Superheavy Skeleta for non-Normal Crossings Divisors

Elliot Gathercole

Lancaster University

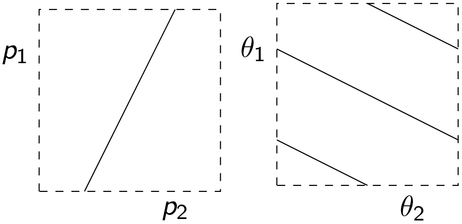
Symplectic Zoominar, October 2024

Motivating Question

Lagrangian Klein bottle in $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega)$

$$K = \{p_1 = 2p_2, 2\theta_1 + \theta_2 = 0\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \tag{1}$$

$$\omega = \sum_i dp_i \wedge d\theta_i \tag{2}$$

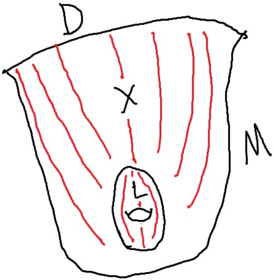


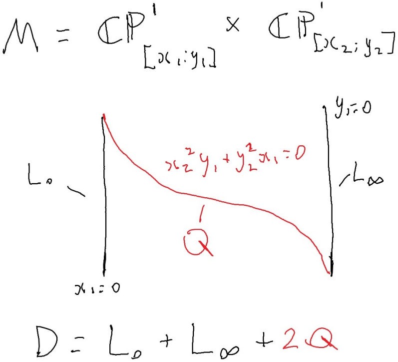
What can be said about K with respect to rigidity?

Theorem

K is superheavy with respect to $1 \in QH^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

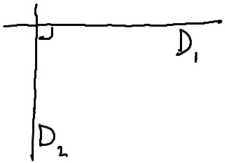
Skeleta of Divisor Complements





$$PD(D) = 2c_1(TM) = [\omega] \tag{3}$$

Orthogonal SC Divisors



Theorem (Borman-Sheridan-Varolgunes 2022)

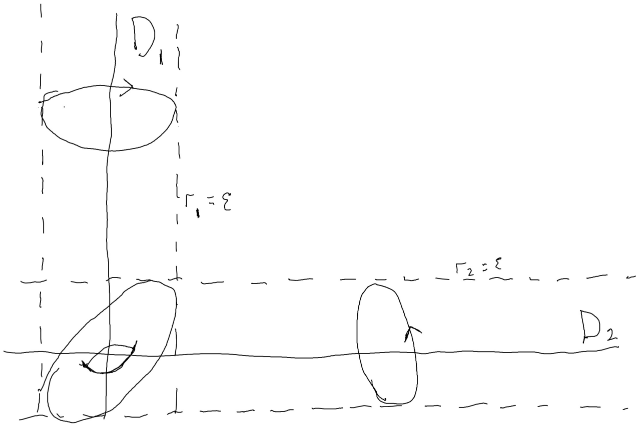
If

$$D = \sum_i \lambda_i D_i \tag{4}$$

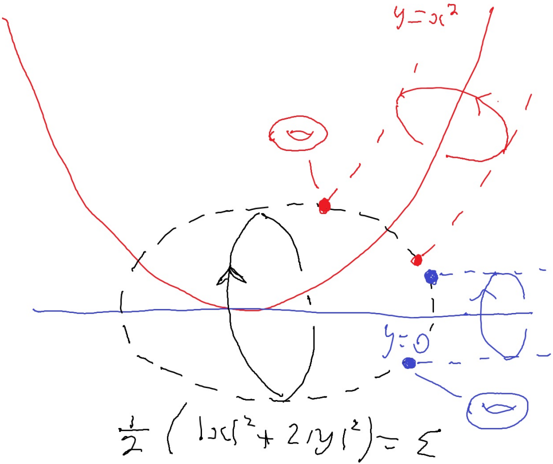
and $\lambda_i \leq 2$ for all i then X has a skeleton which is non-displaceable (in fact, SH-full).

Orthogonal SC Divisors Continued

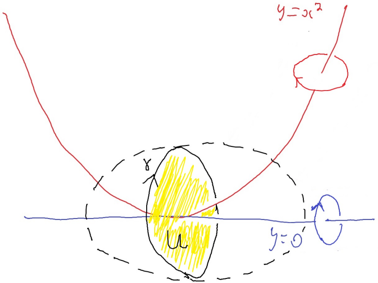
Standard $U(1)$ action on $\nu_{D_i \subset M}$ is generated by a Hamiltonian r_i.



Back to K



Periodic Orbit Properties



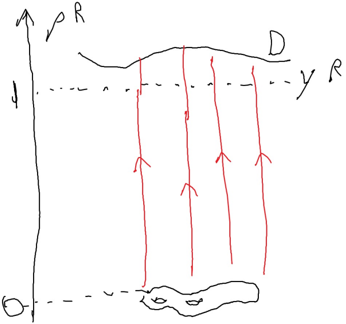
$$\mathcal{A}(\gamma, u) = u \cdot D = u \cdot \{y = 0\} + 2u \cdot \{y = x^2\} = 2 + 2 \cdot 2 = 6 \quad (5)$$

$$CZ(\gamma, u) = CZ(t \mapsto diag(e^{2\pi it}, e^{4\pi it})) = 6 \quad (6)$$

This equality is analogous to the condition $\lambda_i = 2$.

Special Hamiltonian

Can construct a family of Hamiltonians on M (not smooth at L) such that orbits come from local circle actions.



$$d\rho^R(Z) = \rho^R \tag{7}$$

Superheaviness

Definition

A subset $K \subset M$ is superheavy if $\zeta(H) \leq \sup_K H$ for all Hamiltonians H.

Using properties of ρ, we have that $\zeta(h \circ \rho^R) \leq h(0)$ if $h'' \geq 0$. We can bound other Hamiltonians by those of this form to prove superheaviness.

Theorem (Mak, C.Y., Sun, Y. and Varolgunes, U. 2024)

If a compact set $K \subset M$ is superheavy, then it is SH-full.

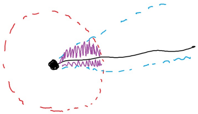
Generalisations

We have proven quite a strong statement about K, but can we apply this to other situations?

In general, we need:

- A stratification of D into embedded symplectic manifolds

- A radial Hamiltonian cutting out each stratum, which generates a circle action
- The radial Hamiltonians to commute when one stratum is contained in the closure of another.



Main Result

In general, as in SC case, result will depend on σ_{crit}.

$$\sigma_{crit} = \max\left(1 - \frac{\mathcal{A}(\gamma, u)}{CZ(\gamma, u)}\right) \tag{8}$$

(where this maximum is over 1-periodic orbits γ of the Hamiltonian circle actions, with respect to small caps u).

Theorem

The subset $K_{crit} = \{\rho^0 \leq \sigma_{crit}\}$ is superheavy. In particular if $\sigma_{crit} = 0$, the skeleton L is superheavy.

Quasihomogeneous Divisors

One situation generalising our initial example: $dim_{\mathbb{C}} M = 2$, and D is locally modelled by the vanishing of quasihomogeneous polynomials

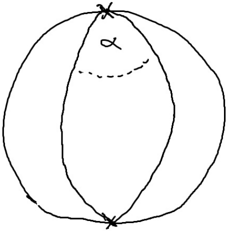
$$f(x^a, y^b) = t^N f(x, y) \tag{9}$$

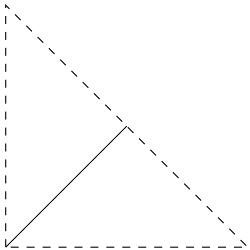
for $t \in \mathbb{C}^*$, where $a, b, N \in \mathbb{N}$.
Radial Hamiltonians for singular points are constructed using the obvious $U(1)$-action, and radial Hamiltonians for the smooth parts of D using a Moser argument. This sinularity contributes $\frac{N-2(a+b)}{N}$ to σ_{crit}.
This also works for surfaces D with ADE type singularities.

Take $M = \mathbb{CP}^2$, $D = aQ_+ + (3-a)Q_-$, and

$$Q_\pm = \{z_1 z_2 \pm z_0^2 = 0\}. \tag{10}$$

L is (approximately, up to Hamiltonian isotopy) a union of two $\mathbb{RP}^2$s, which correspond to arcs in the symplectic reduction.



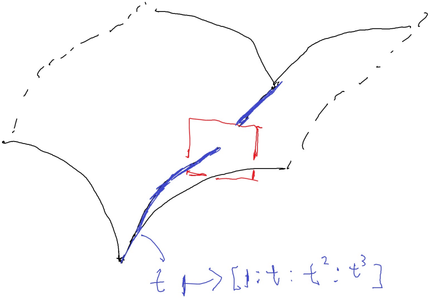


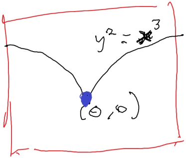
$$\alpha = \frac{2\pi a}{3} \tag{11}$$

$$\sigma_{crit} = 0 \iff 1 \le a \le 2 \iff \frac{2\pi}{3} \le \alpha \le \frac{4\pi}{3} \tag{12}$$

Chiang Lagrangian

Take $M = \mathbb{CP}^3$, D the cubic discriminant locus. Then $X = SL(2,\mathbb{C})/D_6$, $L = SU(2)/D_6$ (Chiang Lagrangian).





$$\sigma_{crit} = \frac{12-10}{12} = \frac{1}{6} \tag{13}$$

Get a neighbourhood of L of volume $\frac{1}{6^3}$.

Thank you!