Instantaneous Hamiltonian displaceability and arbitrary symplectic squeezability for critically negligible sets

Symplectic Zoominar

Yann Guggisberg

Utrecht University

November 15, 2024

Contents

Introduction

- This talk is based on joint work with Fabian Ziltener [GZ24]

Introduction

- This talk is based on joint work with Fabian Ziltener [GZ24]
- What is the smallest "size" of a subset of a symplectic manifold that "carries some symplectic geometry"?

Introduction

- This talk is based on joint work with Fabian Ziltener [GZ24]
- What is the smallest "size" of a subset of a symplectic manifold that "carries some symplectic geometry"?
- In this talk, I will interpret "size" as Hausdorff dimension, and "carries symplectic geometry" will be related to the displacement energy and the (non-)squeezability of the subset.

Introduction

- This talk is based on joint work with Fabian Ziltener [GZ24]
- What is the smallest "size" of a subset of a symplectic manifold that "carries some symplectic geometry"?
- In this talk, I will interpret "size" as Hausdorff dimension, and "carries symplectic geometry" will be related to the displacement energy and the (non-)squeezability of the subset.
- Question 1: What is the smallest Hausdorff dimension of a subset of a symplectic manifold that has positive displacement energy?

Hausdorff dimension

Definition

Let (X, d) be a metric space and $s \in [0, \infty)$. We call (X, d) *s-(Hausdorff-) negligible* if for every $\varepsilon > 0$ there exists a countable collection $\mathcal{S}$ of subsets of X that covers X, such that

$$\sum_{A \in \mathcal{S}} (\mathrm{diam}\, A)^s < \varepsilon.$$

We define the *Hausdorff dimension* of (X, d) as

$$\inf \{ s \in [0, \infty) \,|\, (X, d) \text{ is } s\text{-negligible.} \}$$

Examples

- Examples: An m-dimensional submanifold of $\mathbb{R}^n$ has Hausdorff dimension m.

Examples

- Examples: An m-dimensional submanifold of $\mathbb{R}^n$ has Hausdorff dimension m.
- The Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.

Examples

- Examples: An m-dimensional submanifold of $\mathbb{R}^n$ has Hausdorff dimension m.
- The Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.
- Space filling curves have the same Hausdorff dimension as the space they fill.

Displacement energy

Let (M, ω) be a symplectic manifold. For a Hamiltonian function $H \in C^\infty([0,1] \times M, \mathbb{R})$, we denote by φ_H^t the corresponding Hamiltonian flow. We denote the set of "well-behaved" Hamiltonian functions by $\mathcal{H}(M, \omega)$ the set

$$\left\{ H \in C^\infty([0,1] \times M, \mathbb{R}) \,|\, \forall t \in [0,1], \varphi_H^t : M \to M \text{ is a diffeo} \right\}$$

Displacement energy

Definition

We define $\|\cdot\| : \mathcal{H}(M,\omega) \to [0,\infty]$, $\|H\| := \int_0^1 (\sup_M H^t - \inf_M H^t)dt$. We define the *displacement energy* of a subset $A \subseteq M$ to be

$$e(A) := \inf \left\{ \|H\| \mid H \in \mathcal{H}(M,\omega), \varphi_H^1(A) \cap A = \emptyset \right\}.$$

Back to question 1

Question 1: What is the smallest Hausdorff dimension of a subset of a symplectic manifold that has positive displacement energy?

Back to question 1

Question 1: What is the smallest Hausdorff dimension of a subset of a symplectic manifold that has positive displacement energy?

Theorem (Chekanov, [Che98])

Let (M^{2n}, ω) be geometrically bounded, then every closed Lagrangian submanifold of M has positive displacement energy.

This gives an upper bound of n for the answer to question 1.

Instantaneous C^∞-displaceability

Definition

We call two subsets A and B of (M, ω) *instantaneously C^∞-displaceable* if every weak C^∞-neighborhood of the identity contains a Hamiltonian diffeomorphism φ such that $\varphi(A) \cap B = \emptyset$.

Instantaneous C^∞-displaceability

Definition

We call two subsets A and B of (M, ω) *instantaneously C^∞-displaceable* if every weak C^∞-neighborhood of the identity contains a Hamiltonian diffeomorphism φ such that $\varphi(A) \cap B = \emptyset$.

In particular, if A is instantaneously displaceable from itself, it has displacement energy 0.

Instantaneous C^∞-displaceability

Definition

We call two subsets A and B of (M, ω) *instantaneously C^∞-displaceable* if every weak C^∞-neighborhood of the identity contains a Hamiltonian diffeomorphism φ such that $\varphi(A) \cap B = \emptyset$.

In particular, if A is instantaneously displaceable from itself, it has displacement energy 0.

Definition

We call a metric space (X, d) *countably m-rectifiable* if there exists a surjective locally Lipschitz map $f : \mathbb{R}^m \supseteq A \to X$.

Instantaneous C^∞-displaceability

Theorem (Ziltener, G.)

Let $A \subseteq \mathbb{R}^{2n}$ be a countably m-rectifiable subset for some $m \in \{0, \dots, 2n\}$, and $B \subseteq \mathbb{R}^{2n}$ a $(2n-m)$-negligible subset. Then there exists a linear function $H : \mathbb{R}^{2n} \to \mathbb{R}$, such that for almost every $t \in \mathbb{R}$, we have $\varphi_H^t(A) \cap B = \emptyset$.

Instantaneous C^∞-displaceability

Theorem (Ziltener, G.)

Let $A \subseteq \mathbb{R}^{2n}$ be a countably m-rectifiable subset for some $m \in \{0, \dots, 2n\}$, and $B \subseteq \mathbb{R}^{2n}$ a $(2n-m)$-negligible subset. Then there exists a linear function $H : \mathbb{R}^{2n} \to \mathbb{R}$, such that for almost every $t \in \mathbb{R}$, we have $\varphi_H^t(A) \cap B = \emptyset$.

Corollary

Let $A \subseteq \mathbb{R}^{2n}$ be a bounded countably n-rectifiable and n-negligible subset. Then A has zero displacement energy.

Here, we need to add the boundedness assumption to be able to cut-off the Hamitonian before infinity and get small energy.

Instantaneous C^∞-displaceability

Theorem (Ziltener, G.)

Let $A \subseteq \mathbb{R}^{2n}$ be a countably m-rectifiable subset for some $m \in \{0, \ldots, 2n\}$, and $B \subseteq \mathbb{R}^{2n}$ a $(2n-m)$-negligible subset. Then there exists a linear function $H : \mathbb{R}^{2n} \to \mathbb{R}$, such that for almost every $t \in \mathbb{R}$, we have $\varphi_H^t(A) \cap B = \emptyset$.

Corollary

Let $A \subseteq \mathbb{R}^{2n}$ be a bounded countably n-rectifiable and n-negligible subset. Then A has zero displacement energy.

Here, we need to add the boundedness assumption to be able to cut-off the Hamitonian before infinity and get small energy. This gives a lower bound of n for the answer to question 1, with the additional assumption of countable n-rectifiability.

Squeezability

Definition

Let $(M,\omega),(M',\omega')$ be symplectic manifolds of dimension $2n$ and $S \subseteq M$ a compact subset. We say that S *symplectically embeds* into (M',ω') if there exists an open neighborhood U of S in M and a symplectic embedding $\varphi : (U,\omega|_U) \hookrightarrow (M',\omega')$.
We say that S is *arbitrarily (symplectically) squeezable* if it symplectically embeds into every open neighborhood of the origin in $\mathbb{R}^{2n}$.

Squeezability

Definition

Let $(M,\omega),(M',\omega')$ be symplectic manifolds of dimension $2n$ and $S \subseteq M$ a compact subset. We say that S *symplectically embeds* into (M',ω') if there exists an open neighborhood U of S in M and a symplectic embedding $\varphi : (U, \omega|_U) \hookrightarrow (M',\omega')$.
We say that S is *arbitrarily (symplectically) squeezable* if it symplectically embeds into every open neighborhood of the origin in $\mathbb{R}^{2n}$.

Question 2: What is the smallest Hausdorff dimension of a not arbitrarily squeezable subset of $\mathbb{R}^{2n}$?

Known results

It follows from Gromov's isosymplectic embedding theorem that closed symplectic submanifolds of $\mathbb{R}^{2n}$ of codimension at least 2 can be arbitrarily squeezed.

Known results

It follows from Gromov's isosymplectic embedding theorem that closed symplectic submanifolds of $\mathbb{R}^{2n}$ of codimension at least 2 can be arbitrarily squeezed.

Theorem (Symplectic hedgehog theorem, Hermann, [Her98])

Let $n \geq 2$ and let $\mathbb{T}^n$ denote the standard Lagrangian torus in $\mathbb{R}^{2n}$ then the set

$$[0,1] \cdot \mathbb{T}^n := \{cx \mid c \in [0,1], x \in \mathbb{T}^n\}$$

does not symplectically embed into the open symplectic cylinder of radius 1. Note that the above set has Hausdorff dimension $n+1$.

Known results

Theorem (Swoboda, Ziltener, [SZ13])

For every $n \geq 2$ there exists a compact n-Hausdorff dimensional subset of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$ that does not symplectically embed into the open cylinder of radius 1.
This set can be constructed as the union of a closed Lagrangian and the image of a map $f : S^2 \to \mathbb{R}^{2n}$.

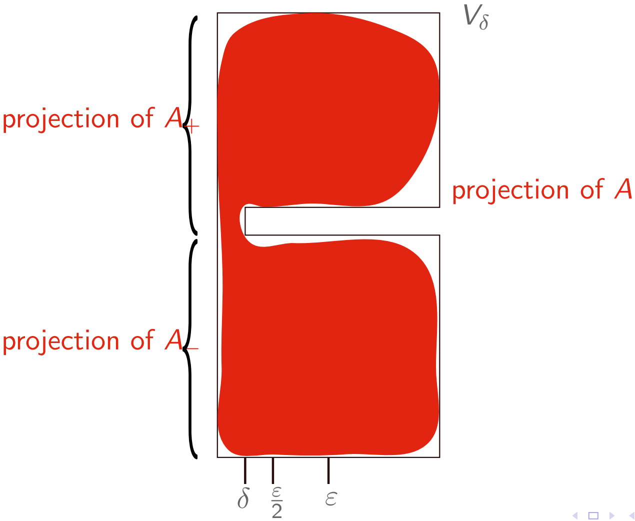
Arbitrary squeezability

Using the displacement result from before, we get the following.

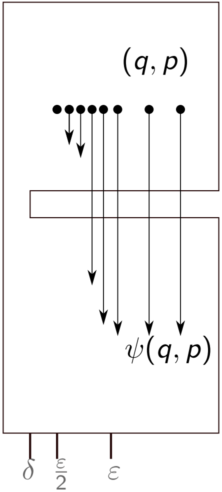
Theorem (Ziltener, G.)

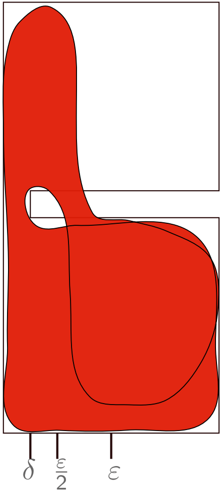
Every compact, countably n-rectifiable and n-negligible subset $A \subseteq \mathbb{R}^{2n}$ is arbitrarily symplectically squeezable.

Idea of the proof



Idea of the proof





Application: barriers

- Lagrangian barriers are Lagrangian submanifolds that always intersect a symplectically embedded ball of a big enough radius. Hence, the Gromov width of the manifold without this Lagrangian is smaller than the Gromov width of the manifold we started with. This notion was coined by Paul Biran [Bir01]. See also [SSVZ24], [BS24], [OS24].

Application: barriers

- Lagrangian barriers are Lagrangian submanifolds that always intersect a symplectically embedded ball of a big enough radius. Hence, the Gromov width of the manifold without this Lagrangian is smaller than the Gromov width of the manifold we started with. This notion was coined by Paul Biran [Bir01]. See also [SSVZ24], [BS24], [OS24].
- The same idea has also been investigated for symplectic submanifolds, namely unions of symplectic planes in [HKHO24].

Application: barriers

- Lagrangian barriers are Lagrangian submanifolds that always intersect a symplectically embedded ball of a big enough radius. Hence, the Gromov width of the manifold without this Lagrangian is smaller than the Gromov width of the manifold we started with. This notion was coined by Paul Biran [Bir01]. See also [SSVZ24], [BS24], [OS24].
- The same idea has also been investigated for symplectic submanifolds, namely unions of symplectic planes in [HKHO24].
- Applying a similar "folding" construction to a closed subset A of the unit ball that has the correct rectifiability and negligibility properties, it should be possible to show that A is not a barrier, in the sense that $w(B_1) = w(B_1 \setminus A) = \pi$, where w denotes the Gromov width.

Thank you

References I

P. Biran, *Lagrangian barriers and symplectic embeddings*, Geom. Funct. Anal. **11** (2001), no. 3, 407–464. MR 1844078

Joé Brendel and Felix Schlenk, *Pinwheels as Lagrangian barriers*, Commun. Contemp. Math. **26** (2024), no. 5, Paper No. 2350020, 21. MR 4731311

Yu. V. Chekanov, *Lagrangian intersections, symplectic energy, and areas of holomorphic curves*, Duke Math. J. **95** (1998), no. 1, 213–226. MR 1646550

Yann Guggisberg and Fabian Ziltener, *Instantaneous hamiltonian displaceability and arbitrary symplectic squeezability for critically negligible sets*, 2024.

David Hermann, *Non-equivalence of symplectic capacities for open sets with restricted contact type boundary*.

References II

- Pazit Haim-Kislev, Richard Hind, and Yaron Ostrover, *On the existence of symplectic barriers*, Selecta Math. (N.S.) **30** (2024), no. 4, Paper No. 65, 11. MR 4768771
- Emmanuel Opshtein and Felix Schlenk, *Liouville polarizations and the rigidity of their lagrangian skeleta in dimension* 4, 2024.
- Kevin Sackel, Antoine Song, Umut Varolgunes, and Jonathan J. Zhu, *On certain quantifications of Gromov's nonsqueezing theorem*, Geom. Topol. **28** (2024), no. 3, 1113–1152, With an appendix by Joé Brendel. MR 4746410
- Jan Swoboda and Fabian Ziltener, *A symplectically non-squeezable small set and the regular coisotropic capacity*, J. Symplectic Geom. **11** (2013), no. 4, 509–523. MR 3117057