Symplectic Orbifold Gromov-Witten Invariants
(work in progress).

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Our aim is to construct Gromov-Witten invariants of symplectic orbifolds.

This was done over $\mathbb{Q}$ by Chen and Ruan in arXiv:0103156, but we wish to define more general counts (e.g. for $K$-theory).

We also wish to present Moduli spaces of holomorphic curves in terms of Global-Kuranishi Charts.

This is part of a larger project with Ritter in which we will attempt to prove a version of the Crepant resolution conjecture relating Gromov-Witten invariants of birational orbifolds.

There is also ongoing work by Mak, Seyfaddini and Smith for global quotient orbifolds.
We think of an orbifold $X$ as a ‘manifold’, except that the charts are locally modelled on open subsets of $\mathbb{R}^n$ quotiented by a finite linear group action.

So, locally, there is a coordinate chart $V \subset \mathbb{R}^n$ together with a linear group action of a finite group $\Gamma$ on $V$ and a map $V/\Gamma \to X$ which is a homeomorphism onto its image.
Suppose $G$ is a compact Lie group acting on a smooth manifold $M$ with finite stabilizers.

Then the quotient $X = [M/G]$ is naturally an orbifold.

(Slice theorem): For each point $x \in M$, there is a $G_x$-equivariant submanifold $S_x \subset M$ containing $x$ and a $G$-equivariant neighborhood $U_x \subset M$ of $x$ so that the following map is a $G$-equivariant diffeomorphism:

$$G \times_{G_x} S_x \to U_x.$$
After shrinking the slice $S_x \subset M$, we can assume that $S_x$ has a global coordinate system with $G_x$ acting linearly.

Then $(S_x, G_x)$ is our induced orbifold chart centered at $x$.

The set theoretic quotient $M/G$ is called the underlying coarse moduli space which we will write as $X$.

Theorem (Pardon): Every smooth orbifold is a quotient $[M/G]$. 

orbifolds
Morphisms of Orbifolds

Let $[M_1/G_1]$ and $[M_2/G_2]$ be orbifolds.

An HS (Hilsum-Skandalis) morphism between these orbifolds is a diagram:

$$
\begin{array}{ccc}
    P & \xrightarrow{f} & M_2 \\
    \downarrow{\pi} & \searrow{G_1-\text{equiv}} & \\
    M_1 & \rightarrow & \end{array}
$$

where $P$ is a smooth manifold admitting a $G_1 \times G_2$-action and with $\pi$ a principal $G_2$-bundle.

Really, it is an equivalence class of such diagrams.

Locally, there are charts $(V_1, \Gamma_1), (V_2, \Gamma_2)$ and a map $\Gamma_1 \rightarrow \Gamma_2$ and a $\Gamma_1$-equivariant map $V_1 \rightarrow V_2$. 
A symplectic orbifold is a smooth orbifold $X$ together with a closed non-degenerate 2-form $\omega$ on it.

We can define compatible almost complex structures $J$ on such symplectic orbifolds.

The spaces of such $J$'s is contractible.

Let us fix $(X, \omega, J)$ and $\beta \in H_2(X; \mathbb{Z})$. 

We can define a complex orbifold to be an orbifold with an integrable almost complex structure.

A twisted nodal curve $\Sigma$ is a space of the form $\tilde{\Sigma}/\sim$ where $\Sigma$ is a one dimensional complex orbifold and where $\sim$ identifies a finite collection of distinct pairs of points $p \sim q$ so that the following balancing condition holds:

- $p$ admits an orbifold chart with coordinate $z$ and where $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, z) \rightarrow e^{2\pi im/k}z$ and
- $q$ admits an orbifold chart with coordinate $w$ where $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, w) \rightarrow e^{-2\pi im/k}w$.

We call $\tilde{\Sigma}$ the normalization of $\Sigma$ and the points that we have identified are called the nodes.
So, near a node, a twisted nodal curve looks like

\[ \{xy = 0\}/(\mathbb{Z}/k\mathbb{Z}) \subset \mathbb{C}^2/(\mathbb{Z}/k\mathbb{Z}) \]

where the group action is \((g, (x, y)) \rightarrow (gx, g^{-1}y)\) where \(g = e^{2i\pi m/k}\).

The reason for the balancing condition is it allows the node to be smoothed. Locally

\[ \{xy = t\}/(\mathbb{Z}/k\mathbb{Z}), \quad t \in \mathbb{C} \]

is the smoothing of the nodal curve \(t = 0\).

A marking on a twisted nodal curve \(\Sigma\) is a collection of distinct points \(p_1, \cdots, p_h\) on \(\Sigma\) disjoint from the nodes and containing all the points with nontrivial stabilizers.
We call $\Sigma = (\Sigma, p_1, \cdots, p_h)$ a **twisted nodal curve with $h$ marked points**.

A **twisted nodal curve** $u : \Sigma \to X$ is an $HS$-morphism from the normalization $\tilde{\Sigma}$ of $\Sigma$ to $X$ so that

- the induced map of stabilizer groups $G_\sigma \to G_{u(\sigma)}$ is injective for each $\sigma \in \tilde{\Sigma}$
- and which descends to a continuous map $\Sigma \to X$ of coarse moduli spaces.

The **genus** of $u$ is the arithmetic genus of the underlying coarse moduli space of its domain $\Sigma$. 
A map $u : \Sigma \to X$ from a twisted nodal curve is *stable* if it has finitely many automorphisms:

$$
\begin{array}{c}
\Sigma \\
\downarrow \phi \\
\Sigma
\end{array} \quad \xrightarrow{u} \quad \begin{array}{c}
X \\
\downarrow \phi
\end{array}
$$

If the domain has marked points, the automorphism $\phi$ must fix these marked points.

We let $\mathcal{M}_{g,h,\beta}(X)$ be the moduli space of stable $J$-holomorphic maps from genus $g$ twisted nodal curves to $X$ representing $\beta$. 
Example:  $M = pt$,  $G = \mathbb{Z}/2$. So $\mathcal{M}_{g,h,0}$ is the moduli space of twisted nodal curves together with a $\mathbb{Z}/2$ principal bundle.

We wish to put a fundamental class on $\mathcal{M}_{g,h,\beta}(X)$ so that we can integrate pullbacks of cohomology classes from the inertia stack against it to give Gromov-Witten invariants.

A global Kuranishi chart is a tuple $(G, T, E, s)$ where $G$ is a Lie group acting semi-freely on a manifold $T$ and $E$ is a $G$-vector bundle over $T$ with a $G$-equivariant section $s$.

We call $T$ the thickening and $E$ the obstruction bundle.

Such a global Kuranishi chart models $\mathcal{M}_{g,h,\beta}(X)$ if this moduli space is homeomorphic to $s^{-1}(0)/G$. 
The fundamental class is given by \([T/G] \cap s^*(Th(E))\) where 
\([T/G]\) is the fundamental class in \(G\)-equivariant homology 
and \(Th(E)\) is the Thom class of the obstruction bundle in 
\(G\)-equivariant cohomology.

Here, we need an appropriate orientation for \(T - g\) and \(E\).

So, how do we construct such a global Kuranishi chart for 
\(\mathcal{M}_{g,h,\beta}(X)\)?
Genus Zero Manifold Case

- Let us start in the simpler setting where $X$ is a smooth manifold and the genus is zero.
- The following construction is due to Abouzaid-M-Smith.
- We let $\mathcal{F}_{h,d}$ be the moduli space of genus zero degree $d$ curves with $h$ marked points mapping to $\mathbb{P}^d$ whose image is not contained in a hyperplane.
- This is a smooth quasi-projective variety.
- We let $C_{h,d} \to \mathcal{F}_{h,d}$ be the corresponding universal curve and $C_{h,d}^\circ$ the complement of its nodes.
Genus Zero Manifold Case

Let $Y_{h,d} \rightarrow C_{h,d} \times X$ be the vector bundle whose fiber over a point $(p, x)$ is the space of anti-holomorphic maps from the tangent bundle of the fiber of $C_{h,d}$ at $p$ to $T_x X$.

A finite dimensional approximation scheme is a sequence $(V_\mu, \lambda_\mu)_{\mu \in \mathbb{N}}$ of $PU(d + 1)$-equivariant maps $\lambda_\mu : V_\mu \rightarrow C_\infty^C(Y_{h,d})$ from a $PU(d + 1)$ representation $V_\mu$ so that the union of their images is dense, $V_\mu \subset V_{\mu + 1}$ and $\lambda_{\mu + 1}|_{V_\mu} = \lambda_\mu$ for each $\mu$.

We define the pre-thickened moduli space $T^{pre}$ to be the space of tuples $(u, \phi, e)$ where $\phi \in F$, $u : C|_\phi \rightarrow X$ is a stable map and $e \in V_\mu$ so that

$$\overline{\partial}Ju(p) = \lambda_\mu(e)(p, u(x)), \quad \forall \ p \in C^o|_\phi \times X$$

The topology on this space is induced from the Hausdorff topology on graphs in $C \times X$ as well as the topology on $V_\mu$. 
A naive guess for the obstruction bundle is $V_{\mu}$ with the section sending $(u, \phi, e)$ to $e$ since setting $e = 0$ gives $J$-holomorphic curves.

This would be fine if our group $G$ is $\text{PGL}_{\mathbb{C}}(d + 1)$ - however this does not work since $\lambda_{\mu}$ cannot be made to be $\text{PGL}_{\mathbb{C}}(d + 1)$-equivariant.

So we need to reduce the group $\text{PGL}_{\mathbb{C}}(d + 1)$ to $\text{PU}(d + 1)$.

First, choose a Hermitian line bundle $L \to X$ whose curvature form $\Omega_L$ tames $J$. 
A framed curve is a triple \((u, \Sigma, F)\) where \(u: \Sigma \to X\) is a smooth map representing \(\beta\) and \(F = (f_0, \cdots, f_d)\) is a basis of \(H^0(u^*L)\) where \(d = c_1(L)(\beta) + 1\).

The basis \(F\) induces a map

\[
\phi_F: \Sigma \to \mathbb{P}^d, \quad \phi_F(\sigma) = [\tau f_0(\sigma), \cdots, \tau f_d(\sigma)]
\]

where \(\tau\) is a trivialization \(\tau: L|_\sigma \cong \mathbb{C}\).

Hence we have an identification \(\psi_F: \Sigma \xrightarrow{\cong} C|_{\phi_F}\).
Let $\mathcal{H}_{d+1}$ be the space of $(d + 1) \times (d + 1)$ Hermitian matrices.

We have an identification

$$\exp : \mathcal{H}_+ \overset{\mathbb{R}}{\to} PGL_{d+1}(\mathbb{C})/PU(d + 1).$$

We define $A_F := \exp^{-1} B$ where $B$ is the matrix with $i,j$ entry

$$\int_\Sigma \langle f_i, f_j \rangle \Omega_L.$$
We define the *thickening* $T$ to be the space of isomorphism classes of tuples $(u, \Sigma, F, e)$ so that $(u \circ \psi_F^{-1}, \phi_F, e) \in T^{pre}$.

The group $G = PU(d + 1)$ acts on $T$ via postcomposition in $\mathbb{P}^N$.

The obstruction bundle $E$ has fiber $\mathcal{H}_+ \times V_\mu$.

The section $s$ sends $(u, \Sigma, F, e)$ to $(A_F, e)$.

So, $(G, T, E, s)$ is our Global Kuranishi chart.
We wish to generalize this to higher genus with $X$ an orbifold rather than a manifold.

There are two problems.

The first problem is that twisted nodal curves with at least one orbifold point don’t map to $\mathbb{P}^d$.

The second problem is that line bundles of a given degree on a higher genus curve aren’t unique.
Let us deal with the first problem.

We will use work of Ross and Thomas.

Instead of looking at moduli spaces of curve mapping to projective space, we use weighted projective space
\[ \mathbb{P}(w_0, \cdots, w_d) = (\mathbb{C}^{d+1} - 0)/\sim, \]
\[ (z_0, \cdots, z_d) \sim (t^{w_0}z_0, \cdots, t^{w_d}z_d) \text{ for each } t \in \mathbb{C}^*. \]
Let $Y$ be a complex compact orbifold with only cyclic quotient singularities.

In our case, we are only interested in one dimensional complex orbifolds corresponding to normalizations of twisted nodal curves.

A line bundle $L$ over $Y$ is \textit{locally ample} if for each $y \in Y$, the stabilizer of $y$ acts faithfully on the fiber $L|_y$.

It is \textit{globally positive} if $L^N$ is the pullback of an ample line bundle from the coarse moduli space $\overline{Y}$ where $N$ is the least common multiple of all the stabilizers of all the points on $Y$. 
L is *orbi-ample* if it is locally ample and globally positive.

Let $n_i := |H^0(L^i)|$ for each $i ∈ \mathbb{N}$.

A *k-framing* of $L$ is a tuple

$$(f_{ij})_{i=k\cdots,2k, j=0\cdots,n_i}$$

where $f_{ij}, j = 1, \cdots, n_i$ is a basis for $H^0(L^i)$ for each $i = k, \cdots, 2k$. 
Define

\[ \mathbb{P}_k(L) := \mathbb{P}(k, \cdots, k, k + 1, \cdots, k + 1, \cdots, 2k, \cdots, 2k) \]

where there are exactly \( n_i \) copies of \( k + i \) for each \( i = 1, \cdots, N \).

Define the map \( \phi_F : Y \to \mathbb{P}_k(L) \) sending \( y \in Y \) to \( \left[ \tau f_{ij}(s) \right]_{i=k, \cdots, 2k, j=0, \cdots, n_i} \) where \( \tau \) is any trivialization \( \tau : L|_s \overset{\sim}{\longrightarrow} \mathbb{C} \).

**Theorem.** (Ross, Thomas). \( \phi_F \) is an embedding for \( k \) large if \( L \) is orbi-ample.
Let \((X, \omega)\) be a symplectic orbifold with compatible almost complex structure \(J\) and let \(\beta \in H_2(X; \mathbb{Z})\).

Choose a locally ample orbi-vector bundle \(W \to X\) (this exists by Pardon’s result).

Choose a Hermitian line bundle \(L\) on \(X\) which is a pullback from the coarse moduli space whose curvature for \(\Omega_L\) tames \(J\).
Abramovich and Vistoli have constructed moduli spaces of twisted nodal curves mapping to smooth DM stacks (i.e. complex orbifolds).

For any weighted projective spaces $\mathbb{P}$, define $\mathcal{F} := \mathcal{F}_{g,h,D}(\mathbb{P})$ to be the moduli space of stable twisted nodal curves $u$ of degree at most $D$ satisfying $H^1(u^*O(1)) = 0$ and $u$ is automorphism free.

This is a smooth quasi-projective variety with universal curve $\mathcal{C} := \mathcal{C}_{g,h,D}(\mathbb{P}) \to \mathcal{F}_{g,h,D}(\mathbb{P})$. 
Let $k \gg 1$.

We define $\mathcal{F}_F$ be the space of tuples $(\phi, u, R)$ where

- $\phi \in \mathcal{F}$,
- $u : C|_\phi \rightarrow X$ is a twisted nodal curve and
- $R = (R_{ij})_{i=1,\ldots,2k.j=1,\ldots,n_i}$ is a $k$-framing of $W_u := K_{C|_\phi} \otimes (u^* W \otimes L_i)$.

The topology is the Hausdorff topology induced by graphs of $R_{ij}$ on the coarse moduli space of $C \times W_u \sum_{i=k}^{2k} n_i$. 
Let $\mathcal{C}^o \subset \mathcal{C}$ be the complement of the nodes and marked points.

Let $Y \to \mathcal{C}^o \times X$ be the vector bundle whose fiber over a point $(p,x)$ is the space of anti-holomorphic maps from the tangent space at $p$ of the fiber of $\mathcal{C}^o$ to $T_xX$.

Choose a finite dimensional approximation scheme $(\lambda_\mu, W_\mu)_{\mu \in \mathbb{N}}, \lambda_\mu : W_\mu \to C_c^\infty(Y)$ for $Y$. 
Definition: The pre-thickened moduli space $\mathcal{T}^{pre}$ is the space of tuples $((\phi, u, R), e) \in \mathcal{F}_F \times V_\mu$ satisfying:

$$\bar{\partial}_J u(p) = \lambda_\mu(e)(p, u(x)).$$

For $k, \mu \gg 1$, we get have that $\mathcal{T}^{pre}$ is a topological manifold (it has a $C^1_{loc}$ structure, when enables us to put a smooth structure on an enlargement of it).
For each \(((\phi, u, R), e) \in \mathcal{T}^{pre}\), define
\[ L_u := K_{C|\phi}(p_1, \cdots, p_h) \otimes u^* L \]
where \(K_{C|\phi}\) is the canonical bundle.

We define the **thickened moduli space** \(\mathcal{T}\) to be the space of tuples \((\phi, u, R, e, F)\) where \((\phi, u, R, e) \in \mathcal{T}^{pre}\) and \(F\) is a \(k\)-framing of \(L_u\).

We now need to construct the obstruction bundle.
Define \( \mathbb{P}_T := \mathbb{P}_k(L_u) \) for some \((\phi, u, R, e, F)\) in \( T \) (this does not depend on the point in \( T \) for \( k \gg 1 \) after shrinking).

This is the weighted projective space that our framing \( F \) maps to. So, we get a natural map \( \phi_T : C|_\phi \to \mathbb{P}_T \) and hence a map \( T \to \mathcal{F}_{g,h,D}(\mathbb{P}_T), \; D \gg 1 \).

Define \( \mathcal{F}_{\mathbb{P}^2_T} \) to be an appropriate moduli space of maps to \( \mathbb{P}^2_T \) and let \( \Delta_T \) be the normal bundle of the diagonal map \( \mathcal{F}_{g,h,D}(\mathbb{P}_T) \to \mathcal{F}_{g,h,D}(\mathbb{P}^2_T) \).

We can pull back this diagonal bundle to \( T \). Call it \( \tilde{\Delta}_T \).
► Our obstruction bundle $\mathcal{E}$ is then $\tilde{\Delta}_\mathcal{J} \times H_{\mathbb{P}_\mathcal{J}} \times H_W \times V_\mu$.
► The first component tells us how far away the bundle $\phi^* O(1)$ is from $L_u$. In other words, how far apart is $\phi$ and $\phi_F$ (which is an element of $\mathcal{F}_{\mathbb{P}^2_\mathcal{J}}$ and hence, via a metric maps to $\tilde{\Delta}_\mathcal{J}$).
► The second component $H_{\mathbb{P}_\mathcal{J}}$ is $G_{\mathcal{J}}^C / G_\mathcal{J}$ where $G_\mathcal{J}$ is the automorphism group of our weighted projective space $\mathbb{P}_\mathcal{J}$ and $G_\mathcal{J}$ is its maximal compact subgroup. It tells us how far our framing $F$ is from being orthogonal.
► The third component $G_W^C / G_W$ tells us how far $R$ is from being orthogonal. It is defined analogously. Here $G_W$ is a product of unitary groups.
► The last component tells us how far our map $u$ is from being holomorphic.
Theorem (in progress, M-Ritter). \((G_{\mathcal{G}} \times G_{\mathcal{W}}, \mathcal{I}, \mathcal{E}, s)\) is a Global-Kuranishi chart for \(\mathcal{M}_{g,h,\beta}(X)\). It is unique up to a series of standard operations and their inverses: stabilization, group enlargement and germ equivalence.

The thickening also admits a \(C^1_{\text{loc}}\)-structure, which means up to stabilization, it admits a smooth structure.

If we deform \(J\), then we get cobordant global Kuranishi charts, which shows that our Gromov-Witten counts are in fact invariants of the symplectic form.