New invariants for Lagrangian tori

joint with

Richard Hind and **Grisha Mikhalkin**

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1. Questions

Take a **real** knot in $\mathbb{R}^3$: knot ν such that $\mathcal{N}_1(\nu)$ is embedded.

Example: circle $S^1(r)$ in $\mathbb{R}^2$ real $\Leftrightarrow$ $r \geq 1$

Question 1 (outer radius)

What is the smallest r such that ν can be isotoped through real knots to some ν' with $\nu' \subset B^3(r)$?

Example: If ν is the unknot: $r = 2$

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Question 2 (cage size) Assume that ν is real and an unknot.

What is the smallest c such that ν is isotopic to $S^1(1)$ through real knots that all lie in $B^3(c)$?

Nothing is known about these questions.

(see, however, the Fáry–Milnor theorem and Pardon's work on distortion)

Lagrangian versions:

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normalization: 2-monotone

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normalization: 2-monotone

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Question 1' (outer radius)
What is the smallest $\rho = \pi r^2$ such that L can be Hamiltonianly isotoped into $B^{2n}(\rho)$?

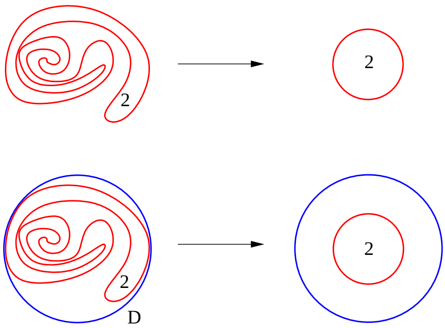
Question 2' (cage size) Assume that L is Hamiltonian isotopic to T_{Cliff} in $\mathbb{R}^{2n}$.
What is the smallest c such that L is isotopic to T_{Cliff} through a Hamiltonian isotopy supported in $B^{2n}(c)$?

In contrast to real knots in $\mathbb{R}^3$, something can be said on these questions:

For $\boxed{n=1}$: "Of course" (Schoenfliess + Moser):

Question 1': $\rho = 2$

Question 2': $c(L) = \inf\{a \mid L \subset D^2(a)\}$

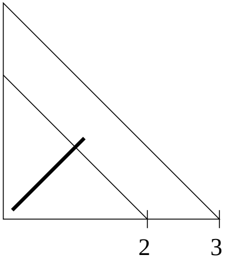


from now on: $\boxed{n=2}$ (for higher dimensions: take products)

An earlier result:

Cieliebak–Mohnke 2018: always $\rho(L) \geq 2$

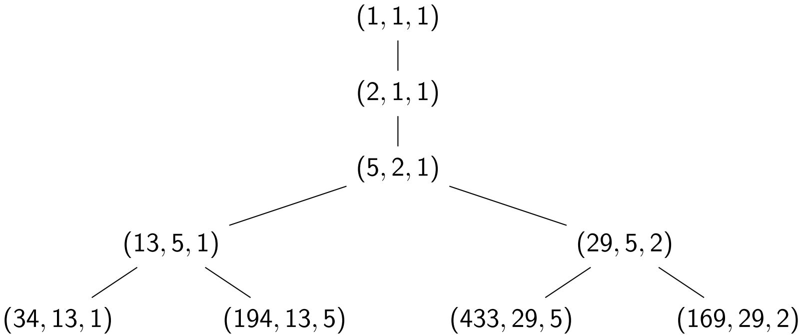
This is sharp for the Clifford and the Chekanov torus:



2. Examples of tori

Goal: find many L in $B^4(3)$

Vianna, Galkin–Mikhalkin: *For every Markov triple* (a, b, c) *there exists a monotone Lagrangian torus* $L_{a,b,c}$ *in* $\mathbb{C}\mathrm{P}^2(3)$*, and different triples yield tori that are not Hamiltonian isotopic.*



The Markov tree, encoding the solutions of $a^2 + b^2 + c^2 = 3abc$

For a survey on this and other non-metrical aspects of Lagrangian knots, see L. Polterovich and F. Schlenk, *Lagrangian knots and unknots – an essay,* Celebratio volume for Yasha Eliashberg.

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Wellschinger 2007: *Let S be an embedded symplectic sphere in $\mathbb{C}\mathrm{P}^2$ of degree 1. Then there exists a Hamiltonian isotopy φ of $\mathbb{C}\mathrm{P}^2$ such that $\varphi(S) \cap L = \emptyset$.*

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Question: What exactly does one get from this?
I.e.: are there different spheres one can remove? In different homology classes of $\mathbb{C}P^2 \setminus L$, or in the same homology class but not isotopic?

Theorem 1 (i) *Exactly 3 homology classes of* $\mathbb{C}\mathrm{P}^2 \setminus L$ *can be represented by embedded symplectic spheres of degree 1.*

(ii) *Any two such symplectic spheres in the same homology class are Hamiltonian isotopic in* $\mathbb{C}\mathrm{P}^2 \setminus L$.

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Corollary For every Markov triple (a, b, c) we obtain 1 or 2 or 3 Hamiltonian isotopy classes of monotone tori $L_{a,b,c}(S^j)$ in $B^4(3)$:

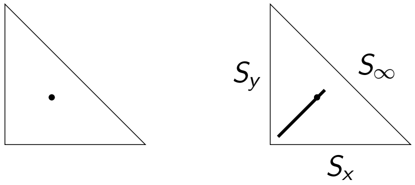
1 if $(a, b, c) = (1, 1, 1)$,

2 if $(a, b, c) = (2, 1, 1)$,

3 if $a > b > c$.

Discussion

for $(a, b, c) = (1, 1, 1)$:



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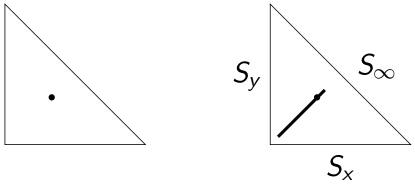

for $(a, b, c) = (2, 1, 1)$: The Chekanov torus T_{Chek} includes in $\mathbb{C}\mathrm{P}^2$ as $L_{2,1,1}$ (**Oakley–Usher, Wu**)

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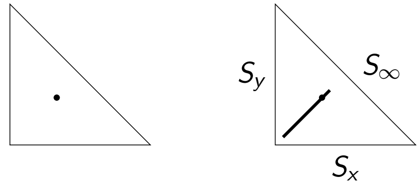

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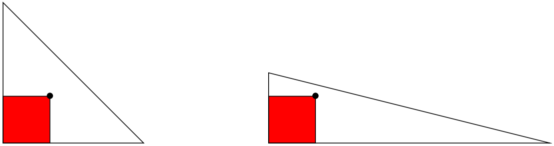
The other two tori $L_{2,1,1}(S_x)$ and $L_{2,1,1}(S_y)$ are Hamiltonian isotopic (by $[z_0 : z_1 : z_2] \mapsto [z_1 : z_0 : z_2]$).

This torus $L_{2,1,1}(S_x)$ in $B^4(3)$ is Ham. isotopic neither to T_{Cliff} nor to T_{Chek}:

- It includes in $\mathbb{C}\mathrm{P}^2$ as $L_{2,1,1}$, so is not T_{Cliff}.
- In $\mathbb{C}^2$, $L_{2,1,1}(S_x)$ is Hamiltonian isotopic to T_{Cliff}, since both are the singular stratum of a cube $D^2(1) \times D^2(1)$:



So already in $\mathbb{C}^2$, $L_{2,1,1}(S_x)$ is not Ham. isotopic to T_{Chek}.

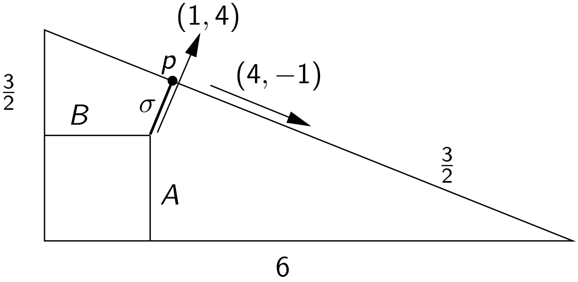
What is its cage number? (Shall see: $\in [3, 3.6]$)

Ideas of proofs

Theorem 1 is proven by neck-stretching.

Here, for the special case of Vianna-tori:
Why are there (at most) 3 homology classes represented?

For $(a, b, c) = (2, 1, 1)$:



Take circles in class $(1,4) \in H_1(T^2)$ over $\sigma \setminus \{p\}$ and one point over p. Get symplectic disc D over σ.

$C := [D]$, in the same way $A, B \in H_2(\mathbb{C}\mathrm{P}^2, L)$

$$4A + B + C \text{ is an absolute class: } = n[\mathbb{C}\mathrm{P}^1]$$

areas: $4 + 1 + 1 = n3$, so $n = 2$:

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Now let $S \subset \mathbb{C}\mathrm{P}^2 \setminus L$ be an embedded sphere of degree 1.

The divisor classes A^*, B^*, C^* generate $H_2(\mathbb{C}\mathrm{P}^2 \setminus L; \mathbb{Q})$, and are dual to A, B, C.

$$[S] = \alpha A^* + \beta B^* + \gamma C^*, \quad \alpha, \beta, \gamma \in \mathbb{Q}.$$

Choose an ω-tame J such that S is J-holomorphic. The disc D representing C has Maslov index 2, so C can be represented by a J-disc, so

$$C \cdot [S] = \gamma \geq 0. \quad \text{Similarly } \alpha, \beta \geq 0. \quad \text{So } \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$$

Applying $[\omega]$ to

$$[S] = \alpha A^* + \beta B^* + \gamma C^*$$

get

$$3 \;=\; \alpha 6 + \beta \tfrac{3}{2} + \gamma \tfrac{3}{2}.$$

So $\alpha = 0$.

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In general, get the equation

$$abc \; = \; \beta b^2 + \gamma c^2, \qquad \beta, \gamma \in \mathbb{Z}_{\geq 0}.$$

Some congruence arithmetics and Markov numberology yields 3 solutions.
E.g. for $(5,2,1)$: $10 = 4\beta + \gamma$, $\qquad (2,2)$, $(1,6)$, $(0,10)$

For $a > b > c$ the three tori $L_{a,b,c}(S^j)$ in $B^4(3)$ are not Ham. isotopic:

Lemma *Let* $\phi \in \mathrm{Symp}(\mathbb{C}\mathrm{P}^2)$ *such that* $\phi(L_{a,b,c}) = L_{a,b,c}$. *Then* $\phi_* \colon H_1(L_{a,b,c}) \to H_1(L_{a,b,c})$ *is the identity map.*

Proof: E.g. by versal deformations and displacement energy profile, cf. Brendel

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Assume now that $\exists$ a compactly supported symplectomorphism

$$\phi\colon B^4(3) \to B^4(3) \quad \text{with} \quad \phi(L_{a,b,c}(S_1)) = L_{a,b,c}(S_2).$$

Then get

$$\varphi\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2 \quad \text{with} \quad \varphi(L_{a,b,c}) = L_{a,b,c} \text{ and } \varphi(S_1) = S_2.$$

Take any disc D with boundary on $L_{a,b,c}$. By the lemma:

$$\varphi_*[D] = [D] + E \text{ with } E \text{ an absolute class} \in H_2(\mathbb{C}\mathrm{P}^2).$$

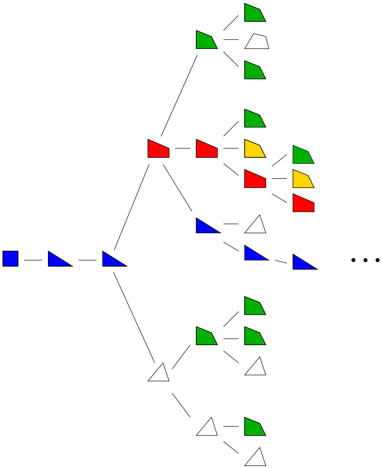
$\mathrm{area}(E) = 0$, so $E = 0$, so

$$\varphi_*[D] = [D] \ \text{ in } H_2(\mathbb{C}\mathrm{P}^2, L).$$

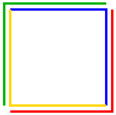
$$[D] \cdot [S_1] \;=\; \varphi_*[D] \cdot \varphi_*[S_1] \;=\; [D] \cdot [S_2] : \ \text{contradiction}$$

3. Variation of the theme: in $S^2 \times S^2$ and $D^2 \times D^2$

For $S^2 \times S^2$, have triangular and quadrilateral bases of ATFs:



Over each base polygon P lives a monotone torus L_P.
Now there are 4 Hamiltonian divisor classes in $(S^2 \times S^2) \setminus L_P$:



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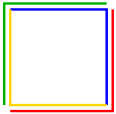

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Theorem *Assume that* $P \neq \square$. *Then*

(i) *If* P *is a quadrilateral, then* $\sigma(L_P)$ *is not Ham. isotopic to* L_P.

(ii) *From a generic triangle, we obtain 4 Ham. isotopy classes of monotone tori in* $D(2) \times D(2)$.

(iii) *From a quadrilateral, we obtain 8 Ham. isotopy classes of monotone tori in* $D(2) \times D(2)$.

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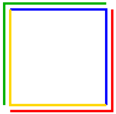

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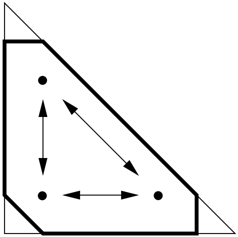
For the proof need:

Lemma *Suppose $P \neq \square$. If $A \in \mathrm{GL}(2; \mathbb{Z})$ takes P to P, then $A = \mathtt{id}$.*

ad (i): Examples of non-monotone tori that are symplectomorphic but not Hamiltonian isotopic were known before (Cho, Brendel):



4. Outer radius

Recall that for a monotone torus $L \subset B^4(3)$:

$$\rho_B(L) \;=\; \inf\{c \mid \varphi(L) \subset B^4(c) \,\text{for some}\, \varphi \in \mathrm{Ham}(B^4(3))\}$$

Also define

$$\rho_{\mathbb{C}P^2}(L) \;=\; \inf\{c \mid \varphi(L) \subset B^4(c) \subset \mathbb{C}P^2 \;\text{for some}\, \varphi \in \mathrm{Ham}(\mathbb{C}P^2)\}$$

Then

$$2 \leq \rho_{\mathbb{C}P^2}(L) \leq \rho_B(L) < 3.$$

$\leq$ and $<$ cannot be improved, in general. $\leq$ can be strict.

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In (a, b, c), if $c = 1$, then $a = g_{n+1}$ and $b = g_n$, where

$$g_1 = 2,\; g_2 = 5,\; g_3 = 13,\; g_4 = 34,\; \ldots$$

are the odd-index Fibonacci numbers.

Set $L_n := L_{g_{n+1},g_n,1}(S_n^c)$.

Theorem 1 $\rho_B(L_n) = 3 - \dfrac{1}{g_n g_{n+1}}$

Proposition $\rho_{\mathbb{C}\mathrm{P}^2}(L_n) \geq 3 - \dfrac{1}{g_n(g_{n+1} - 2g_n)}$

Corollary The invariant $\rho_{\mathbb{C}\mathrm{P}^2}$ distinguishes the tori $L_{g_{n+1},g_n,1} \subset \mathbb{C}\mathrm{P}^2$.

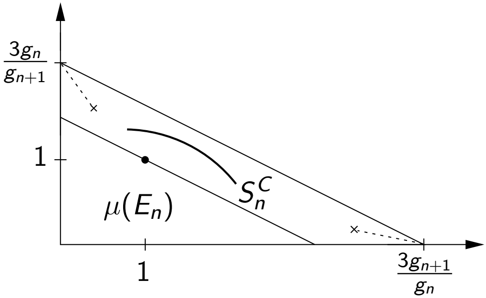
Theorem 2 Assume that $a \geq 5$. Then

$$\rho_{\mathbb{C}\mathrm{P}^2}(L_{(a,b,c)}) > 3\left(1 - \frac{1}{b^2}\right)$$

"Recall": The 3 spheres in $\mathbb{C}P^2 \setminus L_{g_{n+1},g_n,1}$ are in class

$$\beta B^* + \gamma C^*, \qquad \beta = 0, 1, 2.$$

$S_n^C :=$ the one with $\beta = 0$.



In fact, S_n^c can be chosen disjoint from $\overline{E}_n$
(Casals–Vianna, or use that the space of $\overline{E}_n \stackrel{s}{\hookrightarrow} B^4(3)$ is connected)

Proof of $\rho_B(L_n) \leq 3 - \dfrac{1}{g_n g_{n+1}}$:

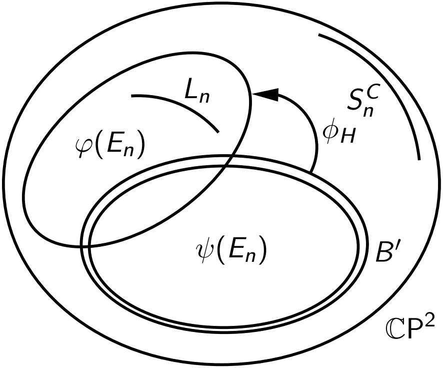


Figure: Construction of a small ball containing L_n

5. Cage size

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For now:

Proposition *There exists an increasing sequence $\alpha_n \in [3,4)$ s.t.*

$$c(L_{g_{n+1},g_n,1}(S_n^C)) \leq \alpha_n.$$

For instance, $\alpha_0 = 3$, $\alpha_1 \approx 3.552$, and α_n converges to ≈ 3.877.