Floer homology with DG coefficients. Applications to cotangent bundles

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Morse homology with DG coefficients. ArXiv 2308.06104

Floer homology with DG coefficients. Applications to cotangent bundles. ArXiv 2404.07953

Motivation #1

A fruitful line of investigation to detect symplectic phenomena is to compare classical invariants (differential topology) to their quantum counterparts (holomorphic curves)

Quantum invariant $\xrightarrow{\sim}$ Floer homology

(reference object)

(filtered $\rightsquigarrow$ spectral invariants)

comparison $\big\Vert$?

Classical invariant

(algebraic topology)

<u>Examples</u>: Quantum homology (Arnold conjecture, Quasi-morphisms)

Quantum Steenrod squares (pseudo-rotations)

Symplectic homology (Weinstein conjecture)

etc.

Quantum invariant $\overset{\sim}{\rule{3em}{0.4pt}}$ Floer homology

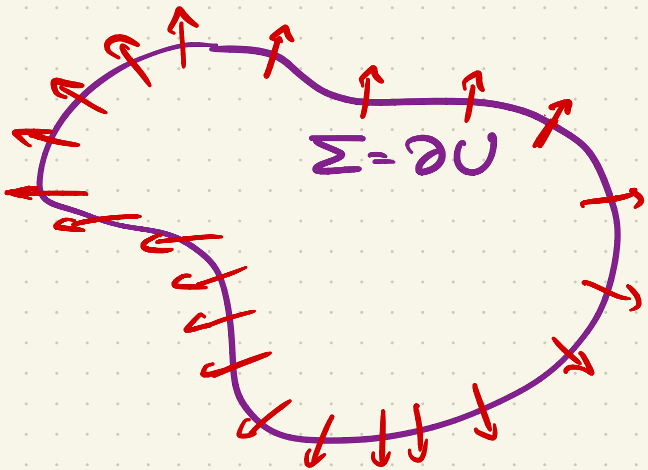
(reference object) (filtered $\rightsquigarrow$ spectral invariants)

comparison $\Vert$?

Classical invariant

(algebraic topology)

Example: Weinstein conjecture in $\mathbb{R}^{2n}$ using symplectic homology



local Liouville vector field Z

$$\mathcal{L}_Z \omega = \omega$$

$$Z \pitchfork \Sigma$$

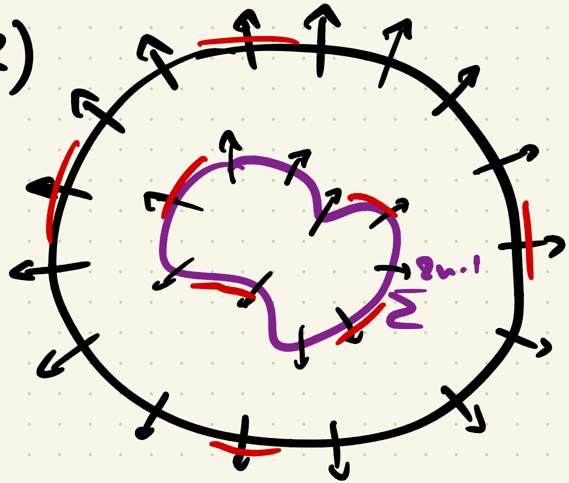
Thm (Viterbo): $\Sigma^{2n-1} \subset \mathbb{R}^{2n}$ closed hypersurface of contact type carries at least one closed characteristic

$\equiv$ closed integral curve of the line distribution $\ker \omega|_\Sigma$

$\equiv$ closed orbit of Hamiltonian $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ for which Σ is regular level set

Example: Weinstein conjecture in $\mathbb{R}^{2n}$ using symplectic homology (II)

$B^{2n}(R)$



"Counts" / "Sees" periodic orbits on the boundary via Floer cylinders / holomorphic curves

Classical invariant : singular homology

Quantum invariant : symplectic homology

$$
\begin{tikzcd}
H^{n-*}(B^{2n}(R)) \arrow[r] \arrow[d, "\text{injective}"] & SH_*(B^{2n}(R)) \arrow[d, dashed] \\
H^{n-*}(\operatorname{int}\Sigma) \arrow[r] & SH_*(\operatorname{int}\Sigma)
\end{tikzcd}
$$

$H^{n-*}(B^{2n}(R)) \neq 0$, $SH_*(B^{2n}(R)) = 0$

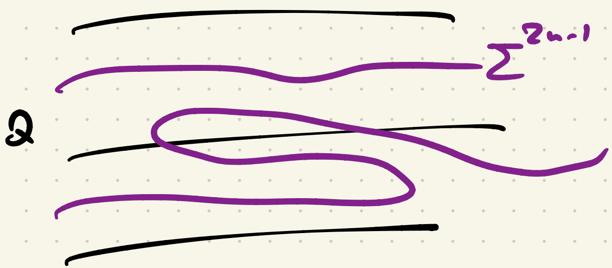
When applicable, this method produces contractible closed characteristics.

What about cotangent bundles?

Q^n smooth closed manifold $\longrightarrow$ T^*Q symplectic manifold

$$\omega_{can} = \sum_{i=1}^{n} dp_i \wedge dq_i$$

$D^*Q = \{|p| \leq 1\}$



Thm (Hofer-Viterbo) $\Sigma^{2n-1} \subset T^*Q$ closed hypersurface of contact type.
If $\mathrm{int}(\Sigma)$ contains the 0-section, then Σ carries at least one closed characteristic.

What about cotangent bundles? (II)

Previous scheme does not work (and cannot work, as it would provide closed characteristics that are contractible in T^*Q)

$$\begin{tikzcd}
H^{n-*}(D^*Q) = H_*(Q) \arrow[r] \arrow[d] & SH_*(D^*Q) = H_*(\mathcal{L}Q) \arrow[d, dashed] \\
H^{n-*}(\mathrm{nbd}\,\Sigma) \arrow[r] & SH_*(\mathrm{nbd}\,\Sigma)
\end{tikzcd}$$

can we make this sufficiently non-zero?

can we make this non-injective?

free loop space $= \mathrm{Map}(S^1, Q)$.

Would need $SH_*(D^*Q) = H_*(\mathcal{L}Q)$ "smaller" than $H_*(Q)$ $\rightsquigarrow$ **DG coefficients!**

Theorem (criterion for almost existence)

(W, ω) Liouville domain $\qquad (D^*Q)$

$U \subset W$ domain w. smooth bdry $\qquad (\mathrm{int}(\Sigma))$

Assume

$\exists\, \alpha \in H^{n-*}(W)$ s.t.

$$\begin{tikzcd} H^{n-*}(W) \arrow[r] \arrow[d] & SH_*(W) \\ H^{n-*}(U) & \end{tikzcd}$$

$$\alpha \longmapsto 0$$

$$\alpha \longmapsto \neq 0$$

Then almost every energy level near Σ carries a contractible closed characteristic.

("almost existence", Hofer-Struwe)

<u>Theorem 1</u> (BDHO) Q^n closed orientable.

Given S closed orientable n-manifold

and a map $\varphi : S \longrightarrow T^*Q$,

let $d = \deg(\pi \circ \varphi)$ and assume that:

$$\begin{tikzcd} & T^*Q \arrow[d, "\pi"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\pi\circ\varphi"'] & Q \end{tikzcd}$$

$K(\pi,1)$:
$\pi_i Q = 0$, $i \geq 2$

$\mathbb{Q}$-$K(\pi,1)$:
$\pi_i Q$ torsion, $i \geq 2$.

1) Either Q is not a $K(\pi,1)$ and $d = \pm 1$,

2) Or Q is not a rational $K(\pi,1)$ and $d \neq 0$.

Then almost existence holds for any hypersurface $\Sigma \subset T^*Q$

s.t. $\operatorname{int} \Sigma \supset \varphi(S)$.

Overlaps/Generalizes Hofer-Viterbo, Ritter, Irie, Frauenfelder-Pajitnov, Albers-Frauenfelder-O. etc.

Theorem 2 (BDHO) Q^n closed orientable

Assume Q is not a rational $K(\pi,1)$.

Then almost existence holds for any hypersurface $\Sigma \subset T^*Q$

s.t. $\pi : \mathrm{int}(\Sigma) \longrightarrow Q$ induces nonzero map

$$H_n(\mathrm{int}\,\Sigma; \mathbb{Z}) \longrightarrow H_n(Q;\mathbb{Z}).$$

Corollary (BDHO; contractible almost existence in exact magnetic cotangent bundles)

Q closed orientable, $\sigma \in \Omega^1(Q)$.

$\rightsquigarrow \omega_\sigma = \omega_{can} + \pi^* d\sigma$ "magnetic" symplectic form on T^*Q.

The previous statements hold for ω_σ

(use symplectomorphism $\underset{\omega_\sigma}{(p,q)} \longmapsto \underset{\omega_{can}}{(p+\sigma(q), q)}$)

E.g.
$Q \neq K(\pi, 1)$
$\operatorname{int}\Sigma \supset Q$

Overlaps/Generalizes/Complements Contreras, Contreras-Macarini-Paternain, Groman-Merry, Benedetti-Ritter …


Morst
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Morse homology with DG coefficients / Motivation #2

Our definition revisits and reinterprets the "enriched Floer complex" defined by Barraud-Cornea in 2007.

Definitions: X path-connected topological space, $\star$ basepoint

ΩX = Moore based loops $\big(\gamma : [0,L] \to X$

$\gamma(0) = \gamma(L) = \star\big)$

$$\Omega X \times \Omega X \xrightarrow[\text{concat.}]{} \Omega X \quad \text{strictly associative}$$

$C_*(\Omega X)$: dga of cubical chains on ΩX

(we use cubical chains because they are better adapted to products)

<u>Definition</u> : A <u>DG local system</u> / <u>derived local system</u> / <u>infinity local system</u> on X is a right $C_*(\Omega X)$-module (dg).

Main source of DG local systems: fibrations

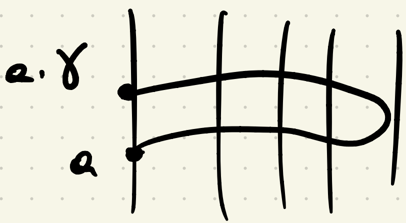
Given
$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] \\ & X \end{tikzcd}$$
Hurewicz fibration, it admits (up to replacing it with a naturally equivalent one)

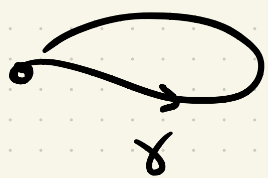
a transitive lifting function, unique up to htpy.

(Hurewicz, Fadell, Guggenheim, Dyer - Kahn '50-'60)

$\longrightarrow$ i.e. right ΩX-module structure on F: $\quad F \times \Omega X \longrightarrow F$

Think of such a lifting function as holonomy





/ analogous to deck transformation action for covering spaces

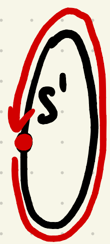
/ if $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ smooth fibration, any connection defines such a lifting function.

<u>Consequence</u>: $C_*F \otimes C_*\Omega X \longrightarrow C_*F$ right $C_*\Omega X$-module.

Example:

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

Hopf fibration



effect of holonomy

loop in ΩS^2

$$H_*(S^1) \otimes H_*(\Omega S^2) \longrightarrow H_* S^1$$

$$\langle pt, [S^1] \rangle \qquad \mathbb{Z}[x] \qquad |x| = 1$$

$$pt \otimes x \longmapsto [S^1]$$

$$\alpha \otimes pt \longmapsto \alpha$$

The rest is zero for degree reasons.

Twisted tensor product (Brown '59)

$F \hookrightarrow E \longrightarrow X$ Hurewicz fibration.

Q: Is it possible to give a model for $C_* E$ as

$$C_* F \otimes C_* X \circlearrowleft \partial_\otimes + \text{perturbation}$$

endowed with a "twisted" differential?

A: (Brown) YES, and we need 2 ingredients:

- $C_* \Omega X$ - module structure on $C_* F$
- a map $\varphi : C_* X \longrightarrow C_{*-1} \Omega X$ (Brown's universal twisting cocycle)

that satisfies a Maurer-Cartan equation

Morse theory with DG coefficients

X manifold, $f : X \longrightarrow \mathbb{R}$ Morse

ξ Morse-Smale negative gradient-like $\vec{v}$ field

$\rightsquigarrow$ $\overline{M}(x,y)$ compactified moduli spaces of Morse trajectories

Pick orientations o_x of $W^u(x)$, $x \in \mathrm{Crit} f$ $\rightsquigarrow$ $\overline{M}(x,y)$ oriented

$$\boxed{\partial \overline{M}(x,y) = \bigcup_{z \in \mathrm{Crit}(f)} (-1)^{|x|-|z|} \, \overline{M}(x,z) \times \overline{M}(z,y)}$$

<u>Barraud-Cornea</u> : $\exists$ collection $\{s_{x,y}\}$

$$s_{x,y} \in C_{|x|-|y|-1}\left(\overline{M}(x,y), \partial\overline{M}(x,y)\right)$$

representative of the fundamental class

s.t.

$$\boxed{\partial s_{x,y} = \sum_z (-1)^{|x|-|z|} s_{x,z} \times s_{z,y}}$$

The moduli spaces $\bar{M}(x,y)$ evaluate naturally into $P_{x\to y}X$

$$\bar{M}(x,y) \xrightarrow{ev} P_{x\to y}X \longrightarrow \Omega X$$

space of Moore paths $x \to y$

$$\begin{tikzcd} x \arrow[d] \\ y \end{tikzcd}$$

$\longleftarrow$ parametrized by level sets of f, an interval $[0, f(x)-f(y)]$

$\longrightarrow$ $m_{x,y} = ev_* s_{x,y} \in C_{|x|-|y|-1} P_{x\to y}X$ satisfy

concatenation!

$$\boxed{\partial m_{x,y} = \sum_z (-1)^{|x|-|z|} m_{x,z} \cdot m_{z,y}}$$

This can be further converted into a collection

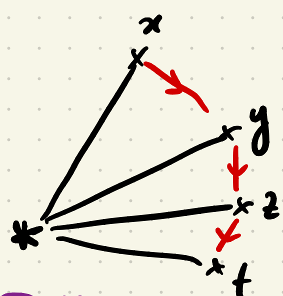
$$\{ m_{x,y} \in C_{|x|-|y|-1} \Omega X \}$$

that satisfies

$$\partial m_{x,y} = \sum_z (-1)^{|x|-|z|} m_{x,z} \cdot m_{z,y}$$

"Barraud-Cornea universal twisting cocycle"

(Choose embedded tree that connects the basepoint $*$ to Crit f, then collapse it)



tree $\subset X$

$P_{x \to y} X \to \Omega(X/_{\text{tree}}) \simeq \Omega X$

<u>Definition</u>: Let $\mathcal{F}$ a $C_*(\Omega X)$-module. The <u>Morse complex with coefficients in $\mathcal{F}$</u> is

$$C_*(X;\mathcal{F}) = \mathcal{F} \otimes \langle \mathrm{Crit} f \rangle$$

with differential

$$\partial(\alpha \otimes x) = \partial\alpha \otimes x + (-1)^{|\alpha|} \sum_y \alpha \cdot m_{x,y} \otimes y$$

Reinterprets Barraud-Cornea. This complex was also defined by Charette with Z/2 coefficients in 2017.

Summary of the construction:

Think of all this as Morse homology with coefficients in a fibration.

$$\{\overline{\mathcal{M}}(x,y)\} \xrightarrow[\text{of fundam. classes}]{\text{representatives}} \{s_{x,y} \in C_{|x|-|y|-1}\,\overline{\mathcal{M}}(x,y)\}$$

$$\xrightarrow[\text{path/loop spaces}]{\text{evaluate into}} \{m_{x,y} \in C_{|x|-|y|-1}\,\Omega X\}$$

universal twisting cocycle

$$\xrightarrow[\text{use } \mathcal{F} \in C_*\Omega X\text{-mod}]{} C_*(X;\mathcal{F}) = \mathcal{F} \otimes \mathrm{Crit}\, f$$

$$\partial(\alpha \otimes x) = \partial\alpha \otimes x + (-1)^{|\alpha|} \sum_y \alpha \cdot m_{x,y} \otimes y$$

Theorem (Charette, BDHO)

Let $F \hookrightarrow E \longrightarrow X$ be a Hurewicz fibration,

and let $\mathcal{F} = C_* F$ the corresponding DG local system.

We have an isomorphism

$$H_*(C_*(X; \mathcal{F})) \simeq H_*(E).$$

Point of view: Barraud-Cornea cocycle is Morse counterpart of Brown cocycle.

both are $\mathrm{Tor}^{C_*\Omega X}(\mathcal{F}, \mathbb{Z})$.


Flow
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Hamiltonian Floer homology with DG coefficients

(W, ω) symplectically aspherical, $H : S^1 \times W \to \mathbb{R}$ Hamiltonian

Morse theory for the action functional A_H defined on

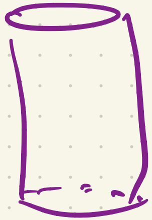
$$X = \mathcal{L}W$$

DG local system $\mathcal{F}$: $C_* \Omega \mathcal{L} W$ - module

(work componentwise on $\mathcal{L}W$, say on $\mathcal{L}_0 W$)

contractible

$$\{\overline{M}(x,y) : x,y \in \mathcal{P}_{cr}(H)\} \xrightarrow[\text{fundam. classes}]{\text{repres. of}} \{s_{x,y} \in C_{|x|-|y|-1}\overline{M}(x,y)\}$$



$$\xrightarrow[\text{into paths/loops in } X=\mathcal{L}_0 W]{\text{evaluate}} \{m_{x,y} \in C_{|x|-|y|-1}\Omega\mathcal{L}_0 W\}$$

Barraud-Cornea universal twisting cocycle

Think of this as Floer theory with coefficients in a fibration

$$\xrightarrow[C_*\Omega\mathcal{L}_0 W\text{-module}]{\text{use } \mathcal{F}} FC_*(H;\mathcal{F}) = \mathcal{F} \otimes \langle \mathcal{P}_{cr}(H) \rangle$$

$$\partial(\alpha \otimes x) = \partial\alpha \otimes x + (-1)^{|\alpha|} \sum_y \alpha \cdot m_{x,y} \otimes y$$

Reinterprets/Generalizes Barraud-Cornea. Related work: Abouzaid, Zhou, Rezchikov.

Two computations (BDHO)

1. DG PSS isomorphism $\quad (W,\omega)$ closed sympl. aspherical

$F \hookrightarrow E$ fibration, $\mathcal{F} = C_* F$

$$
\begin{tikzcd}
F \arrow[r, hook] \arrow[rr, bend left, equal] & i^*E \arrow[r, hook] \arrow[d] & E \arrow[d] \\
& W \arrow[r, hook, "i"] & \mathcal{L}_0 W
\end{tikzcd}
$$

$$
\left| \; FH_*(H;\mathcal{F}) \simeq H_*(W; i^*\mathcal{F}) \simeq H_*(E|_W) \; \right|
$$

Two computations (BDHO)

2. DG Viterbo isomorphism $\quad Q^n$ closed manifold

$$\begin{array}{ccc} F \hookrightarrow & E & \text{fibration}, \quad \mathcal{F} = C_* F \\ & \downarrow & \\ & \mathcal{L}Q & \end{array}$$

$$\begin{array}{c} \mathcal{L}T^*Q \\ \downarrow \pi \\ \mathcal{L}Q \end{array}$$

$$\boxed{SH_*(T^*Q; \pi^*\mathcal{F} \otimes \eta) \simeq H_*(\mathcal{L}Q; \mathcal{F}) \simeq H_*(E)}$$

spin + orientation local system

Proof of Theorem 1 ($d = \pm 1$)

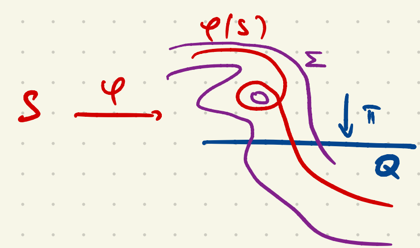
Statement: Q^n closed oriented $\neq K(\pi, 1)$

Contractible almost existence holds for any hypersurface

$$\Sigma \subset T^*Q \text{ s.t. } \operatorname{int} \Sigma \supset \varphi(S).$$

where $\varphi : S \longrightarrow T^*Q$ is any map s.t. $\deg(\pi \circ \varphi) = \pm 1$.

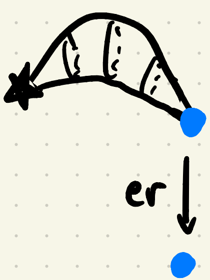
(closed oriented n-dimensional)

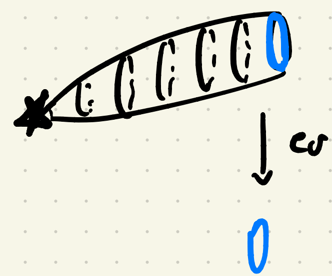


<u>Proof</u> : Consider the "universal" fibration over $\mathcal{L}_0 Q$ + restriction to Q (path-loop)

$$
\begin{tikzcd}
\Omega \mathcal{L}_0 Q \arrow[d] & \Omega \mathcal{L}_0 Q \arrow[d] \\
\Omega^2 Q \arrow[r, hook] \arrow[d] & P_* \mathcal{L}_0 Q \arrow[d] \\
Q \arrow[r, hook, "i"] & \mathcal{L}_0 Q
\end{tikzcd}
$$

$\mathcal{F} = C_* \Omega \mathcal{L}_0 Q$ as module over itself





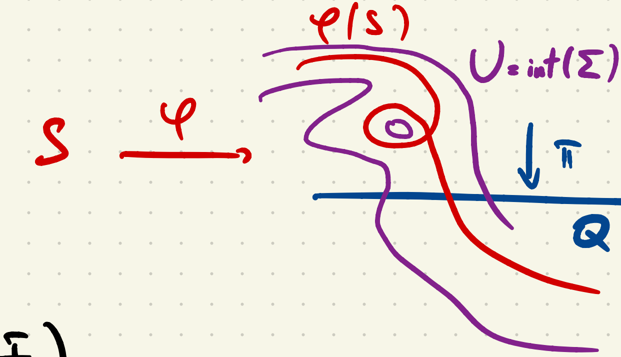
$$\begin{tikzcd}
H_*(Q; i^*\mathcal{F}) \arrow[r, "\text{if } Q \neq K(\pi_1,1)\text{, this is not injective}"'] \arrow[d, "\pi_! \text{ is injective if } d = \pm 1"] & SH_*^{\text{controllable}}(D^*Q; \mathcal{F} \otimes \eta) \\
H_{*+n}(U, \partial U; \pi^* i^* \mathcal{F}) &
\end{tikzcd}$$

$$H_*(Q; i^*\mathcal{F}) = H_*(\mathcal{P}_* \mathcal{L}_0 Q|_Q) = H_*(\Omega^2 Q)$$

$$SH_*^{\text{controllable}}(D^*Q; \mathcal{F} \otimes \eta) = H_*(\mathcal{L}_0 Q; \bar{\mathcal{F}}) = H_*(\mathcal{J}_* \mathcal{L}_0 Q) = H_*(pt)$$

$$\boxed{\Omega^2 Q \simeq pt \quad \text{iff } Q = K(\pi_1, 1)}$$

Injectivity of $\pi_!$:

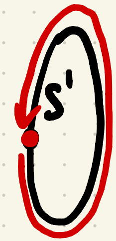


$$
\begin{tikzcd}
 & H_{*+n}(U, \partial U; \pi^* i^* \mathcal{F}) \arrow[dl, "\varphi_!"'] & \\
H_*(S; (\pi\circ\varphi)^* i^* \mathcal{F}) & & H_*(Q; i^* \mathcal{F}) \arrow[ul, "\pi_!"'] \arrow[ll, "(\pi\circ\varphi)_!"]
\end{tikzcd}
$$

$$\boxed{(\pi\circ\varphi)_* (\pi\circ\varphi)_! = d \cdot \mathrm{Id}, \quad d = \deg(\pi\circ\varphi : S \to Q)}$$

<u>Summary</u> :

- Morse / Floer homologies with DG coefficients capture "higher dimensional twists" / fibrations
- the fibres can be infinite-dimensional (and they actually <u>are</u> in applications)
- filtered theories + full package (differential, continuation maps, htpies)
- lie in between homology and homotopy
- computability ++



THANK YOU !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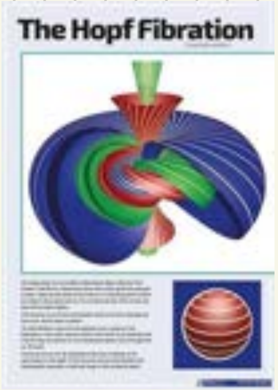

The Hopf Fibration Poster by Edmund Harris


Hopf fibration - Wikipedia