Non-commutative Cartier isomorphism and quantum cohomology

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Quantum connection

Let M be a closed symplectic manifold which is monotone,

$$[\omega_M] = c_1(M) \in H^2(M; \mathbb{R}).$$

The quantum connection on $H^*(M; \mathbb{C})[q^{\pm 1}]$ differentiates with respect to the variable q:

$$\nabla_q x = \partial_q x + q^{-1}([\omega_M] *_q x),$$

where $*_q$ is the small quantum product,

$$x *_q y = x *^{(0)} y + q\, x *^{(1)} y + q^2\, x *^{(2)} y + \cdots$$

Local monodromy theorem

Let $\bar{S}$ be a complex, smooth projective curve and let

$$S = \bar{S} - \{p_1, \cdots, p_r\}$$

be the complement of a finite number of points. Suppose we have a proper smooth algebraic family $W : Y \longrightarrow S$ with fibers of complex dimension n.

Theorem (Griffiths, Grothendieck, Katz)

- *The local monodromy about each missing point p_i has eigenvalues which are roots of unity and Jordan blocks of size at most $n + 1$.*
- *The underlying algebraic vector bundle equipped with its Gauss-Manin connection has regular singularities.*

Singularities of connections

Take a formal meromorphic connection:

$$\nabla_Q = \partial_Q + A_Q \quad \text{where} \quad A_Q \in Mat_r(\mathbb{C}((Q))).$$

- ∇_Q has a regular singular point if, by a formal gauge transformation, A_Q can be transformed into $\tilde{A}_Q \in Q^{-1} Mat_r(\mathbb{C}[[Q]])$.
- The connection then has a monodromy given by

$$\exp(-2\pi i \tilde{A}_{-1})$$

which is well-defined up to conjugation.

Pole at $q = \infty$

- For the quantum connection, in all known cases, the pole at $q = \infty$ is irregular.

- The next simplest kind of singularity is a singularity of unramified exponential type, meaning it can be formally gauge transformed into a direct sum:

$$\tilde{\nabla}_Q = \bigoplus_k \tilde{\nabla}_{Q,k}, \qquad \tilde{\nabla}_{Q,k} = \tilde{\nabla}^{\mathrm{reg}}_{Q,k} + \lambda_k Q^{-2} I, \quad \lambda_k \in \mathbb{C}$$

Here $Q = 1/q$.

Main result I

Theorem (P.-Seidel)

Suppose M is a monotone symplectic manifold which carries a smooth "anti-canonical divisor" D.

- *The quantum connection has a singularity of unramified exponential type at $q = \infty$.*
- *The regularized formal monodromies at $q = \infty$ are quasi-unipotent (have eigenvalues which are roots of unity).*

Remark

Using a different approach involving quantum Steenrod operations, Zihong Chen has proven this statement without the assumption that D exists.

Main result II

Theorem (P.-Seidel)

- *Suppose M is a monotone symplectic manifold which carries a smooth "anti-canonical divisor" D. Any Jordan block for an eigenvalue $\neq 1$ of the regularized monodromy is of size $\leq \dim_{\mathbb{C}}(M)$ and any Jordan block for the eigenvalue 1 is of size $\leq \dim_{\mathbb{C}}(M) + 1$.*
- *Let $f = \prod_\lambda (t-\lambda)^{m_\lambda}$ denote the minimal polynomial for $q^{-1}c_1(M) *_q$. Then the size of the Jordan blocks for monodromy in the summand corresponding to λ are bounded by m_k.*

Remark

The second bound arises from a strengthening of Chen's method and there is no obvious relation with the first bound.

Algebraic geometry model

Let's assume $W : Y \to \mathbb{A}^1$ is a proper algebraic function. There is a convenient model for the Gauss-Manin connection given as follows:

$$\mathcal{E}_q = \Omega^*_Y[q],$$
$$d_{\mathcal{E}_q}\theta = d\theta - q\, dW \wedge \theta.$$

Let E^*_q denote the hypercohomology. They carry an endomorphism induced by:

$$\nabla_q : \theta \longmapsto \partial_q \theta - W\theta,$$

Proposition

Restricting E^*_q to $\mathbb{C}^* = \{q \neq 0\}$ yields a connection which has a regular singularity at $q = 0$, and a singularity of unramified exponential type at $q = \infty$.

Holonomic D-modules

- Let W_q be the algebra of differential operators in one variable q, over $\mathbb{C}$. This is generated by q and ∂_q, with the relation

$$[\partial_q, q] = 1. \tag{1}$$

  Left W_q-modules are called D-modules.
- A D-module N_q is called holonomic if it is finitely generated and, for every $x \in N_q$, there is a nonzero $w \in W_q$ such that $wx = 0$.
- Let N_q be a holonomic D-module. Then there is a nonzero $f \in \mathbb{C}[q]$, such that $N_{q,1/f}$ is isomorphic to a D-module coming from a connection.

Fourier-Laplace transform

Take another formal variable t and identify $W_t \cong W_q$ by setting

$$t = -\partial_q, \;\; \partial_t = q. \tag{2}$$

- Given a W_t-module N_t, the Fourier(-Laplace) transform N_q is the same vector space space considered as a module over W_q via (2).

Proposition

Let N_t be a holonomic W_t-module, with only regular singularities (including at $t = \infty$). Then, the Fourier-Laplace tranform N_q is nonsingular on $\mathbb{C}^*$. If we look at the associated connection ∇_q, then it has a regular singular point at $q = 0$, and a singularity of unramified exponential type at $q = \infty$.

Symplectic cohomology

Suppose M carries a smooth (anti-canonical) divisor D.

- Then using a standard model for a tubular neighborhood of D, the complement $X := M \setminus D$ is a convex symplectic manifold.
- The analogue of the Gauss-Manin D-module will be a q-deformations of (S^1-equivariant) symplectic cohomology of X. The most basic of these is

$$SC^*_q(X) := (SC^*(X)[q], d_q).$$

- There are variations on this e.g.

$$SC^*_{q^\pm}(X) := (SC^*(X)[q, q^{-1}], d_q),$$

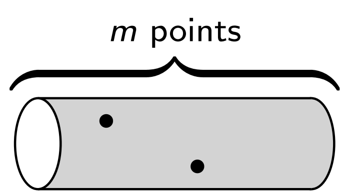
The telescope complex

- Begin with the usual telescope model for symplectic cohomology $SH^*(X)$:

$$SC^*(X) = \big(CF(0) \oplus CF(1) \oplus \cdots\big) \oplus \eta\big(CF(0) \oplus CF(1) \oplus \cdots\big) \tag{3}$$

  where $CF(w)$ are Floer complexes of slope approximately w and η is a variable of degree -1.

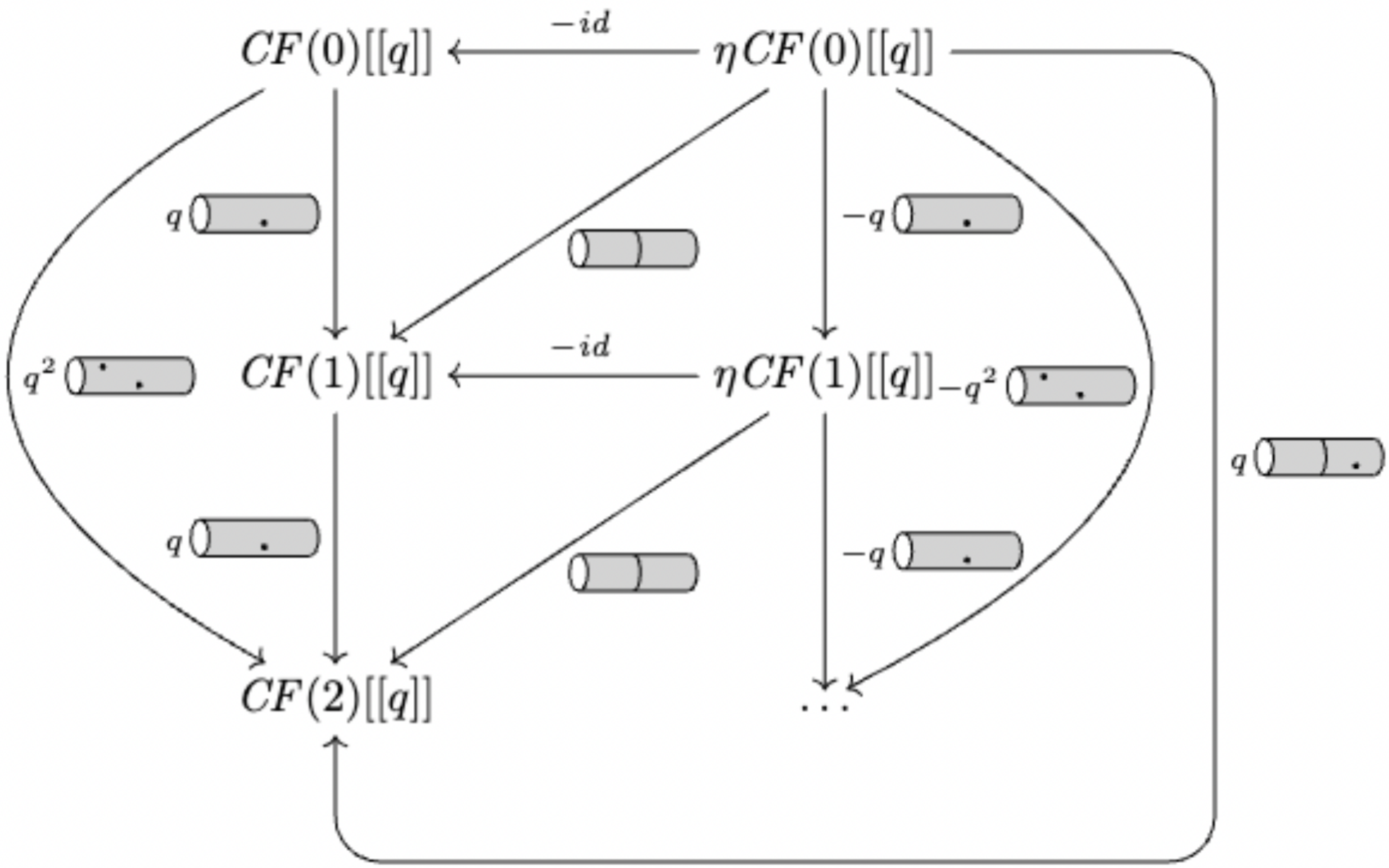
- We deform this by counting curves with m additional marked points that pass through the divisor D. For example, For every $w \geq 0$ and $m > 0$, we have maps:



$$CF^{*+1-2m}(w+m) \xleftarrow[d_m]{} CF^*(w) \tag{4}$$

The telescope complex II

The final product looks like this:



Deformed SH^* versus QH^*

Theorem (Theorem A)

There is a canonical isomorphism

$$PSS^q_{log} : H^*(M)[q] \oplus \bigoplus_{w \geq 1} H^*(D)z^w \cong H^*(SC^*_q(X)),$$

The map is q-linear on the $H^*(M)[q]$ component.

Corollary

After inverting q, we obtain an isomorphsim

$$H^*(M)[q, q^{-1}] \cong H^*(SC^*_{q^{\pm}}(X)).$$

Motivation

- The above theorem can be viewed as a refinement of work of Borman-Sheridan-Varolgunes, who constructed (in the setting of snc anti-canonical divisors D) a spectral sequence

$$SH^*(X)[q^{\pm}] => H^*(M)[q^{\pm}] \quad (5)$$

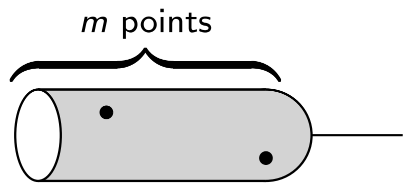
Remark

M. El-Alami and N. Sheridan extended the ideas in [BSV] to prove

$$SH^*_{q^{\pm}}(X) \cong H^*(M)[q^{\pm}].$$

The map, part I

For all $m \geq 0$, one can define maps



$$CF^{*-2m}(m) \xleftarrow{\quad s_m \quad} CM^*(M) \tag{6}$$

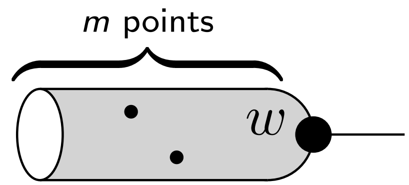
The first component of the PSS^q_{log} map is given by:

$$PSS^{q,(0)}_{log} = \sum_m q^m s_m : CM^*(M) \longrightarrow SC^*_q. \tag{7}$$

which is then extended q-linearly.

The map, part II

Fix $w \geq 1$. For all $m \geq 0$, we can look at moduli spaces with tangency conditions:



$$CF^{*-2m}(w+m) \xleftarrow{\quad s_{w,m} \quad} CM^*(D)z^w \tag{8}$$

The w-th component of the PSS^q_{log} map is given by:

$$PSS^{q,(w)}_{log} = \sum_m q^m s_{w,m} : CM^*(D)z^w \longrightarrow SC^*_q. \tag{9}$$

$k[q]$-module structure

The induced q module structure on

$$H^*(M)[q] \oplus \bigoplus_{w \geq 1} H^*(D)z^w$$

is interesting and seems to be determined by relative GW invariants. For a class $\alpha \in H^*(D)$:

- $q \cdot (\alpha z^1) = i_*(\alpha) \in H^*(M)$.

Conjecture

$q \cdot (\alpha z^2) = (c_1(ND) \cup \alpha)z^1 + qGW_{0,2}^{(1)}(i_*(\alpha))$.

- $q \cdot (\alpha z^3) = (???)$.

$SC^*_{S^1}(X)$ and $SC^*(X)_{q,u}$

- There is an S^1-equivariant version of symplectic cohomology which incorporates "loop rotation equivariance". We have deformed S^1-equivariant theory

$$SC^*(X)_{q,u} := (SC^*_{S^1}(X)[q], d_q).$$

as well as variants e.g. $SC^*(X)_{q,u^{\pm}}$.

- $SH^*(X)_{q,u^{\pm}}$ carries a connection ∇_q. Let's define the symplectic Gauss-Manin D-module to be

$$(SH^*(X)_{q,u^{\pm}}, \nabla_q). \tag{10}$$

Equivariant comparison

- There is an isomorphism:

$$PSS^{q}_{log,eq} : H^*(M)[u,q] \bigoplus_{w \geq 1} H^*(D)[u]z^w \cong SH^*(X)_{q,u}. \quad (11)$$

  It is given by counting thimbles with "angle decorations" and additional marked points.

- The map is $k[u,q]$-linear on the $H^*(M)[u,q]$ component. After inverting q we obtain:

$$PSS^{q}_{log,eq} : H^*(M)[u,q^{\pm}] \cong SH^*(X)_{u,q^{\pm}}. \quad (12)$$

Katz' approach

Take a field $\mathbb{F}$ of characteristic $p > 0$ and S a curve over $\mathbb{F}$. The p-curvature $\psi(\nabla)$ of a flat connection (E, ∇) is the map $Der_{\mathbb{F}}(S) \longrightarrow End_{\mathbb{F}}(E, E)$ given by

$$\psi(\nabla) = (\nabla_{\delta})^{p} - \nabla_{\delta^{[p]}}. \tag{13}$$

In the specific case of a rational connection in one variable,

$$\nabla_{\partial_z} = \partial_z + A, \quad A \in Mat_r(\mathbb{F}(z)) \tag{14}$$

The p-curvature is the z-linear endomorphism

$$\psi(\nabla_{\partial_z}) = \nabla^{p}_{\partial_z} \in Mat_r(\mathbb{F}(z)). \tag{15}$$

Katz' theorem

Let $R \subset \bar{\mathbb{Q}}$ be the ring obtained by starting with the integers in some algebraic number field, and inverting finitely many elements. Let ∇_{∂_z} be a connection defined over R, which means that $A \in Mat_r(R[z])$.

Theorem (Katz)

(a) If the reduction mod $\mathfrak{p}$ has nilpotent p-curvature for all primes $\mathfrak{p}$, then the original connection, considered as defined over the complex numbers, has regular singularities, and quasi-unipotent monodromy around each singularity.

(b) More precisely, suppose there is some μ such that the mod $\mathfrak{p}$ reduction satisfies $\nabla_{\partial_z}^{p\mu} = 0$, for all $\mathfrak{p}$. Then the original connection has monodromy whose Jordan blocks have a most size μ.

nc-monodromy theorem

Let R be the algebra of functions on a smooth irreducible curve S in characteristic $p > 0$. Let A be a dg-algebra over a R and A is free as an R-module. The monodromy theorem has the following analogue in non-commutative geometry:

Theorem (Kaledin, Petrov-Vaintrob-Vologodsky)

Let A be a smooth and proper DG algebra over R, and let d be a positive integer such that

$$HH_m^R(A, A) = 0, \quad \textit{for every } m \textit{ with } |m| > d.$$

Then the p-curvature of the Gauss-Manin connection on $HP_*^R(A)$ is nilpotent of exponent $\leq d + 1$.

$\mathcal{W}(X)$ as an nc-fibration

We consider the wrapped Fukaya category $\mathcal{W}(X)$ and view it as linear over $\mathbf{k}[t]$, where $t = s_{1,0}(1)$. Let

$$f(t) = \prod_{\lambda} (t - \lambda)^{m_\lambda} \tag{16}$$

be the minimal polynomial for the operation $q^{-1}c_1(M)*_q$.

Theorem

The wrapped Fukaya category $\mathcal{W}(X)_{\mathbf{k}[t,1/f(t)]} := \mathcal{W}(X) \otimes_{\mathbf{k}[t]} \mathbf{k}[t, 1/f(t)]$ is smooth over $R = \mathbf{k}[t, 1/f(t)]$.

Remark

One should compare this to the result of Ganatra-Pardon-Shende that the wrapped Fukaya category is smooth over $\mathbf{k}$ *(which is an ingredient in the proof).*

We now consider a different localization of $SH^*_{S^1,q}$:

$$SH^*_{S^1,q,1/f}(X) = \mathbf{k}[\nabla_{u\partial_q}, f(\nabla_{u\partial_q})^{-1}] \otimes_{\mathbf{k}[\nabla_{u\partial_q}]} SH^*_{S^1,q}(X). \quad (17)$$

Theorem

There is an identification

$$HP_{\mathbf{k}[t,1/f(t)]}(\mathcal{W}(X)_{\mathbf{k}[t,1/f(t)]}) \cong SH^{*+n}_{S^1,q,1/f}(X). \quad (18)$$

which identifies

$$t \longrightarrow u\nabla_q, \nabla_{GM} \longrightarrow q.$$

Remark

This says that the two D-modules are Fourier-Laplace dual.

Bound on the Jordan blocks

- In Petrov-Vaintrob-Vologodsky, the bound on the exponent is $d + 1$ where $HH^R_*(A) = 0$ for $* > d$.
- In our geometric context, $SH^*(X)$ is concentrated in degrees $[0, 2n - 1]$ (this is easy to see from the periodic flow along the boundary).
- Using this together with the relation between Hochschild homology and deformed symplectic cohomology, we conclude that we can take $d = n - 1$, which implies the bound on the Jordan blocks.

p-curvature of quantum connection

Let $\psi_{u\partial_q} = \psi(\nabla_{u\partial_q})$ denote the p-curvature of the quantum connection. Then one can readily make the following computations:

(a) $q^p \psi_{u\partial_q}(x) = \underbrace{c_1(M) *_q \cdots *_q c_1(M)}_{p} *_q x + O(u)$.

(b) $q^p \psi_{u\partial_q}(x) = (c_1(M)^p - u^{p-1} c_1(M)) \cup x + O(q)$.

(c) $q^p \psi_{u\partial_q}$ commutes with $q\nabla_{u\partial_q}$ (by definition).

quantum Steenrod operations

Set $\mathbf{k} = \mathbb{F}_p$. Introduce another formal variable θ, of degree 1. The quantum Steenrod operation yields, for any $c \in H^*(M, \mathbb{F}_p)[q]$, an endomorphism of degree $p|c|$,

$$Q\Sigma_c : H^*(M)[q, u, \theta] \longrightarrow H^*(M)[q, u, \theta]. \tag{19}$$

Here are some basic properties:

(a')

$$Q\Sigma_c(x) = \underbrace{c *_q \cdots *_q c}_{p} *_q x + O(u, \theta).$$

(b') $Q\Sigma_c(x) = St(c) \cup x + O(q)$, where St is a classical total Steenrod operation. If c is of degree 2 and is the mod p reduction of a $\mathbb{Z}$-valued cohomology class,

$$Q\Sigma_c(x) = (c^p - u^{p-1}c) \cup x + O(q). \tag{20}$$

p-curvature=QSt

Theorem (Seidel-Wilkins)

$Q\Sigma_c$ commutes with $q\nabla_{u\partial_q}$ (here, the connection has been tacitly extended to be θ-linear).

Theorem (P.-Seidel, forthcoming)

For any prime p, we have an equality of operations:

$$q^p \psi_{u\partial_q} = Q\Sigma_{c_1(M)}. \tag{21}$$

Remark

- *The relation between p-curvature and quantum Steenrod operations was observed by Jae Hee Lee.*
- *Chen used a weaker version of this statement (which he attributes to Seidel) to give his alternative proof of the exponential type property.*