Relative symplectic cohomology of pairs

Symplectic Zoominar

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Based on joint work together with
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Contents

Singular cohomology

Singular cohomology of pairs

Singular cohomology is a functor {Top. spaces} → {Ab. groups}

$$X \mapsto H^*(X).$$

It has a 'standard model' on chain level:

Singular cochain complex is a functor {Top. spaces} → {Chain cxs.}

$$X \mapsto C^*(X), \qquad \text{satisfying } H^*(X) = H^*(C^*(X)).$$

From functoriality: If $B \subseteq A \subseteq X$ then $\mathrm{res}_B = \mathrm{res}_B^A \circ \mathrm{res}_A$.

The **singular cohomology of the pair** (X, A) where $A \subseteq X$, is defined by

$$H^*(X, A) = H^*(C^*(X, A))$$

where

$$C^*(X, A) = \ker\left(\mathrm{res}_A \colon C^*(X) \to C^*(A)\right).$$

Excisive couples, Mayer-Vietoris and product

Mayer-Vietoris

(A, B) is excisive $\iff$ it satisfies Mayer-Vietoris:

$$\cdots \to H^*(A \cup B) \to H^*(A) \oplus H^*(B) \to H^*(A \cap B) \to H^{*+1}(A \cup B) \to \cdots$$

is exact.

Cup product

There is a **cup-product** $\smile : H^*(X, A) \times H^*(X, A) \to H^*(X, A)$.

$$(A, B) \text{ excisive couple} \implies \smile : H^*(X, A) \times H^*(X, B) \to H^*(X, A \cup B)$$

Relative symplectic cohomology

Relative symplectic cohomology

- (M, ω) - closed symplectic manifold.

Relative symplectic cohomology (Varolgunes) is a functor

$$\{\text{compact subsets of } M\} \to \{\Lambda\text{-algebras}\}, \qquad K \mapsto SH^*(K),$$

where Λ is the universal Novikov field, and the product was defined by **Tonkonog-Varolgunes**.

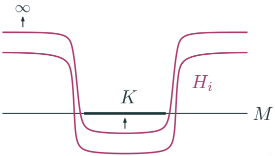
Have: $SH^*(M) = QH^*(M)$ *as an algebra*, and $SH^*(\varnothing) = 0$.

Complex for relative symplectic cohomology

Varolgunes defined a cochain model of relative symplectic cohomology.

Let $K \subseteq M$ compact.

- (H_i) Hamiltonians **adapted to** K:



Any Hamiltonians (H_i) adapted to K define a complex $SC(K)$ satisfying

$$SH^*(K) = H^*(SC(K)), \quad \text{where } SC(K) = \widehat{\varinjlim} CF^*(H_i).$$

The complex $SC(K)$ is only well-defined *up to homotopy equivalence*.

Similarly, there are chain-level restriction maps, defined up to homotopy, and they are functorial only up to homotopy.

Relative symplectic cohomology of pairs

Thus it makes sense to define $SH(M, K)$ using *homotopy kernels*.

A model for the homotopy kernel of $f \colon C \to C'$: $\mathrm{cocone}(f) = C \oplus C'[-1]$.

Therefore we define

$$SC(M, K) = \mathrm{cocone}\,(\mathrm{res} : SC(M) \to SC(K))$$

and show that this object is well-defined up to homotopy.

Thus $SH(M, K) = H(SC(M, K))$ is well-defined and is a functor.

Mayer-Vietoris, descent and commuting sets

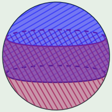
Similarly to singular cohomology, we have a Mayer-Vietoris sequence for A, B if and only if A, B satisfy a property called **descent**.

Question: When are subsets in descent?

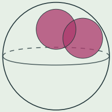
Definition

Say that compact $A, B \subseteq M$ *commute* if there exist Poisson commuting $f, g \in C^\infty(M)$ with $A = f^{-1}(0)$, $B = g^{-1}(0)$.

Example



Non-intersecting boundaries.
Commuting sets.



Boundaries intersect.
The sets don't commute.
(True in dim 2).

Theorem (Varolgunes)

If A, B commute, then A, B are in descent.

Descent and products

For every $K \subseteq M$ we define a product

$$*\colon SH(M,K) \times SH(M,K) \to SH(M,K).$$

Similarly to singular cohomology, for every K, K' we have

$$SH(M,K)\times SH(M,K') \to SH(M,K\cap K')\times SH(M,K\cap K') \xrightarrow{*} SH(M,K\cap K').$$

Theorem (D.-Ganor-Polterovich-Zapolsky)

If K, K' are in descent then we have a lift:

$$*\colon SH(M,K) \times SH(M,K') \to SH(M,K\cup K').$$

Moreover, the product commutes with restriction maps.

Ideal-valued quasi-measure

Ideal valued quasi-measures

(M,ω) - closed symplectic manifold. A - graded unital algebra.

An A-**ideal valued quasi-measure**, (A-IVQM) is an assignment

$$U \subseteq M \text{ open} \mapsto \tau(U) \subseteq A \text{ graded ideal, satisfying:}$$

- Normalization.
- Monotonicity.
- Additivity.
- Symp_0-invariance.
- Vanishing for disp. sets.
- **(quasi-multiplicativity):** $\tau(U) * \tau(U') \subseteq \tau(U \cap U')$, if U and U' commute[a].
- **(intersection):** If U, U' cover M, then $\tau(U \cap U') = \tau(U) \cap \tau(U')$.

[a] U and U' commute if $M \setminus U$ and $M \setminus U'$ commute

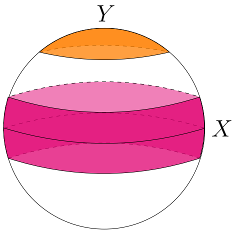
Remark: IVQMs extend to compact sets. $\tau(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \tau(U)$.

Ideal valued quasi-measures – example

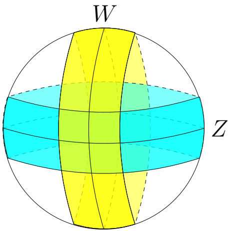
Consider $M = S^2$ of area$= 1$. A - graded unital algebra

$$\tau(U) = \begin{cases} 0, & U \text{ is contained in a smooth} \\ & \text{open disk of area} \leq 1/2, \\ A, & \text{else,} \end{cases}$$

where $U \subseteq S^2$ is a connected open set.



$\tau(X) = A, \ \tau(Y) = 0$



$\tau(Z) = \tau(W) = A$

$\tau(Z \cap W) = 0$

$\tau(Z) * \tau(W) \not\subseteq \tau(Z \cap W)$

Z, W do not commute

Construction of IVQM

Using relative symplectic cohomology we construct an IVQM.

Definition

$\forall\, U \subseteq M$ open, put $\tau(U) = \ker(\mathrm{res}\colon SH(M) \to SH(M \setminus U))$.
τ is called the ***quantum cohomology IVQM*** *of (M, ω).*

Theorem (D.-Ganor-Polterovich-Zapolsky)

τ is a $QH^*(M)$-IVQM on (M, ω).

Remark

For quasi-multiplicativity we use product on relative cohomology of pairs.

Example

The quantum cohomology IVQM of S^2 was described on the last slide for $A = QH^*(S^2)$.

Computational examples

Example

1. On $(\mathbb{T}^{2n} = \mathbb{T}^n(p) \times \mathbb{T}^n(q), dp \wedge dq)$, for $1 \leq k \leq n$, consider the coisotropic torus
$$T = \{p \in \mathbb{T}^n \ : \ p_1 = \cdots = p_k = 0\} \times \mathbb{T}^n(q).$$
A calculation shows
$$\tau(T) = H^*(\mathbb{T}^{2n}) \cdot \langle [dp_1 \wedge \ldots \wedge dp_k] \rangle$$
2. On $\mathbb{T}^{2n} \times S^2$, the Künneth formula implies
$$\tau_{\mathbb{T}^{2n} \times S^2}(T \times L) \supseteq \tau_{\mathbb{T}^{2n}}(T) \otimes \tau_{S^2}(L) = \tau_{\mathbb{T}^{2n}}(T) \otimes QH^*(S^2).$$

Applications to symplectic rigidity

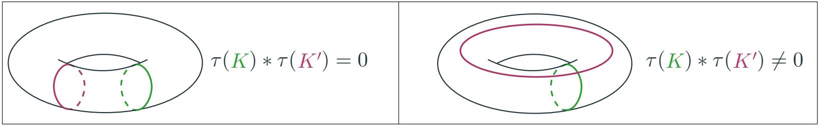
Non-displaceability of two sets

- (M,ω) - a closed symp. manifold.
- τ - The *QH*-IVQM on M.

Proposition

Let $K, K' \subseteq M$ be compact sets. If $\tau(K) * \tau(K') \neq 0$ then K is Symp_0*-non-displaceable from K'.*

Examples:



Proof: Assume $\exists\, \phi \in \mathrm{Symp}_0$ displaces K from K': $\phi(K) \cap K' = \varnothing$.

Then $\phi(K)$ and K' commute.

(Quasi-multiplicativity) $\implies \tau\big(\phi(K)\big) * \tau(K') \subseteq \tau\big(\phi(K) \cap K'\big) = \tau(\varnothing) = 0.$

(Invariance) $\implies \tau(\phi(K)) = \tau(K)$, hence

$0 \neq \tau(K) * \tau(K') \subseteq 0.$ Contradiction! □

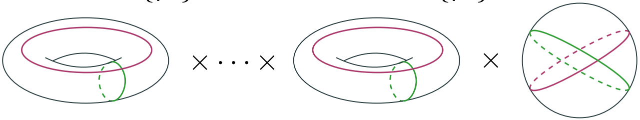
Nontrivial symplectic example

Consider $(M = \mathbb{T}^{2n} \times S^2, \omega_0 \oplus \sigma)$ where

$(\mathbb{T}^{2n} = \mathbb{T}^n(p) \times \mathbb{T}^n(q),\ \omega_0 = dp \wedge dq)$, and $\sigma =$standard area form.

Let $L, L' \subseteq S^2$ be equators, and consider

$$K = \mathbb{T}^n \times \{\mathrm{pt}\} \times L \quad \text{and} \quad K' = \{\mathrm{pt}\} \times \mathbb{T}^n \times L'.$$



Theorem (D.-Ganor-Polterovich-Zapolsky)

K is Symp_0-non-displaceable from K'.

Proof: $\tau(K) * \tau(K') \supseteq H^*(\mathbb{T}^{2n}) \cdot [\omega_0^n] \otimes QH^*(S^2) \neq 0.$ □

Note: They are smoothly displaceable.

Remark: Ham-non displaceability was proven in 2018, by Kawasaki arXiv:1811.00527.

Involutive Maps

- (M, ω) - a closed symplectic manifold.

Definition

A smooth map $f \colon M \to Y$ is called *involutive* if $\{f^* F, f^* G\} = 0$ for all $F, G \in C^\infty(Y)$.

Example

- A map $f = (f_1, f_2, \ldots, f_N) : M \to \mathbb{R}^N$ is involutive if and only if $\{f_i, f_j\} = 0$ for all i, j.
- On $(\mathbb{T}^{2n} = \mathbb{T}^n(p) \times \mathbb{T}^n(q), dp \wedge dq)$, the map $f \colon \mathbb{T}^{2n} \to \mathbb{T}^n$, $(p, q) \mapsto q$ is involutive.

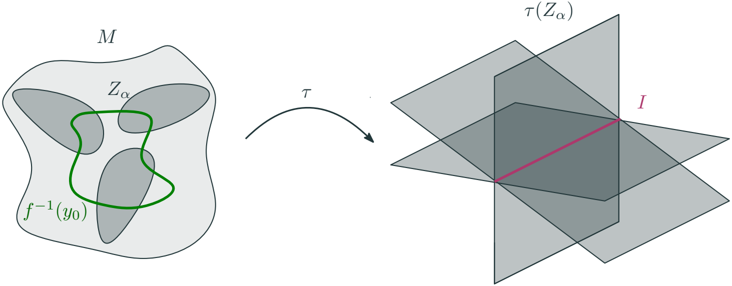
Symplectic center fiber theorem

(M, ω) - closed symplectic manifold.

A - graded unital algebra. τ an A-IVQM on (M, ω).

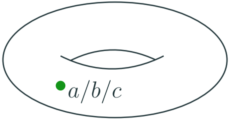
f: $M \to Y$ - involutive map. $\dim Y = d$.

Symplectic central fiber theorem: I - graded ideal with $I^{d+1} \neq 0$. Then $\exists\, y_0 \in Y$ s.t. $f^{-1}(y_0)$ intersects all the compact sets $Z \subseteq M$ with $I \subseteq \tau(Z)$.



IVQMs - symplectic centerpoint – concrete example

Torus $\mathbb{T}^6$, with coordinates $(p_i, q_i) \in \mathbb{T}^2$, $\omega = \sum dp \wedge dq$.
For $a, b, c \in \mathbb{T}^2$ consider coisotropic subtori:



$$T_1(a) = \{(\mathbf{p}, \mathbf{q}) \mid (q_1, q_2) = a\},$$
$$T_2(b) = \{(\mathbf{p}, \mathbf{q}) \mid (p_1, p_3) = b\},$$
$$T_3(c) = \{(\mathbf{p}, \mathbf{q}) \mid (p_2, q_3) = c\}.$$

Set $T(a, b, c) = T_1(a) \cup T_2(b) \cup T_3(c)$

Theorem (D–Ganor–Polterovich–Zapolsky)

Every involutive map $\mathbb{T}^6 \times S^2 \to Y^2$ has a fiber intersecting all sets of the form:

$$T(a, b, c) \times \textit{equator}.$$

An *equator* in S^2 is any loop dividing S^2 into two disks of equal area.

Virtual commutation

Question: Given two subsets $A, B \subseteq M$, can we make them commute?

Definition: Subsets $A, B \subseteq M$ **virtually commute** if $\exists \varphi \in \mathrm{Symp}_0(M)$ such that A and $\varphi(B)$ commute.

Example

Every two closed disks $D, D' \subset S^2$ virtually commute.

Virtually commuting - Examples

Proposition: Let (M, ω) be a symplectic manifold, let K be a compact subset of M and let L be an isotropic, closed and connected submanifold of M. If L, K commute then either $L \cap K = \varnothing$ or $L \subseteq K$.

Example

Consider $(\mathbb{T}^{2n} = \mathbb{T}^n(p) \times \mathbb{T}^n(q),\, dp \wedge dq)$.

Denote

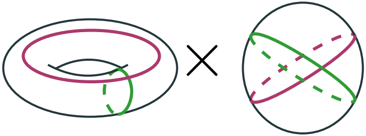
$$A = \{\mathrm{pt}\} \times \mathbb{T}^n(q) \times L, \;\; B = \mathbb{T}^n(p) \times \{\mathrm{pt}\} \times L \subset \mathbb{T}^{2n} \times S^2,$$

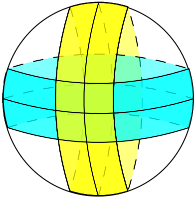
where L is an equator.

Then A, B **do not virtually commute**.

Indeed, topological reasons implies that $\nexists\, \varphi \in \mathrm{Symp}_0$ s.t. $\varphi(B) \subseteq A$. Additionally, using **IVQM** we proved that $\nexists\, \varphi \in \mathrm{Symp}_0$ s.t. $A \cap \varphi(B) = \varnothing$.

B is a closed connected Lagrangian submanifold, thus the Proposition implies that A, B do not virtually commute.





Questions?