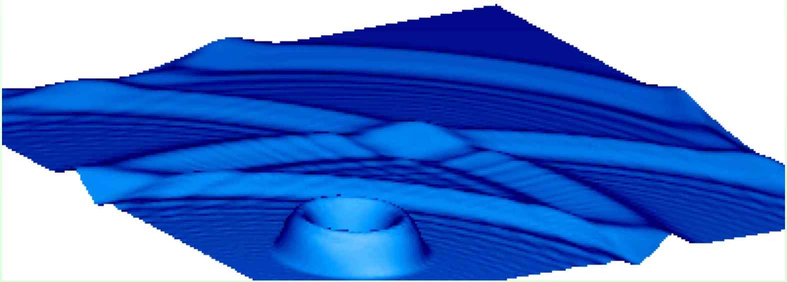

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Another example: Spherical pendulum (finite dimension).

Today I will focus on FINITE dimension.

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Let's see some MOTIVATIONS to study

Integrable Systems

and learn about some distinguished

Historical Figures

who have been influential in this area and the connections with other areas.

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[2 of 71] MOTIVATIONS to study integrable systems

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Even more, let's recall what Jürgen Moser told us:

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"It is impossible to even touch on the many ramifications that have evolved from the study of integrable systems."

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Intelligent studies and deep insight opened up to a novel field impinging on differential geometry, algebraic geometry, and mathematical physics, including applications in communication of fiber optics."

[5 of 71] INTEGRABLE SYSTEMS: LEADING FIGURES

Integrable systems are studied from many viewpoints. A lot of authors have made pioneering contributions: **Vladimir Arnold, Hans Duistermaat, Lars H. Eliasson, Anatoly Fomenko, Victor Guillemin, Nigel Hitchin, Andréi Kolmogorov, Sofya Kovalévskaya, Robert Langlands, Jürgen Moser, Emmy Noether, Nicolai Reshetikhin, Karen Uhlenbeck** and **Alan Weinstein**, among many others.



H. Duistermaat 1942-2010 K. Uhlenbeck 1942 N. Hitchin 1946

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"for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics."

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Today we focus on **finite dimensional integrable (Hamiltonian) systems** from the point of view of symplectic geometry and topology.

How are they defined?

We see it next.

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[7 of 71] Classical Integrable Systems: DEFINITION

Let (M^{2n}, ω) be $2n$-dimensional symplectic manifold.
A smooth function

$$F = (f_1, \ldots, f_n) \colon M^{2n} \to \mathbb{R}^n$$

is an $\boxed{\textbf{\textit{integrable system}}}$ if two conditions hold:

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Extra hypothesis today: the fibers $F^{-1}(c)$ are compact $\forall c$. Also, throughout talk, manifolds are $\boxed{\text{ASSUMED TO BE CONNECTED}}$.

[8 of 71] Historical note about integrable systems

1. Historically the term **integrable** comes from considering a symplectic manifold M^{2n} (**PHASE SPACE**) and a function (ENERGY FUNCTION or **HAMILTONIAN**) $H : M^{2n} \to \mathbb{R}$.

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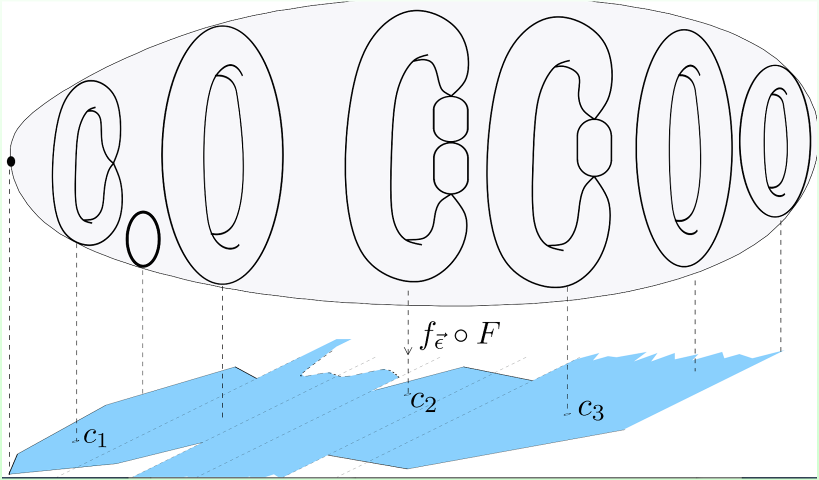
5. **The modern viewpoint is**: to consider all integrals simultaneously $(H, f_2, \ldots, f_n): M^{2n} \to \mathbb{R}^n$.
6. **Generalizations**: you can have more than $n-1$ integrals by relaxing conditions: **superintegrable systems** etc.

[9 of 71] Typical integrable system

What does a **typical integrable system** $F\colon M^4 \to \mathbb{R}^2$ look like?



Keep in mind this picture, we'll COME BACK TO IT later.

$**$

Next I am going to present four examples of

Integrable systems

which are very well known, and with

EXPLICIT FORMULAS.

$**$

[10 of 71] COMPLEX PROJECTIVE SPACE

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It comes endowed with **Fubini-Study symplectic form** (I skip formula). This **integrable system**

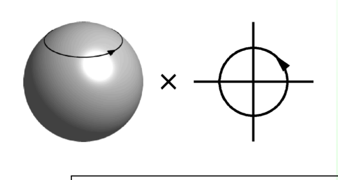
$$F = (f_1, \dots, f_n) \colon \mathbb{C}P^n \to \mathbb{R}^n$$

is given by the formula

$$F([z_0 : z_1 : \ \dots : z_n]) = \left(\frac{|z_1|^2}{\sum_{i=0}^n |z_i|^2}, \dots, \frac{|z_n|^2}{\sum_{i=0}^n |z_i|^2} \right)$$

and is induced by the rotational action of a torus on $\mathbb{C}^{2n+1}$.

[11 of 71] COUPLED SPIN-OSCILLATOR, i.e. JAYNES-CUMMINGS MODEL (1963)



Another example is the $\boxed{\text{JAYNES-CUMMINGS MODEL}}$, which is the integrable system

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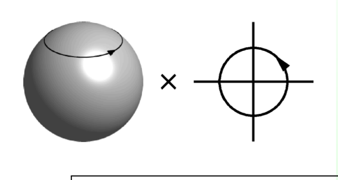

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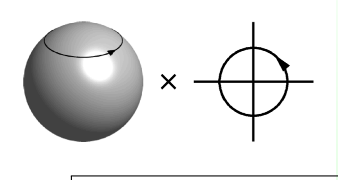

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where (x, y, z) are coordinates on S^2, (u, v) on $\mathbb{R}^2$, and $S^2 \times \mathbb{R}^2$ is endowed with $\mathrm{d}\theta \wedge \mathrm{d}h + \mathrm{d}u \wedge \mathrm{d}v$.

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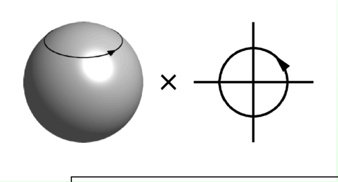

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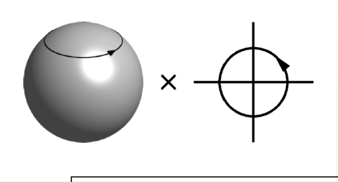

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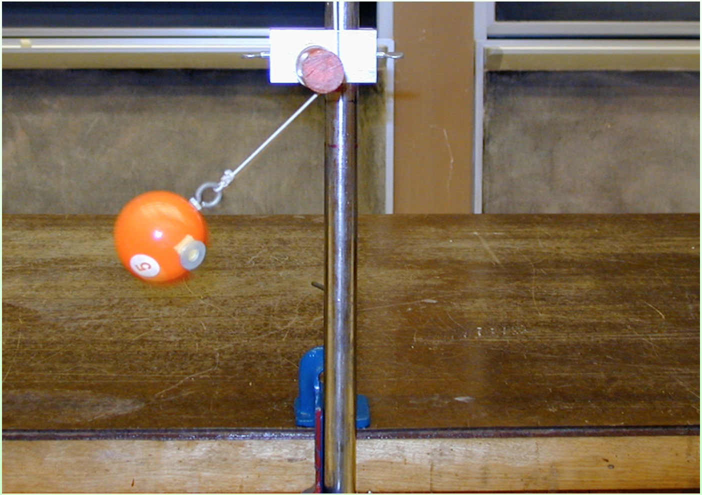
One of the $\boxed{\text{most famous integrable systems}}$ is the **pendulum**, which goes back to the XVII century. How does one describe it mathematically?



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$$(f_1, f_2)\colon \mathrm{T}^*S^2 \to \mathbb{R}^2,$$

where f_1 is the **sum of kinetic energy plus potential**

$$f_1(\theta, \varphi, \xi_\theta, \xi_\varphi) = \frac{1}{2}\left(\xi_\theta^2 + \frac{1}{\sin^2\theta}\xi_\varphi^2\right) + \cos\theta.$$

and the **first integral** is

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Here (θ,φ) are spherical angles – with φ being rotation angle about vertical axis and θ measuring angle from north pole – and (ξ_θ,ξ_φ) are cotangent conjugate variables.

[13 of 71] COUPLED ANGULAR MOMENTA

Lastly, it is impossible not to mention a system as important as the **COUPLED ANGULAR MOMENTA** of **Sadovskií-Zhilinskií**. It is an integrable system essential in physics.

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The integrable system is

$$F_t = (f_1, f_{2,t}) \colon S^2 \times S^2 \to \mathbb{R}^2$$

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On $S^2 \times S^2$ the symplectic form is the product form $-(R_1\omega_{S^2} \oplus R_2\omega_{S^2})$, where ω_{S^2} is the standard area form on S^2.

****

Next let's see a great

CHALLENGE IN THE FIELD.

****

[14 of 71] Challenge for XXI Century

In mathematics, equivalent objects are called ***isomorphic***.

When are two integrable systems F_1, F_2 isomorphic?

[14 of 71] Challenge for XXI Century

In mathematics, equivalent objects are called ***isomorphic***.

When are two integrable systems F_1, F_2 isomorphic?

- Being ***isomorphic*** means there is diffeomorphism of underlying manifolds which exchanges the systems and symplectic forms.
- ***Invariant***: object/property that isomorphic systems share.

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We want to understand something difficult (INTEGRABLE SYSTEMS) in terms of something "easy" (INVARIANTS)!

[15 of 71] What we know about previous challenge

1. It has quantum applications, to inverse problems.

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****

To address this challenge, it is necessary to understand integrable systems in:

- **neighborhood of regular/singular point**, and
- **neighborhood of regular/singular orbit/fiber**.

But ... we know little ... with the exception of a result by **Arnold and Mineur.**

****

[16 of 71] Arnold-Liouville-Mineur Theorem 1935, 1960

If $X_{f_1}(m), \ldots, X_{f_n}(m)$ are linearly independent, m is ***regular***.

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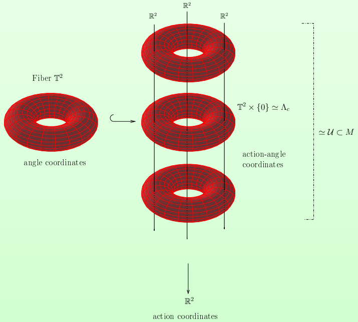
If $X_{f_1}(m),\ldots,X_{f_n}(m)$ are linearly independent, m is ***regular***.

$$\boxed{\textbf{The famous Action-Angle Theorem}}$$

says that a fiber with only regular points is n-torus $\mathbb{T}^n$ in $\mathrm{T}^*\mathbb{T}^n$ and in neighborhood $F(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)=(\xi_1,\ldots,\xi_n)$ (assuming preimages of compact sets are compact).



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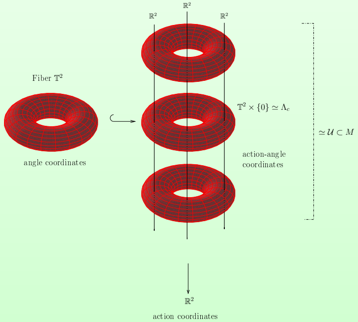
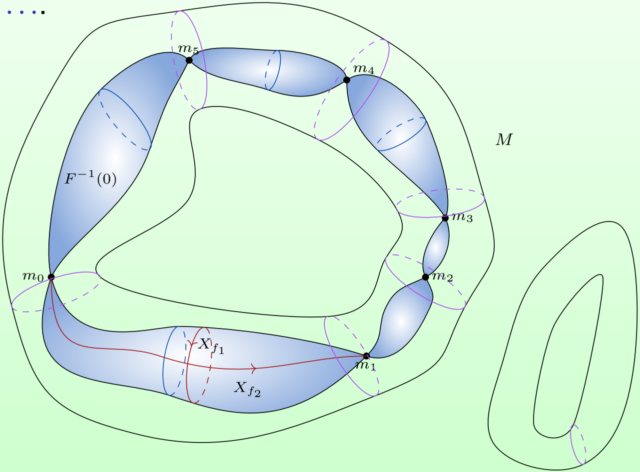


Global version by Duistermaat (**Comm. Pure Appl. Math 1980**).

[17 of 71] Singular fiber

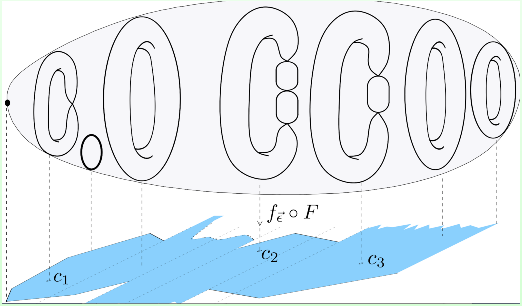
... But the **most important information** about an integrable system can be found in its ***singularities***: the m where $X_{f_1}(m), \ldots, X_{f_n}(m)$ are linearly dependent. For example $m_0, m_1, \ldots$.



[18 of 71] Remember this picture?

Remember our earlier figure of typical integrable system $F\colon (M^4,\omega) \to \mathbb{R}^2$? It is singular Lagrangian fibration with:

- $\boxed{\text{Regular fibers}}$: 2-dimensional tori.
- $\boxed{\text{Fibers with singularities}}$: circles, points, pinched tori.



We do not understand (most of) their symplectic invariants **with exceptions**: so called toric and semitoric cases (in a moment).

$***$

The singularities correspond to

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critical points of $F\colon M^{2n} \to \mathbb{R}^n$,

that is, the m such that $\mathrm{d}_m F$ has rank $< n$. Expanding "in power series",

What are the local models of F?

Very few, and without higher order terms. Let's see it.

****

[19 of 71] Deep analytic theorem by Eliasson (see also Colin de Verdière, Rüssmann, Vey, Vũ Ngọc, Wacheux, Zung ...)

It is known since 1984 that an integrable system without hyperbolic singularities is given

in neighborhood of each singularity $m = (0, \dots, 0)$,

in symplectic coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ by models:

$$(Q_1, Q_2, \dots, \dots)$$

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3. ***Focus-Focus***: $Q_i = (x_i \xi_{i+1} - x_{i+1} \xi_i,\ x_i \xi_i + x_{i+1} \xi_{i+1})$.

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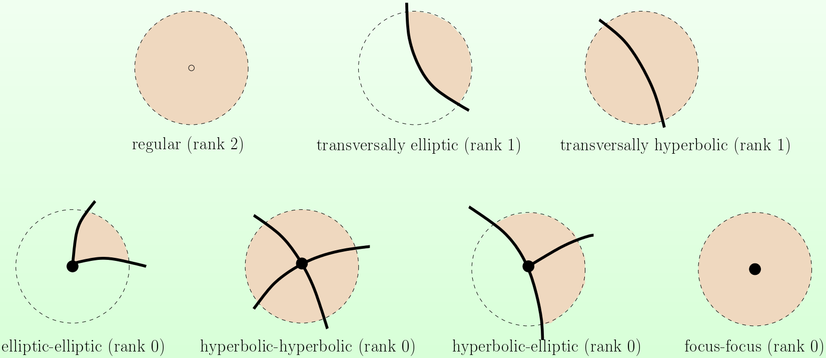
There is version with hyperbolic singular points ($Q_i = x_i\xi_i$, only other possible model). Singularities: assumed non-degenerate.

Images of singularities in dimension 4:



This concludes PART I (Integrable Systems).

Recommended articles for this part:

1. Hamiltonian and symplectic symmetries: an introduction, **Bulletin of the American Mathematical Society** 2017.
2. Symplectic theory of completely integrable Hamiltonian systems (with S. Vũ Ngọc) **Bulletin of the American Mathematical Society** 2011.

****

Let's solve CHALLENGE in two important cases:

TORIC AND SEMITORIC.

We will also see related results/challenges and applications to Quantum Geometry.

****

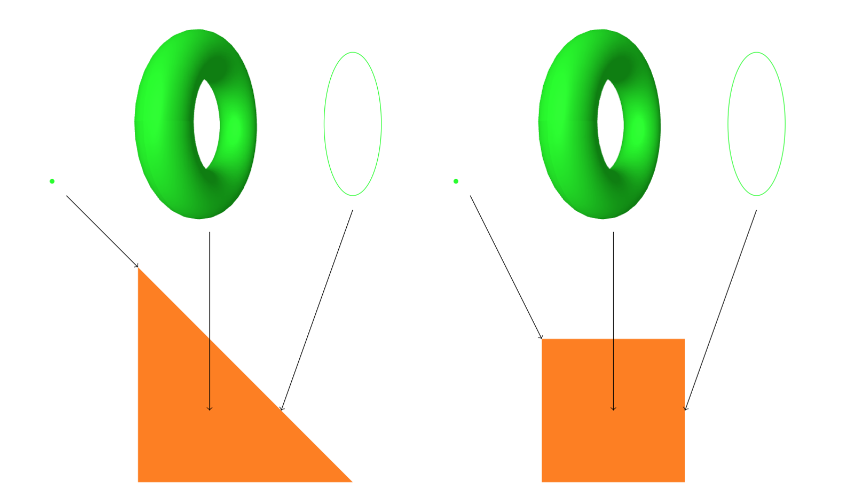
****

PART 2: TORIC SYSTEMS

(approx. 30 minutes)

****

Image and fibers of typical toric system on compact symplectic 4-manifold, viewed as map $M \to \mathbb{R}^2$:



THERE ARE NO PINCHED TORI !

[20 of 71] TORIC SYSTEMS: Atiyah ... 1980s

An integrable system

$$F = (f_1, \ldots, f_n) \colon M^{2n} \longrightarrow \mathbb{R}^n$$

is $\boxed{\textit{toric}}$ if all the vector fields $X_{f_1}, \ldots, X_{f_n}$ generate periodic flows.

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The **fundamental example** is

$$F = (f_1, \ldots, f_n) \colon \mathbb{C}P^n \to \mathbb{R}^n$$

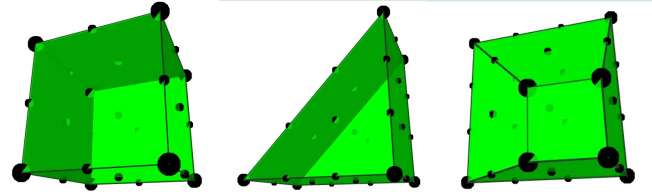
$$F([z_0 \colon z_1 \colon \ldots \colon z_n]) = \left(\frac{|z_1|^2}{\sum_{i=0}^n |z_i|^2}, \ldots, \frac{|z_n|^2}{\sum_{i=0}^n |z_i|^2} \right).$$

It is induced by the rotational torus action on $\mathbb{C}^{2n+1}$.

[21 of 71] TORIC SYSTEMS ARE CONNECTED TO POLYTOPES

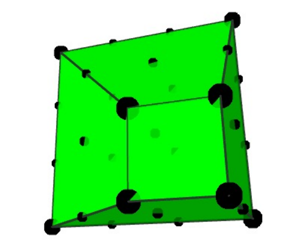
Theorem (Atiyah **Bulletin London Mathematical Society 1982**, Guillemin-Sternberg **Inventiones Mathematicae 1982**)

If M^{2n} is compact and F is toric, the ***classical spectrum*** *$F(M^{2n})$ is convex polytope in $\mathbb{R}^n$ (simple, rational, smooth).*



[22 of 71] Delzant polytopes

In honor of **Thomas Delzant**, the polytopes we just saw carry his name.



Definition (DELZANT POLYTOPE)

A **Delzant polytope** in $\mathbb{R}^n$ is a simple, rational n-polytope such that the primitive edge-direction vectors at each vertex are basis of $\mathbb{Z}^n$.

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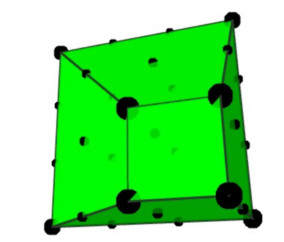

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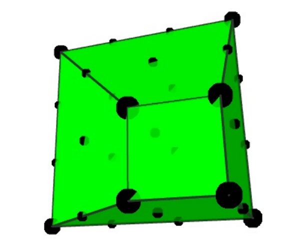

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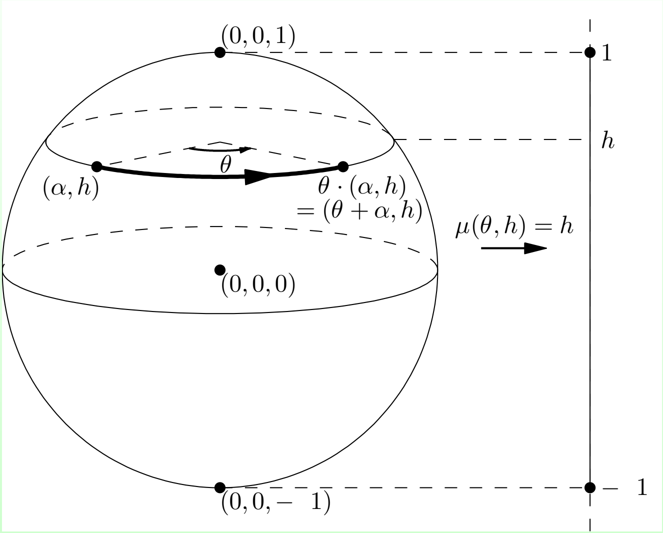
Notation: $\mathscr{D}(n) \equiv$ set of Delzant n-polytopes in $\mathbb{R}^n$.

[23 of 71] Toy toric system

Toy toric system $\mu = f_1 : S^2 \to \mathbb{R}$.



[24 of 71] Classification of toric systems

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Teorema (Delzant **Bull Soc Math France 1988**)

The correspondence which sends

$$\underbrace{[F\colon (M^{2n},\omega)\to\mathbb{R}^n]}_{\text{isomorphism class of toric system}} \qquad \textit{to} \qquad \underbrace{F(M)}_{\text{polytope}}\subset\mathbb{R}^n$$

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is a bijection between the isomorphism classes of toric systems and the set of simple, rational, smooth polytopes.

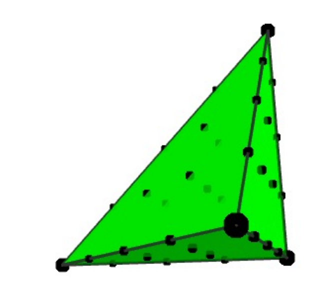
Around the influence of this theorem, and related ideas, **strong research groups developed at different universities, including Harvard, MIT and UC Berkeley**.

 Bridge between two mathematical worlds!

Thanks to these impressive theorems,

PROBLEMS ABOUT TORIC SYSTEMS

can be posed as



PROBLEMS ABOUT POLYTOPES.

These THEOREMS give $\boxed{\text{bridge between two worlds}}$.

[26 of 71] Why translating from manifolds to polytopes?

Disadvantages:

- The combinatorial problem may not be any easier.

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Disadvantages:

- The combinatorial problem may not be any easier.
- Translating problem back and forth is technical.

Advantages:

- Since 1990s, major developments about polytopes.
- Some problems in geometry, once translated, appear to be more doable (seem more "concrete").

[27 of 71] Peculiarity of toric systems

The conclusions of these theorems are so strong because the

singularities of toric systems are the simplest ones

since **they cannot have focus-focus nor hyperbolic models**:

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$$F(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = \Big(\underbrace{\frac{x_1^2 + \xi_1^2}{2}, \ldots, \frac{x_k^2 + \xi_k^2}{2}}_{\text{elliptic type}}, \xi_{k+1}, \ldots, \xi_n\Big).$$

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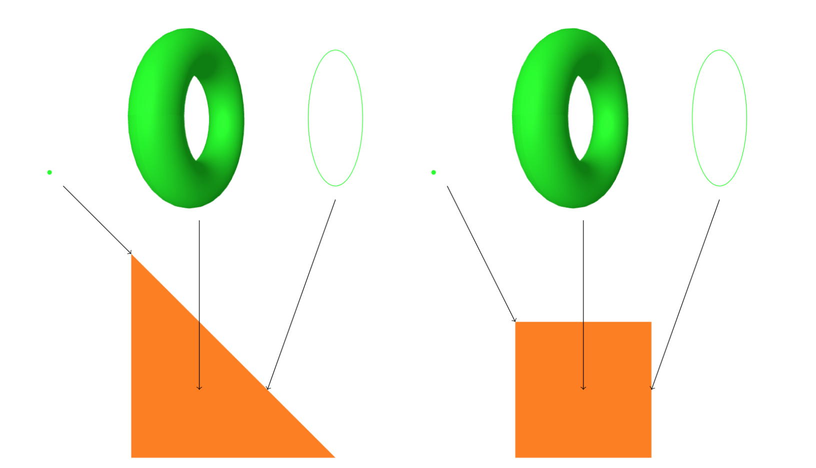
2. The fibers are diffeomorphic to tori $\mathbb{T}^{n-k}$, $0 \leq k \leq n$.

[28 of 71] Fibers of toric systems

LEFT: Toric system on $\mathbb{C}P^2$.

RIGHT: On $\mathbb{C}P^1 \times \mathbb{C}P^1$.



[29 of 71] More classifications of toric systems

The previous theorems have extensions to a variety of cases, **where the symplectic manifold is replaced by a more general or complicated space**. For example, see following works (and references therein) about:

1. *Toric systems on* $\boxed{\textit{LOG-SYMPLECTIC}}$ *manifolds* ***(Poisson manifolds with symplectic structure away from hypersurfaces with certain configurations):***

   Li-Gualtieri-Pelayo-Ratiu, **Math. Annalen 2017**.

2. *Toric systems on* $\boxed{\textit{NON COMPACT}}$ *manifolds:*

   Karshon-Lerman, **SIGMA 2015**.

3. *Toric systems on* $\boxed{\textit{ORBIFOLDS}}$*:*

   Lerman-Tolman, **Trans. Amer. Math. Soc. 1997**.

****

Toric integrable systems provide useful examples, and many problems about them are still open, including famous conjectures.

What are some of these problems?

We see next a small sample ...

****

****

A Natural Question:

What is the structure (geometric, topological etc) of the set

$$\mathscr{M}(2n)$$

of isomorphism classes of (compact connected) toric integrable systems?

****

****

THIS CAN BE ANSWERED USING $\boxed{\text{POLYTOPE THEORY!}}$

Let's see how next.

****

[30 of 71] Metrics and topologies

Definition (SYMMETRIC AND HAUSDORFF DISTANCE)

Let $\mathscr{C}(n)$ be set of compact convex sets in $\mathbb{R}^n$. Define

$$\delta^V, \delta^H : \mathscr{C}(n) \times \mathscr{C}(n) \longrightarrow \mathbb{R}_{\geq 0},$$

$$\text{SYMMETRIC} : \delta^V(P,Q) = \mathrm{Vol}(P \setminus Q) + \mathrm{Vol}(Q \setminus P),$$

$$\text{HAUSDORFF} : \delta^H(P,Q) = \max\{\max_{x \in P} \mathrm{dist}(x,Q), \max_{y \in Q} \mathrm{dist}(y,P)\}.$$

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- $\mathscr{C}_{\mathrm{p}}(n) \subset \mathscr{C}(n)$: set of proper ones (non-empty interior).
- On $\mathscr{C}_{\mathrm{p}}(n)$, δ^H and δ^V are **not equivalent** distances, but induce **same topology**. We endow $\mathscr{D}(n)$ with it.

[30 of 71] Metrics and topologies

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$$\delta^V, \delta^H : \mathscr{C}(n) \times \mathscr{C}(n) \longrightarrow \mathbb{R}_{\geq 0},$$

SYMMETRIC : $\delta^V(P,Q) = \mathrm{Vol}(P \setminus Q) + \mathrm{Vol}(Q \setminus P)$,

HAUSDORFF : $\delta^H(P,Q) = \max\{\max_{x \in P} \mathrm{dist}(x,Q), \max_{y \in Q} \mathrm{dist}(y,P)\}$.

- $\mathscr{C}_{\mathrm{p}}(n) \subset \mathscr{C}(n)$: set of proper ones (non-empty interior).
- On $\mathscr{C}_{\mathrm{p}}(n)$, δ^H and δ^V are **not equivalent** distances, but induce **same topology**. We endow $\mathscr{D}(n)$ with it.
- Endow $\mathscr{M}(2n)$ with metrics/topology by $\mathscr{M}(2n) \equiv \mathscr{D}(n)$.

[31 of 71] Connectivity of moduli spaces

Theorem (Pelayo-Pires-Ratiu-Sabatini **Geom. Dedicata** 2014, Pelayo-Santos 2023 **arxiv:2303.02369**)

$\mathscr{M}(2n)$ *is* $\boxed{\textbf{\textit{path-connected}}}$ *and* $\boxed{\textbf{\textit{dense in }} \mathscr{C}_{\mathrm{p}}(n)}$.

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- Same paper solves problems by **Fujita**, **Kitabeppu**, **Mitsuishi**.
- $\boxed{\text{Most of paper is "metric geometry"}}$ concerning both δ^V, δ^H, and contains a number of other results (technical for talk).

****

Another natural question:

What happens with blow-ups of toric integrable systems?

THIS CAN ALSO BE ANSWERED USING POLYTOPES.

How? We see it next.

****

[32 of 71] Reflexive polytopes

Lattice polytope: polytope with integer vertices.

Definition (REFLEXIVE POLYTOPE)

A **reflexive polytope** is a lattice polytope such that every facet-supporting hyperplane is of the form

$$u_F \cdot x = 1,$$

where u_F is the primitive vector normal to the facet.

[33 of 71] Properties of reflexive polytopes

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[33 of 71] Properties of reflexive polytopes

1. Equivalently, a reflexive polytope is a lattice polytope whose dual also has integer vertices.
2. Every reflexive polytope **has the origin as the unique interior lattice point**. Hence, for reflexive polytopes $\mathrm{AGL}(n, \mathbb{Z})$-equivalence is the same as $\mathrm{GL}(n, \mathbb{Z})$-equivalence.

[34 of 71] Monotone polytopes and monotone blow-ups

Definition (MONOTONE POLYTOPE AND BLOW-UP)

- **Monotone polytope**: Delzant and reflexive.

dim	1	2	3	4	5	6	7	8	9
#	1	5	18	124	866	7622	72256	749892	8229721

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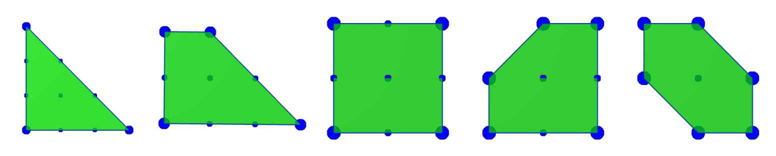
- **Monotone blow-up** in monotone polytope: blow-up ("chop off vertex") that results in monotone polytope.

These are the only five 2-dimensional monotone polygons (up to $\mathrm{GL}(2,\mathbb{Z})$ equivalence):



[35 of 71] The monotone simplex

1. n-**simplex**: convex hull of its $n+1$ vertices.

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2. The **smooth unimodular** n-**simplex** is:

$$\Delta_n := \left\{ x \in \mathbb{R}^n \,\middle|\, x_i \geq 0 \; \forall i \quad \text{and} \quad \sum_{i=1}^{n} x_i \leq 1 \right\}$$

[35 of 71] The monotone simplex

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3. Only **monotone** n-**simplex** (mod $\mathrm{GL}(n, \mathbb{Z})$):

$$-\mathbf{1} + (n+1)\Delta_n \simeq (n+1)\Delta_n$$

[36 of 71] McDuff's Question

McDuff (**Geometry and Topology 2011**): Is there a monotone polytope Δ of dimension $n > 2$ for which:

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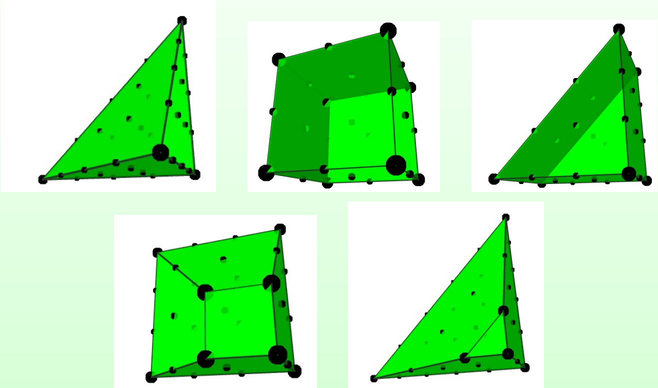
McDuff (**Geometry and Topology 2011**): Is there a monotone polytope Δ of dimension $n > 2$ for which:

1. one can make at least two monotone and disjoint blow ups of points,
2. or, more generally, of any two faces of codimension > 2?

[37 of 71] Monotone polytopes in dimension three

These are the five **maximal** monotone 3-polytopes: simplex, cube, triangular prism, slanted cube, and slanted prism.



Maximal is with respect to inclusion. In dimension 3 they coincide with those that cannot be obtained as blow-up of another monotone polytope.

[38 of 71] Full answer to McDuff's Question

Theorem (Polyhedral version of result by Bonavero 2003, arXiv:2308.03085)

Only monotone polytopes that admit one monotone blow-up at a point are the **monotone n-simplex $(n+1)\Delta_n$** *and the* **blow-up of a codimension-two face of it**.

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Theorem (Polyhedral version of result by Bonavero 2003, **arXiv:2308.03085**)

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Theorem (Pelayo-Santos 2023, **arXiv:2308.03085**)

1. *No monotone polytope of dimension* $n > 2$ *admits* $\boxed{\textit{two}}$ *monotone disjoint blow-ups at points.*

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1. *No monotone polytope of dimension $n > 2$ admits* $\boxed{\textbf{two}}$ *monotone disjoint blow-ups at points.*
2. *The monotone n-simplex $(n+1)\Delta_n$ admits disjoint blow-ups at faces F_1, F_2 if and only if F_1, F_2 are disjoint and* $\boxed{\textbf{their codimensions add up to } n+2 \textbf{ or } n+1}$.

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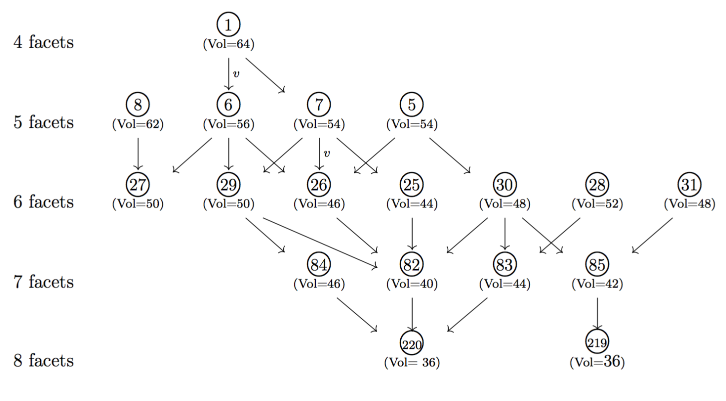
   So if $n \geq 4$, the monotone n-simplex admits two disjoint monotone blow-ups at faces of codimensions > 2.

[39 of 71] Blow-ups among monotone 3-polytopes.

THE 18 MONOTONE 3-POLYTOPES (arranged in rows according to number of 2-faces) and relations through blow-ups:



- Arrows represent monotone blow-ups.
- Arrows with v: only two blow-ups at vertices.
- Each blow-up adds 1 to number of facets.

****

The previous results concern

POLYTOPES,

but what do they tell us about

TORIC SYSTEMS?

****

[40 of 71] TORIC SYSTEMS = Symplectic toric manifolds

In symplectic and algebraic geometry,

TORIC SYSTEMS

$$F = (f_1, \ldots, f_n) \colon (M^{2n}, \omega) \to \mathbb{R}^n$$

are usually called

SYMPLECTIC TORIC MANIFOLDS

$$(M^{2n}, \omega, \mathbb{T}^n)$$

to emphasize: $\boxed{F \text{ induces Hamiltonian } n\text{-torus action}}$ on (M^{2n}, ω), by concatenation of Hamiltonian flows of $f_1, \ldots, f_n$. Depending on context I use both names.

[41 of 71] Monotone symplectic manifolds

Definition (MONOTONE SYMPLECTIC MANIFOLD)

A compact symplectic (M^{2n}, ω) is **monotone** if there is $\lambda > 0$ with $[\omega] = \lambda \mathrm{c}_1(M^{2n})$.

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- By rescaling, we may assume $\lambda = 1$.

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Notes:

- By rescaling, we may assume $\lambda = 1$.
- Δ is monotone $\iff$ $\Delta =$ momentum polytope of monotone symplectic toric manifold (modulo $\lambda = 1$ and translation).
- That is, via the Delzant Correspondence: **Monotone symplectic toric $2n$-manifolds $\equiv$ monotone n-polytopes**.

[42 of 71] Symplectic translations

The previous results have $\boxed{\text{symplectic formulations}}$:

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2. $\mathbb{C}P^n$ *admits two disjoint monotone toric blow-ups if and only if exceptional divisors of blow-ups have complex dimensions adding up to* $n-2$ *or* $n-1$.

****

There is a famous conjecture from the 1980s by Günter Ewald

concerning monotone geometry,

and a generalization by Benjamin Nill from 2009. In fact, these ideas

have symplectic implications!

Let's see what we can say ...

****

[43 of 71] Ewald's Conjecture

Ewald's Conjecture 1988

If P is monotone n-polytope in $\mathbb{R}^n$ then the set

$$\mathbb{Z}^n \cap P \cap -P = \text{"symmetric integer points"}$$

contains unimodular basis of $\mathbb{Z}^n$, i.e. the standard basis up to $\mathrm{GL}(n,\mathbb{Z})$.

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Nill (2009) asked if this generalization might hold:

General Ewald's Conjecture 2009

If P is lattice Delzant n-polytope with origin in interior, then $\mathbb{Z}^n \cap P \cap -P$ contains unimodular basis of $\mathbb{Z}^n$.

[44 of 71] Progress on these conjectures

1. Up to $n = 7$, **Ewald's conjecture** was proven by **Øbro** ($\sim$ 2007) using computational software.

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3. The paper contains a proof of **Generalized Ewald Conjecture** for $n = 2$.

4. The paper also has proof of **Generalized Ewald Conjecture** for $n = 3$ up to a minor hypothesis. The case $n \geq 4$ is open.

****

This is all very interesting, but many of you are probably wondering

Does $\boxed{\text{Ewald's Conjecture}}$ have applications in symplectic geometry?

The answer is **YES**. I learned of the connection in work of Dusa McDuff.

****

[45 of 71] Why do symplectic geometers care about Ewald's Conjecture?

Ewald's Conjecture is $\boxed{\text{connected}}$ with the problem of when fibers of monotone symplectic toric manifolds:

$$F^{-1}(c), \;\; c \in F(M^{2n}),$$

are displaceable by Hamiltonian isotopy:

Biran-Entov-Polterovich, Comm. Contemp. Math. 2004,

Cho, IMRN 2004,

McDuff, Geom. Top. 2011.

Being displaceable at **manifold** level can be studied with **polytopes** via ideas related to the conjecture.

[46 of 71] Some examples of this connection

In view of **J. Brendel** (**J. Sympl. Geom. 2023**), the work by **Crespo-Pelayo-Santos** on Ewald's Conjecture implies that if $F(M^{2n})$ belongs to a certain (large, explicit) class of polytopes $\mathscr{U}$, then:

the Chekanov torus can be embedded into the monotone symplectic toric manifold M^{2n} to yield an exotic Lagrangian which is not real.

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This statement corresponds to **Brendel**'s result, and our contribution is to give $\mathscr{U}$ (Thanks to J. Brendel for letting us know about connection).

[47 of 71] Conclusions

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2. Perhaps a similar translation can also be done for more general integrable systems.
3. This has given rise to new problems in combinatorics.
4. From the viewpoint of symplectic geometry, the answers to the combinatorial problems may have crucial consequences.

****

Ewald's Conjecture is a challenge.
It has symplectic implications. Some symplectic problems can be solved as problems about polytopes. What are key problems about polytopes which can be solved as symplectic problems?

****

****

I conclude PART 2 speaking about **quantum integrable systems.**

The hope is that from

Quantum (Spectral) Information

we can extract a lot of

Classical Information.

****

****

First let's see what we mean by

Quantum Integrable System

****

[49 de 71] Quantum integrable systems

Let $(\mathscr{H}_k)_{k\in\mathbb{N}^*}$ be Hilbert spaces **giving quantization** of $2n$-manifold (M^{2n},ω) (famous method: Kostant-Souriau Geometric Quantization; I learned about it from B. Kostant and RAC colleague P. L. García).

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$\boxed{\textbf{\textit{Quantum integrable system}}}$: n self-adjoint commuting semiclassical operators

$$\psi_1 := (\psi_{1,k})_{k\in\mathbb{N}^*}, \ldots, \psi_n := (\psi_{n,k})_{k\in\mathbb{N}^*}$$

acting on $(\mathscr{H}_k)_{k\in\mathbb{N}^*}$, whose principal symbols form integrable system on M^{2n}.

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acting on $(\mathscr{H}_k)_{k \in \mathbb{N}^*}$, whose principal symbols form integrable system on M^{2n}. Its (semiclassical) **_spectrum_** is the support of joint spectral measure, which if $\mathscr{H}_k$ are finite dimensional is, for $k \in \mathbb{N}^*$,

$$\left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \bigcap_{j=1}^{n} \ker(\psi_{j,k} - \lambda_{j,k} \mathrm{Id}) \neq 0 \right\}.$$

****

To make these notions (quantization, principal symbol) precise is technical, depends on context (compactness etc) and is beyond the scope of talk. For introduction:

- **my 2024 entry speech at Royal Spanish Academy of Sciences (video, in Spanish)**;
- **my 2023 BBVA Foundation Project Opening Lecture in Madrid (video, in English)**.
- My paper in **Bull. Belgian Math. Soc. 2023** and papers with Charles, Vũ Ngọc **A.E.N.S 2023** and with Polterovich, Vũ Ngọc, **Proc. LMS 2014**.

****

[50 of 71] Idea of previous concepts: quantization/symbol

Very roughly (if M^{2n} is compact and connected):

$$\mathscr{H}_\hbar := \mathrm{H}^0(M^{2n}, \mathscr{L}^k), \ \ \hbar = \frac{1}{k},$$

is ***space of holomorphic sections of tensor powers*** $\mathscr{L}^k$ of certain line bundle over M^{2n} and our semiclassical (Berezin-Toeplitz) operators are sequences

$$T := (T_\hbar := \Pi_\hbar f(\cdot, k) \colon \mathscr{H}_\hbar \to \mathscr{H}_\hbar)_{\hbar = 1/k,\, k \in \mathbb{N}^*}$$

where multiplication operator $f(\cdot, k)$ has expansion

$$f_0 + k^{-1} f_1 + k^{-2} f_2 + \cdots$$

for C^∞ topology ($\Pi_\hbar$ surjective orthogonal projector) and f_0 is ***principal symbol*** of T.

[51 of 71] Challenge by Weyl, Bochner and Moser

Inverse Problem for quantum integrable systems
(Origin: Weyl and Bochner at end of XIX century and beginning of XX, Moser in 1970s, Kac in 1960s)

Given the **semiclassical/quantum spectrum**

$$(X_\hbar)_{\hbar>0} \subset \mathbb{R}^d, \quad \hbar = \frac{1}{k}, \quad k \in \mathbb{N}^*$$

of quantum system

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$$T_1 := (T_{1,\hbar})_{\hbar>0}, \ldots, T_d := (T_{d,\hbar})_{\hbar>0},$$

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$$T_1 := (T_{1,\hbar})_{\hbar>0}, \ldots, T_d := (T_{d,\hbar})_{\hbar>0},$$

What information can we obtain about $\boxed{\text{classical system}}$ of principal symbols

$$f_1, \ldots, f_d \text{ of } T_1, \ldots, T_d?$$

****

There is a general principle in quantum mechanics which says something very interesting about the

Weyl-Bochner-Moser problem.

****

[52 de 71] QUANTUM MECHANICAL principle

- **CORRESPONDENCE PRINCIPLE**: behavior of quantum observables converges in high frequency limit (semiclassical, after rescaling) to analogous behavior of classical observables.

[52 de 71] QUANTUM MECHANICAL principle

- CORRESPONDENCE PRINCIPLE: behavior of quantum observables converges in high frequency limit (semiclassical, after rescaling) to analogous behavior of classical observables.
- EXAMPLE: it is "generally" known (depends on operators, compactness etc) that **QUANTUM SPECTRUM** converges to **CLASSICAL SPECTRUM** (image of principal symbols).

[52 de 71] QUANTUM MECHANICAL principle

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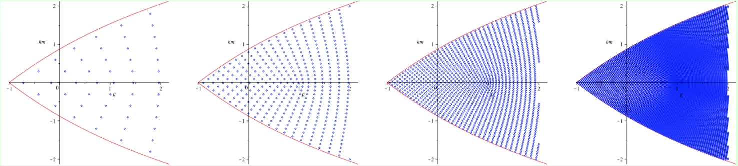

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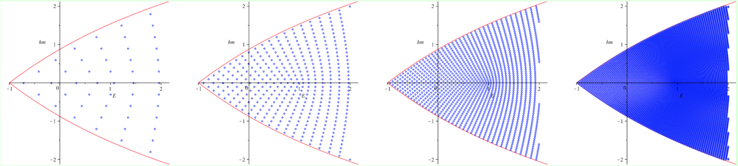


- $\boxed{\text{Pioneering results by Colin de Verdière}}$ (1979, 1980, pseudodifferential operators), extended by **Polterovich, Vũ Ngọc, myself** (**Proc. LMS 2014**) to Berezin-Toeplitz case.

****

So we can in principle recover $F(M)$ of classical system from quantum spectrum! We saw, for **TORIC SYSTEMS**: this image (a polytope) is only invariant. So if we verify convergence we are done.

How about for general integrable systems?

The spectrum is $\boxed{\text{unlikely}}$ to contain all symplectic information of the principal symbols. But since we understand very well **SEMITORIC SYSTEMS** ...
We see what happens with these two cases next.

****

****

Using **Symplectic Geometry**...

... next we solve the inverse quantum problem by Weyl, Bochner and Moser for **Toric Systems**.

****

[53 de 71] Quantum toric systems

A quantum integrable system

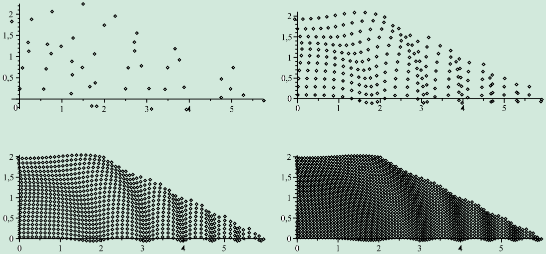
$$\psi_1, \ldots, \psi_n$$

is $\boxed{\textit{toric}}$ if the principal symbols are a toric system.

[54 de 71] Solution to Weyl, Bochner and Moser challenge for toric systems

Teorema (Charles-Pelayo-Vũ Ngọc, **Ann. E.N.S. 2013**)

Let $\psi_1, \ldots, \psi_n$ be a quantum toric system on compact (M^{2n}, ω). Then spectrum of $\psi_1, \ldots, \psi_n$ ***converges to classical spectrum****:*



So the spectrum determines the classical toric system given by (M, ω) and the principal symbols of $\psi_1, \ldots, \psi_n$.

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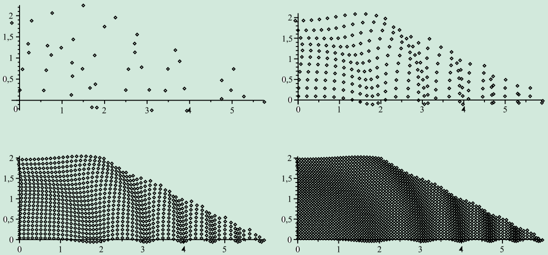


So the spectrum determines the classical toric system given by (M, ω) and the principal symbols of $\psi_1, \ldots, \psi_n$.

Hence, challenge solved in compact toric case! Same paper shows: **toric systems on compact manifolds can be quantized.**

This concludes PART II (Toric Systems).

1. Reduced phase space and toric variety [...] of Delzant spaces (with J.J. Duistermaat), **Math. Proc. Cambr. Phil. Soc.** 2009.
2. Isospectrality for quantum toric integrable systems (with L. Charles, S. Vũ Ngọc), **Ann. Sci. l'École Norm. Sup.** 2013.
3. Semiclassical quantization and spectral limits of h-pseudodifferential and Berezin-Toeplitz operators (with L. Polterovich, S. Vũ Ngọc), **Proc. London Math. Soc.** 2014.
4. The tropical momentum map: a classification of toric log symplectic manifolds (with M. Gualtieri, S. Li, T. Ratiu), **Math. Ann.** 2017.
5. Ewald's Conjecture and integer points in algebraic and symplectic toric geometry (with L. Crespo, F. Santos), **2023. arxiv:2310.10366.**
6. The structure of monotone blow-ups in symplectic toric geometry and a question of McDuff (with F. Santos), **2023. arxiv:2308.03085.**
7. Moduli spaces of Delzant polytopes and symplectic toric manifolds (with F. Santos), **2023. arxiv:2303.02369.**

****

Toric systems are very interesting, however they are rare in physical world. The periodicity of all flows is too restrictive ... what IF

some flow is not periodic? They are semitoric and there are many.

Jaynes-Cummings Model, coupled angular momenta ... let's see it!

****

****

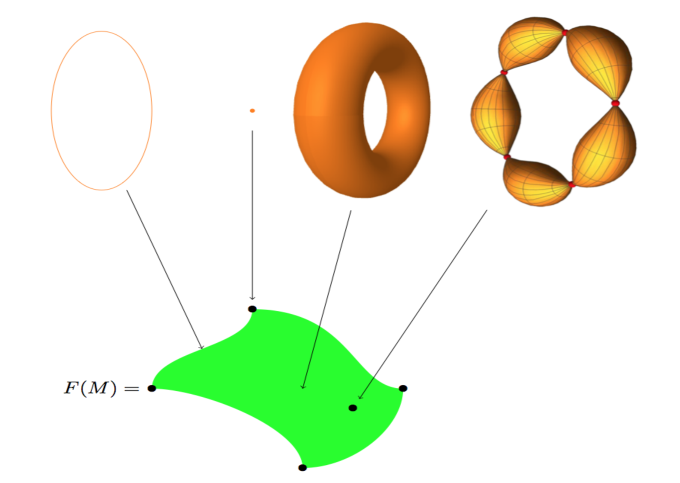
PART 3: SEMITORIC SYSTEMS

(approx. 15 minutes)

****

[55 of 71] Picture to keep in mind

Image and fibers of a typical semitoric system on a compact symplectic 4-manifold, viewed as a map $M^4 \to \mathbb{R}^2$:



[56 of 71] Semitoric systems in DIMENSION 4

An integrable system on a symplectic 4-manifold (M^4, ω):

$F = (f_1, f_2)$ is $\boxed{\textit{semitoric}}$ if vector field X_{f_1} has periodic flow,

but we do not ask anything of X_{f_2}.

 Semitoric systems in DIMENSION 4

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but we do not ask anything of X_{f_2}.

In addition, M^4 **may not be compact** but in order to prove things, for the moment we also require that they satisfy:

- F has no local hyperbolic models;
- preimages of compact sets by f_1 are compact;
- $\boxed{\textbf{Simple}\text{: each fiber of } f_1 \text{ has at most one focus-focus point}}$.

****

The term semitoric is discussed in my paper **Top. App. 2023** in special volumen in honor of my college geometry teacher Prof. J.M.R. Sanjurjo.

In contrast with toric case, <u>many physical models are semitoric</u>, like the ones we saw earlier; let's recall them ...

****

[57 of 71] Two crucial examples we saw earlier

Among the most important **semitoric** integrable systems are:

COUPLED ANGULAR MOMENTA by Sadovskií-Zhilinskií

Let $R_2 > R_1 > 0$. On $M^4 = S^2 \times S^2$ with $(x_1, y_1, z_1, x_2, y_2, z_2)$:

$$\begin{cases} f_1 := R_1 z_1 + R_2 z_2 \\ \\ f_{2,t} := (1-t) z_1 + t(x_1 x_2 + y_1 y_2 + z_1 z_2) \end{cases} \quad \forall t \in [0,1],$$

with symplectic form $-(R_1 \omega_{S^2} \oplus R_2 \omega_{S^2})$.

COUPLED SPIN-OSCILLATOR i.e JAYNES-CUMMINGS MODEL

On $M^4 = S^2 \times \mathbb{R}^2$, $(x,y,z) \sim (\theta, h)$ coordinates on S^2, (u,v) on $\mathbb{R}^2$,

$$\begin{cases} f_1(x,y,z,u,v) = \frac{u^2+v^2}{2} + z \\ \\ f_2(x,y,z,u,v) = \frac{ux+vy}{2} \end{cases}$$

endowed with $\mathrm{d}\theta \wedge \mathrm{d}h + \mathrm{d}u \wedge \mathrm{d}v$.

****

In essence: a semitoric system is like a toric system but **in addition it has very complicated singularities** (of focus-focus type).

How is this reflected in list of invariants?

Before (toric) there was a polygon, now there is a $\boxed{\text{polygon and labels}}$. We see it next.

****

[58 of 71] Symplectic invariants of semitoric systems

Theorem (Pelayo-Vũ Ngọc, **Inventiones Mathematicae 2009**)

*A semitoric system $(f_1,f_2)\colon M^4 \to \mathbb{R}^2$ is **symplectically determined** by polygon Δ constructed from its classical spectrum and points $p_1,\ldots,p_n \in \Delta$ labelled with invariant $k \in \mathbb{Z}$ and invariant $\sum a_{ij}x^iy^j$.*

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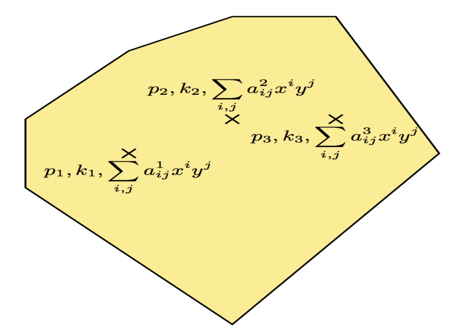

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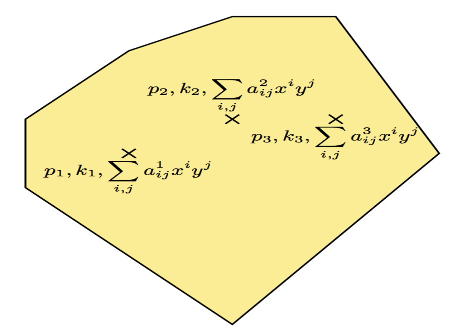

$\boxed{p_1,\ldots,p_n \text{ are focus-focus values}}$, k encodes topology "between" fibers, $\sum_{i,j} a_{ij}x^iy^j$ encodes dynamics of X_{f_1},X_{f_2} near focus-focus fibers.

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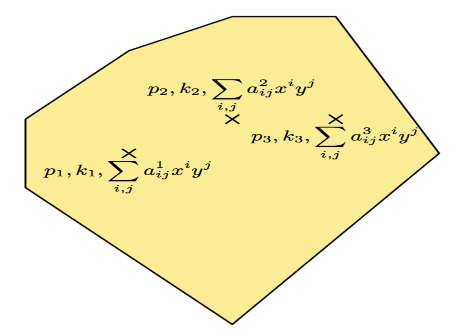

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[59 of 71] Proof in Inventiones Mathematicae 2009

The proof of this theorem:

1. **uses ideas or techniques of** Arnold, Atiyah, Delzant, Dufour, Duistermaat, Eliasson, Guillemin, Miranda, Molino, Sternberg, Toutlet, Zung and others.

2. **is inspired by** **theory of toric integrables systems by** Atiyah, Guillemin, Delzant, Kostant, Sternberg, and **the classifications of the Fomenko School** (Bolsinov, Fomenko, Matveev, Oshemkov, Tabachnikov, Zung, ...).

3. **is related to** **classification of almost-toric systems** (Symington 2002, Leung-Symington 2010, Hohloch-Sabatini-Sepe-Symington 2017).

[60 of 71] Classification of semitoric systems

Inventiones: uniqueness. Following: existence. Together: classification.

[60 of 71] Classification of semitoric systems

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Theorem (Pelayo-Vũ Ngọc, **Acta Mathematica 2011**)

From the

purely combinatorial information of a rational polygon Δ

with certain properties and points $p_1, \ldots, p_n$ *in it, each labelled with an integer* k *and a Taylor series* $\sum_{i,j} a_{ij} x^i y^j$, *one can construct*

symplectic 4***-manifold*** M^4 ***and semitoric system*** (f_1, f_2)***,***

whose invariants are Δ, *the points* $p_1, \ldots, p_n$ *and the labels.*

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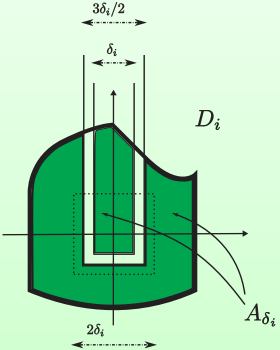
This result is **ANOTHER BRIDGE** between ON ONE END:

- **Symplectic Geometry** (of Integrable Systems) and on the other
  - a mixture of **Combinatorics** (polygon Δ), **Analysis** (Taylor series) and **Topology** (the k).

[61 of 71] Idea of proof, Acta Mathematica 2011

The proof is based on ***new cut and paste techniques***.

One **cuts** Δ into pieces over which there are action-angle variables. **Paste** together pieces and obtain "continuous" system F, to which we have to **glue/attach** fibers of focus-focus models given by Taylor series. We obtain "system", extremely singular on overlaps:



Hardest part: modify the system to make it C^∞-smooth $(\varepsilon - \delta)$.

****

An indication of good health for a subject is that there are many people working on it from different angles. We conclude this part mentioning a few active avenues of research in the "toric/semitoric" worlds:

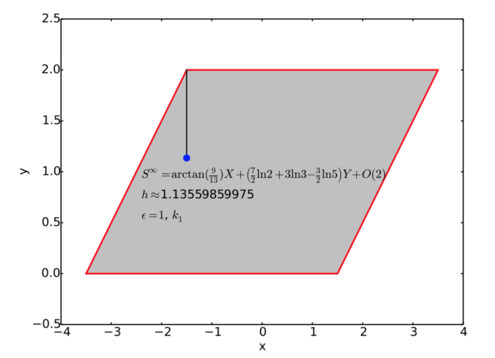
1. **COMPUTATIONS OF INVARIANTS**,
2. **EXTENSIONS OF THEORY**,
3. **MODULI SPACES**.

****

[62 of 71] Example of computation of invariants

Theorem (Le Floch-Pelayo, J. Nonlinear Science 2019)

If $t = 1/2$, $R_1 = 1$, $R_2 = 5/2$, *the invariants of the semitoric system* $F_t\colon S^2 \times S^2 \to \mathbb{R}^2$ *of **coupled angular momenta** are: polygon* Δ, *one point* $p_1 \in \Delta$ *with labels* $k_1 = 0$ *and Taylor series* $\mathrm{S}^\infty = \sum_{i,j} a_{ij} x^i y^j$*:*



Note: Calculations can be done for arbitrary R_1, R_2. If $\Theta = \frac{R_2}{R_1}$,

$$h = \frac{2R_1}{\pi} \arccos\left(\frac{1}{2\sqrt{\Theta}}\right) + \frac{R_1}{\pi}\sqrt{4\Theta - 1} - \frac{2R_1(\Theta-1)}{\pi}\arctan(2\Theta - 1) + \frac{2R_1(\Theta-1)}{\pi}\arctan\left(\frac{(2\Theta^2 - 2\Theta + 1)\sqrt{4\Theta - 1} - 2\Theta^2}{(2\Theta-1)^2}\right).$$

[63 of 71] Works about computations of invariants

Symplectic invariants of semitoric systems are computed in:

1. **Jaynes-Cummings Model/Coupled spin-oscillator**: Pelayo-Vũ Ngọc **Comm. Math. Physics 2012**.
2. **Spherical Pendulum**: Dullin **Journal of Differential Equations 2013**.
3. **Coupled Angular Momenta**: Le Floch-Pelayo **Journal of Nonlinear Science 2019**.
4. **Symplectic classification of coupled spin-oscillators**: Alonso-Dullin-Hohloch **Journal of Geometry and Physics 2019**.
5. **Symplectic classification of coupled angular-momenta**: Alonso-Dullin-Hohloch **Nonlinearity 2020**.
6. **Families with two focus-focus singularities**: Alonso-Hohloch **Journal of Nonlinear Science 2021**.
7. **New interpretations/computations of invariants**: Alonso-Hohloch-Palmer **arxiv:2309.16614, 2023**.

[64 of 71] Example: extension of original theory 2009-2011

Concerning extensions, my PhD students:

- **Joseph Palmer**, PhD University of California, San Diego 2016 (Illinois Urbana-Champaign, USA)
- **Xiudi Tang**, PhD University of California, San Diego 2018 (Beijing Institute Technology, China),

and I, extended classification of semitoric systems to case of **several focus-focus singularities per fiber *(non-simple* case)**.

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Theorem (Palmer-Pelayo-Tang, **arXiv 2019-2023**)

The symplectic invariant of (simple or non-simple) semitoric systems $(f_1,f_2)\colon M^4 \to \mathbb{R}^2$ *is convex polygon* Δ *with* $p_1,\ldots,p_n$ *inside, each with label* $(k_s \in \mathbb{Z},\, \sum_{i,j} a^1_{ij}x^iy^j,\ldots,\sum_{i,j} a^{m_s}_{ij}x^iy^j)$ *with as many series* m_s *as focus-focus points in* $F^{-1}(p_s)$.

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Invariant k_s is more complicated. There are several series per p_s (when there is one pinched point in fiber, series due to **Vũ Ngọc**, **Topology 2003**, general case **Pelayo-Tang**, **J. Fixed Point Theory Appl. 2024**).

[65 of 71] Some papers on extensions of semitoric theory

1. **Complexity 1 spaces**: Karshon-Tolman (several papers), Sepe-Tolman **arxiv:2402.05814, 2024**.
2. **Toric-focus integrable systems**: Ratiu-Wacheux-Zung **Memoirs Amer. Math. Soc. 2023**.
3. **Hyperbolic singularities**:
   - Dullin-Pelayo **J. Nonlinear Science 2016**.
   - Hohloch-Palmer **arXiv:2105.00523, 2021**.
   - Gullentops-Hohloch **arXiv:2209.15631, 2022**.
4. **Dimensions** $2n > 4$; Wacheux, **arXiv:1408.1166, 2014**, affine invariant if $2n = 6$.
5. **Weaker hypothesis**: Pelayo-Ratiu-Vũ Ngọc, **Journal of Symplectic Geometry 2015, Nonlinearity 2017**.
6. **Faithful semitoric systems**: Hohloch-Sabatini-Sepe-Symington **SIGMA 2018**.

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[66 of 71] Moduli spaces related to integrable systems

The results about moduli spaces of integrable systems are technical and we do not discuss them here. I recommend:

1. Pelayo (**Proc. Amer. Math. Soc. 2007**) studies moduli space of toric symplectic ball embeddings.
2. Figalli-Pelayo (**Adv. Geom. 2016**) studies further problem above.
3. Palmer (**J. Geom. Physics 2017**) studies moduli spaces of semitoric systems.
4. Figalli-Palmer-Pelayo (**Ann. SNS. Pisa 2018**) continuity of equivariant capacities on these moduli spaces.
5. (Previously mentioned papers) Pelayo-Pires-Sabatini-Ratiu (**Geom. Dedicata 2014**) by Pelayo-Santos (**arxiv 2023**).

****

SO we understand now classical semitoric systems.

Does this mean we know how to solve the Weyl-Bochner-Moser theorem for them?

In fact **YES**, let's see how.

****

[67 of 71] Solution to Challenge for semitoric systems

A quantum integrable system ψ_1, ψ_2 is $\boxed{\textit{semitoric}}$ if principal symbols form semitoric system.

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Theorem (Pelayo-Vũ Ngoc, **Communications in Mathematical Physics 2012**, **Communications in Mathematical Physics 2014** y Le Floch-Pelayo-Vũ Ngọc, **Mathemastiche Annalen 2016**)

The spectrum of quantum semitoric integrable system

determines all classical invariants

but perhaps k*: that is, determines* Δ, $p_1, \ldots, p_n$ *and the* $\sum_{i,j} a_{ij} x^i y^j$.

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In 2021 Le Floch-Vũ Ngọc (**arXiv:2104.06704**) proved k is determined.
Together with theorem, **challenge solved for semitoric systems!**
That is, spectrum of quantum system determines classical system.

[68 of 71] Idea of proofs for quantum theorems

Idea to prove these quantum theorems is first

$\boxed{\text{detect classical invariants in quantum spectrum.}}$

This involves **Analysis** $\varepsilon - \delta$ (Microlocal) and **Spectral Theory**.
Since we know symplectic geometry, we know what to look for.

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This involves **Analysis** $\varepsilon - \delta$ (Microlocal) and **Spectral Theory**.
Since we know symplectic geometry, we know what to look for.

Once we have invariants, DONE, by previous theorems!

****

Hence, we have constructed bridges

between **Symplectic Geometry** on the one hand and **Combinatorics, Analysis and Topology**

on the other hand, in order to classify classical integrable systems and then solve the quantum inverse problem of **Weyl, Bochner, Moser** in the toric/semitoric case. Still, A LOT LEFT TO DO...

****

****

... INDEED, it is a challenge to obtain symplectic classifications of integrable systems in dimensions 6 **or higher (beyond toric)**. Even in semitoric case. There is too much freedom in singularities/singular fibers. **So it is also a challenge to solve their Weyl-Bochner-Moser inverse spectral problem**.

****

****

Progress on the symplectic geometry of integrable systems in the past 20 years has been remarkable. Despite of this, current methods do not seem powerful enough to break the dimension barrier (more than two degrees of freedom) or deal with general systems. The field can benefit from incorporating techniques from other areas. The recent work on Ewald's Conjecture (which uses polytope theory) is an example.

In a recent (2023) special volume in honor of my college geometry teacher JMR Sanjurjo, I listed open problems. Solving them may require "outside" techniques, next I mention a few.

****

[69 of 71] Open problems

Thanks to L. Polterovich for explanations about the following.

By Floer theory we get non-trivial measure (**Entov-Polterovich, CMH 2006**) on image of integrable system $F: M^{2n} \to \mathbb{R}^n$ which helps detecting its non-displaceable fibers (**Polterovich-Rosen, CRM Monograph 2014** and **Polterovich, ECM 2016**).

Dickstein-Ganor-Polterovich-Zapolsky (2021), **arXiv:2107.10012**, develop categorification of quasi-states: the image of $F: M^{2n} \to \mathbb{R}^n$ contains a structure called IVM (ideal-valued measure).

PROBLEM. Can these or similar techniques be used to construct invariants of integrable systems (eg. foliations replacing polytope)?

[70 of 71] More open problems ...

$\boxed{\textbf{PROBLEM.}}$ Use J-holomorphic techniques to give other proofs of known results about integrable systems, or prove new ones.

[70 of 71] More open problems ...

PROBLEM. Use J-holomorphic techniques to give other proofs of known results about integrable systems, or prove new ones.

PROBLEM. Can displaceable (or non-displaceable) fibers be detected in semiclassical spectrum of quantum integrable system?

[71 of 71] A last challenge ...

IAS Professor **Vladimir Voevodsky** (born 1966) passed away in 2017. He was a Fields Medallist, and one of the most important figures in **Algebraic Geometry and Algebraic Topology**.

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In 2015, Voevodsky, Warren and I published paper taking first steps in $\boxed{p\text{-adic geometry of integrable systems}}$. **Can one carry out the symplectic/quantum geometry I just discussed over p-adics?** **I believe so, and this is a great challenge.**

Papers on p-adic (symplectic) integrable systems

The paper I just mentioned by:

Pelayo-Voevodsky-Warren (Mathematical Structures in Computer Science 2015)

sketches basic ideas of p-adic integrable systems.

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Luis Crespo (Cantabria) and I have been working on

p-ADIC JAYNES-CUMMINGS MODEL.

The basic "real" theory (image of system, fibers, etc) is simple compared to "p-adic" theory, which involves long calculations, often depending on value of p.

Further contributions to integrable systems

The literature on integrable systems is extensive. The work I have presented today has been influenced by ideas of my teachers, students, mentors, collaborators, and many other authors, including:

- Hamiltonian dynamics, symplectic dynamics, analytic methods: **Hofer, Mather, Zehnder** (1970s-).
- Direct/inverse spectral problems: **Charles, Guillemin, Moser, Polterovich, Sarnak, Sjöstrand, Uhlmann, Weinstein, Zelditch, Zworski** (1970s-).
- Fourier integral operators, spectral theory: **Colin de Verdière, Duistermaat, Guillemin, Hörmander, Weinstein** (1970s).
- Global action-angle coordinates: **Duistermaat** (1980).
- Convexity of toric systems, equivariant theory: **Atiyah, Bott, Guillemin, Heckman, Sternberg** (1982, 1988).
- Stable bundles and integrable systems: **Hitchin** (1987).
- Singularities: **Eliasson, Rüssmann, Vey** (1960s-1990s).
- Affine, complex geometry, Lagrangian fibrations: **Auroux, Gross-Siebert, Gualtieri, Seidel** (2003-).

Moser's words

I conclude, with the words of **Jürgen Moser** in a Plenary Talk at the 1998 ICM in Berlin:

"In a time of dangerous specialization we should feel free to use all tools available to us, and use them with proper taste.

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"In a time of dangerous specialization we should feel free to use all tools available to us, and use them with proper taste. To me it seems idle to argue whether to prefer solving of challenging problems, building of abstract structures, or working on applications. Rather we should keep an open mind when we approach new problems, and not forget the unity of mathematics."

This concludes PART III (Semitoric Systems).

Recommended articles for this part:

1. Semitoric integrable systems on symplectic 4-manifolds (with S. Vũ Ngọc), **Inventiones Mathematicae** 2009.
2. Constructing integrable systems of semitoric type (with S. Vũ Ngọc), **Acta Mathematica** 2011.
3. Hamiltonian dynamics and spectral theory for spin-oscillators (with S. Vũ Ngọc), **Comm. Math. Phys.** 2012.
4. Semiclassical inverse spectral theory for singularities of focus-focus type (with S. Vũ Ngọc), **Comm. Math. Phys.** 2014.
5. Inverse spectral theory for semiclassical Jaynes-Cummings systems (with Y. Le Floch, S. Vũ Ngọc), **Math. Ann.** 2016.
6. Symplectic G-capacities and integrable systems (with A. Figalli and J. Palmer), **Ann. Scuola Norm. Sup. Pisa** 2018.
7. Semitoric systems of non-simple type (with J. Palmer and X. Tang), **Preprint 2019, revised 2023. arXiv:1909.03501**.
8. Vũ Ngọc's Conjecture on focus-focus singular fibers with multiple pinched points (with X. Tang), **J. Fixed Point Theory App.** 2024.

HOPE YOU ENJOYED THE TALK. THANKS FOR LISTENING!