Symplectic Zoominars

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Bounding Lagrangian intersections using Floer homotopy theory

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$(M, \omega) \rightsquigarrow$ symplectic $2n$-manifold, i.e.,
$\omega \in \Omega^2(M)$ closed, non-degenerate 2-form.

$(L_0, L_1) \rightsquigarrow$ pair of Lagrangians, i.e.,
$L_i \subset M$ codimension n submanifold s.t. $\omega|_L = 0$.

Question:

How many points does L_0 intersect L_1 in?

The results towards answering this depend on our assumptions on the underlying geometry.

Assumption (A1):

$\langle \omega, \pi_2(M, L_i) = 0 \rangle$.

- (M, ω) is either closed or Liouville,
- (L_0, L_1) Hamiltonian isotopic via a compactly supported Hamiltonian isotopy,
- L_i is connected, closed, & relatively exact.

Arnol'd Conjecture:

Suppose A1, then

$$\# L_0 \cap L_1 \geq \min_{f \in C^\infty(L_i)} \# \mathrm{Crit}(f).$$

If the Arnol'd conjecture is true, then the bound is sharp.

Theorem (Floer 1988): Suppose AI & $L_0 \pitchfork L_1$, then
$$\# L_0 \cap L_1 \geq \sum_j \operatorname{rank} H_j(L_i; \mathbb{Z}/2).$$

$\boxed{\text{What if we want to ditch the transversality hypothesis?}}$

$R \rightsquigarrow$ commutative ring.

$\underline{c}_R(X) \rightsquigarrow R$-cup-length of space X

$$\inf_k \{ \forall \alpha_1, \dots, \alpha_k \in \tilde{H}^*(X; R) : \alpha_1 \cup \dots \cup \alpha_k = 0 \}.$$

Theorem (Hofer 1988, Floer 1989): Suppose AI, then
$$\# L_0 \cap L_1 \geq c_{\mathbb{Z}/2}(L_i).$$

Recent work of Hirschi-Porcelli generalizes this using Floer homotopy.

$\mathcal{R} \rightsquigarrow$ ring spectra (w/ multiplicative cohomology theory $\mathcal{R}^*$).

Theorem (Hirschi-Porcelli 2022): Suppose AI & $\mathcal{R}$-orientability hypotheses, then
$$\# L_0 \cap L_1 \geq c_{\mathcal{R}}(L_i).$$

Note, Hirschi–Porcelli show this bound is strictly stronger than non-Floer homotopy bounds.

Now, what if we want to ditch the Hamiltonian isotopic hypothesis?

Consider the quantum cup product action on Lagrangian Floer cohomology:

$$HF(L_0,L_1;\mathbb{Z}/2)\otimes H^*(L_i;\mathbb{Z}/2) \longrightarrow HF(L_0,L_1;\mathbb{Z}/2)$$
$$\beta\otimes\alpha \longmapsto \beta*\alpha.$$

Counts pseudoholomorphic strips w/ incidence relation on the boundary of the strip

The precise assumption is not too important, it just allows for $\mathbb{Z}/2$-Lagrangian Floer cohomology to exist.

The following is a "quantum cup-length" estimate.

Theorem (B. 2024): Suppose $\boxed{A2}$ & $\exists\ \alpha_1,\dots,\alpha_k\in H^*(L_i;\mathbb{Z}/2)$ together w/ $\beta\in HF(L_0,L_1;\mathbb{Z}/2)$ s.t.

$$\beta*\alpha_1*\cdots*\alpha_k\neq 0,$$

then any compactly supported Hamiltonian deformation of (L_0,L_1) intersects in at least k points of distinct action.

Can we do better using Floer homotopy?

i.e., exists w.r.t. the choice of Λ.

Assumption 3 (A3):

- (M,ω) is Liouville,
- L_i is either closed exact ① or cylindrical at infinity ②,
- (M, L_0, L_1) has a framed brane structure ③.

① L_i closed & $\lambda|_{L_i} = df_i$.

② $\lambda|_{L_i} = df_i$ & L_i looks like $[1,\infty) \times (L_i \cap M_\infty)$ at infinity w/ M_∞ the "contact boundary at infinity".

③ $\exists d \in \mathbb{Z}_{\geq 0}$ & $\Lambda \subset TM \oplus \underline{\mathbb{C}}^d$ totally real subbundle s.t.

(i) $TM \oplus \underline{\mathbb{C}}^d \cong \Lambda \otimes_{\mathbb{R}} \underline{\mathbb{C}}$,

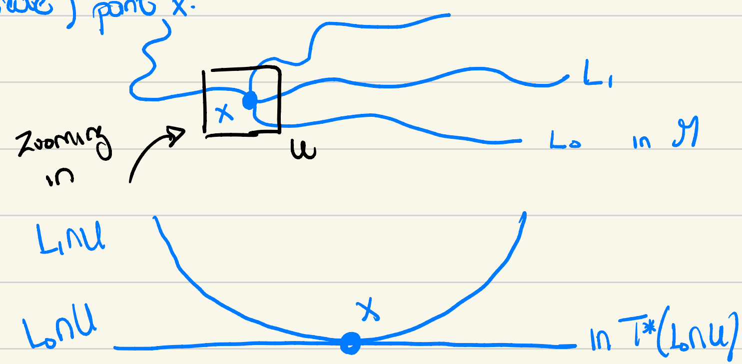
(ii) $TL_i \oplus \underline{\mathbb{R}}^d$ homotopic, through totally real subbundles, to Λ (the homotopy is part of data).

Theorem (Cohen-Jones-Segal 1995, Large 2021)

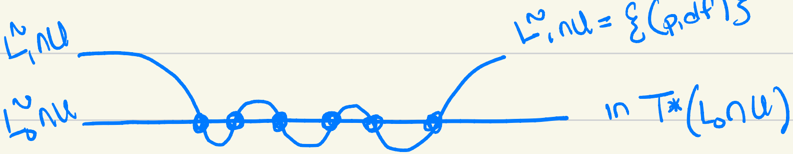
Suppose A3, then the Floer homotopy type $\mathfrak{F}^{\Lambda}_{L_0 L_1}$ exists, i.e., a spectrum whose (co)homology is the Lagrangian Floer (co)homology.

Now, back to intersections!

Suppose for fun (L_0, L_1) intersect in a single (possibly degenerate) point x.



Near x, L_1 looks like $\{(p, df_p)\}$, where $f \in C^\infty(\mathbb{R}^n)$ w/ a single (possible degenerate) critical point corresponding to x. We can do a local deformation to make $L_0 \pitchfork L_1$.



Theorem (Floer 1988)

$\exists m$, $HF^*(L_0, L_1; \mathbb{Z}/2) \cong HM^{*-m}(L_0 \cap \mathcal{U}; f; \mathbb{Z}/2)$.

$$HF^*(\tilde{L}_0, \tilde{L}_1; \mathbb{Z}/2) \qquad HM^{*-m}(\tilde{L}_0 \cap \mathcal{U}; \tilde{f}; \mathbb{Z}/2)$$

We can upgrade this to a statement about Floer homotopy.

Morse homotopy type, i.e., spectra whose (co)homology is Morse (co)homology.

Prop^n (B.):

$$\mathfrak{F}^{\wedge}_{L_0, L_1} \simeq \Sigma^m \boxed{\mathcal{M}^f_{L_0 \cap \mathcal{U}}}.$$

Finally, we know

$$HM^*(L_0 \cap \mathcal{U}; f; \mathbb{Z}/2) \cong \tilde{H}^*(\mathcal{C}_f; \mathbb{Z}/2),$$

where $\mathcal{C}_f$ is a space called the Conley index of f.

Again, we can upgrade (this is essentially folklore).

Prop^n (B.)

$$\mathcal{M}^f_{L_0 \cap \mathcal{U}} \simeq \Sigma^\infty \mathcal{C}_f.$$

The (stable) homotopy type of the Conley index of a smooth function on $\mathbb{R}^n$ w/ a single critical point is quite constrained.

Space w/ homotopy co-product: $X \to X \vee X$.

<u>Theorem</u> (Pears 1994):

$\mathcal{E}_f \simeq S^0$ or $\mathcal{E}_f$ is a $\boxed{\text{Co-H-space.}}$

<u>Prop</u>^{n} (B.):

$$Sq^j : H^p(\mathcal{E}_f ; \mathbb{Z}/2) \longrightarrow H^{p+j}(\mathcal{E}_f ; \mathbb{Z}/2),$$

where Sq^j is j-th Steenrod square, vanishes if $j \geq (n-1)/2$.

Here... if $\mathcal{F}^{\wedge}_{L_0,L_1}$ has a non-zero Sq^j, w/ $j \geq (n-1)/2$, L_0 intersects L_1 in <u>at least</u> 2 points of <u>distinct action</u>.

(If only intersect in one point, we violate $*$).

Now, we generalize by filtering $\mathcal{F}^{\wedge}_{L_0,L_1}$ by action.

<u>Theorem</u> (B.):

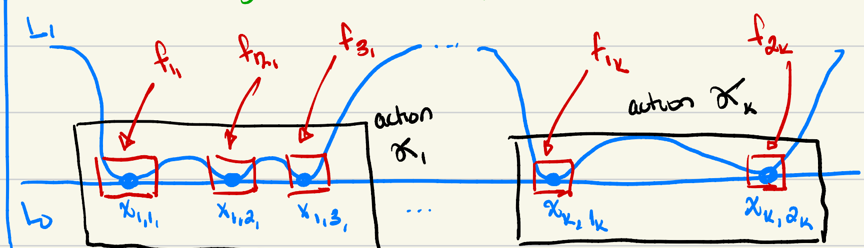
Suppose A3 &

$$L_0 \cap L_1 = \{ \{x_{1,n_1}\}, \{x_{2,n_2}\}, \dots, \{x_{k,n_k}\} \},$$

where $\{x_{j,n_j}\}$ all have action α_j ($\alpha_1 > \dots > \alpha_k$), then $\exists \{d_{n_j}\}$ & cofiber sequences

$$\mathcal{F}^{\wedge, \alpha_{j-1}}_{L_0,L_1} \longrightarrow \boxed{\mathcal{F}^{\wedge, \alpha_j}_{L_0,L_1}} \longrightarrow \bigvee_{n_j} \Sigma^{d_{n_j}} \mathcal{E}_{f_{n_j}}.$$

Fiber homotopy type computed w/ points of action at least α_j.



The main motivation of the previous theorem is the following result.

<u>Theorem</u> (B.):

Suppose A3, $\exists \{r_{js}\} \subset \mathbb{Z}$ w/ $r_{js} \geq (n-1)/2$, &

& $\exists \{\alpha_s\} \subset HF^*(L_0, L_1; \mathbb{Z}/2)$ s.t.

$\forall s \qquad Sq^{r_{k_s}} \cdots Sq^{r_{1s}}(\alpha_s) \neq 0$,

$\forall s' < s$, $\deg \alpha_s - \deg Sq^{r_{k_{s'}}} \cdots Sq^{r_{1s'}}(\alpha_{s'}) \geq n-1$,

then <u>any</u> compactly supported Hamiltonian deformation of (L_0, L_1) intersects in <u>at least</u> $\#\{\alpha_s\} + \sum k_s$ points of <u>distinct action</u>.

Note, sometimes we can use Steenrod squares w/ smaller degree.

i.e., when $n = 14$ we can take $r_{js} \geq 2$. (Instead of ≥ 3.)

Does this bound ever do better than previous results?

Yes! We construct (M, L_0, L_1, Λ), $\dim M = 14$, s.t.

(i) (L_0, L_1) not Hamiltonian isotopic,

(ii) vanishing quantum cup products,

(iii) $\mathcal{F}^{\Lambda}_{L_0 L_1} \simeq \Sigma^{\infty}_{+}(\mathbb{CP}^2 \vee \cdots \vee \Sigma^{8(j-1)}\mathbb{CP}^2)$.

Investigation w/ just $HF^*(L_0, L_1)$ shows any compactly supported Hamiltonian deformation intersects in at least j points of distinct action. Steenrod squares tell us $2j$.

Essentially, just by using degree arguments.

A quick explanation of the construction is the following.

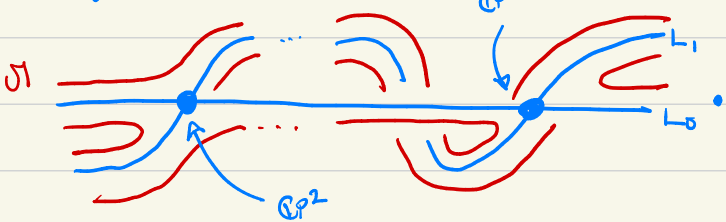
Classically, $\exists$ an embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{R}^7 \hookrightarrow S^7$.

Now, plumb together two T^*S^7's along j disjoint $\mathbb{CP}^2$'s.

The result is a symplectic 14-manifold M w/ (L_0, L_1), $L_i \overset{\text{top}}{\cong} S^7$, s.t.

$$L_0 \cap L_1 = \underbrace{\mathbb{CP}^2 \sqcup \cdots \sqcup \mathbb{CP}^2}_{j\text{-times}}$$

cleanly.



We define Λ by taking the "standard" $\mathbb{R}$-polarizations

$$T(T^*S^7) \cong \begin{cases} \text{horizontal bundle} \otimes_{\mathbb{R}} \mathbb{C}, \\ \text{vertical bundle} \otimes_{\mathbb{R}} \mathbb{C} \end{cases}$$

gluing them together, & stabilizing by appropriate rotations of $\mathbb{R}$'s around the non-trivial loops in $\pi_1(M)$.

This is what gives the shifts by Σ in $\mathcal{F}^{\Lambda}_{L_0 L_1}$.

Thank you!