Kähler compactification of $\mathbb{C}^n$ and Reeb dynamics

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2. When $n = 2$, $(X, D) = (\mathbb{P}^2, \mathbb{P}^1)$, $(Q_2, Q_2 \cap H)$, where Q_2 is a quadratic surface in $\mathbb{P}^3$ and H is a hyperplane tangent to Q_2. $Q_2 \cap H$ is not irreducible. There are many more.
3. When $n = 3$, $(X, D) = (\mathbb{P}^3, \mathbb{P}^3), (Q_3, Q_3 \cap H) \ldots$. Now $Q_3 \cap H$ is the projective cone over a quadratic curve, which is now irreducible.

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We will call $(\mathbb{P}^n, \mathbb{P}^{n-1})$ the standard compactification.

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Question

Classification is too hard, but can we characterize the standard compactification?

Characterizing the standard compactification

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Theorem (Li-Z. 24)

The conjecture is true if X is Kähler.

Remark

Peternell posted a short proof for the n-even case shortly after our post.

Orbifold compactification

We can consider the orbifold analogue where X is a smooth orbifold and D is a smooth suborbifold.

Example

$(X, D) = (\mathbb{P}(1, w_1, \ldots, w_n), \mathbb{P}(w_1, \ldots, w_n))$ for $w_i \in \mathbb{N}_+$.

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The weighted projective space pairs are the only compactification of $\mathbb{C}^n$ if both X, D are both smooth orbifolds.

Remark

D being a suborbifold is crucial, otherwise $(Q_3, Q_3 \cap H \simeq \mathbb{P}(1,1,2))$ is also a compactification.

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But why orbifolds?

Asymptotically conical metric

Definition

(M^{2n-1}, g, α) is called Sasaki if the cone $(R_+ \times M, dr^2 + r^2 g, r^2 \mathrm{d}\alpha + 2r\mathrm{d}r \wedge \alpha)$ is Kähler. We use $(\mathcal{C}, g_0, J_0)$ to denote this Kähler cone.

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Definition

A complete Kähler manifold (W, g, J) is called asymptotically conical (AC) with the asymptotical cone $(\mathcal{C}, g_0, J_0)$ if there exists a compact subset $K \subset W$ and a diffeomorphism $\Phi : \{r > 1\} \to W \setminus K$ such that $\Phi^* g \to g_0$ and $\Phi^* J \to J_0$ as $r \to \infty$.

A detour-contact manifolds and their symplectic fillings

Definition

(M^{2n-1}, ξ) is called a contact manifold, if $\xi \subset TM$ and there exists $\alpha \in \Omega^1(M)$ such that

1. $\xi = \ker \alpha$;
2. $\alpha \wedge (\mathrm{d}\alpha)^{n-1} \neq 0$.

The Reeb vector field R_α is characterized by $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} \mathrm{d}\alpha = 0$.

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Contact manifolds are natural boundaries of symplectic manifolds.

Definition

(W, λ) is a Liouville cobordism from (M_-, ξ_-) to (M_+, ξ_+) if

1. $\lambda \in \Omega^1(W)$ and $\mathrm{d}\lambda$ is a symplectic form;
2. $\partial W = M_- \sqcup M_+$, $\lambda|_{M_\pm}$ are contact forms for $\xi_\pm$.
3. The Liouville vector field defined by $\iota_X \mathrm{d}\lambda = \lambda$ points outward/inward along $M_{+/-}$.

A Liouville cobordism from $\emptyset$ to M is a Liouville filling of M.

Orbifold compactification of asymptotically conical metrics

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Theorem (Li 20, Conlon-Hein 24)

Orbifold compactification appears naturally in the compactification of domains with AC metrics. If $m = 1$, then we add in a divisor $M/\langle R\rangle$. If $m > 1$, we add in a divisor $M/\langle R'\rangle$, where R' is a rational direction close to R.

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Similar examples in higher dimensions were found by Szekelyhidi.

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Theorem (Li-Z. 24)

The above conjecture holds if the Shokurov conjecture holds (for Fano cone singularities). In particular, the above conjecture holds in dimension 3.

Minimal discrepancy and the Shokurov conjecture

Let X be a normal $\mathbb{Q}$-Gorenstein variety and o be an isolated singularity. Consider a resolution $\pi : \widetilde{X} \to X$ of the singularity o, we have

$$K = \pi^* K_X + \sum a_i E_i$$

where E_i are exceptional divisors.

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Conjecture (Shokurov 02)

$\mathrm{md}(o) \leq \dim_{\mathbb{C}} X - 1$ *and when the equality holds, o is a smooth point.*

It is true for up to dimension 3 and some special cases, e.g. complete intersection.

Minimal discrepancy for topologists

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M is naturally a contact manifold, whose contact structure ξ is the CR structure, i.e. the maximal complex subspace of TM.

$\mathbb{Q}$-Gorenstein implies that $c_1^{\mathbb{Q}}(\xi) = 0$. Then we may view $c_1^{\mathbb{Q}}(\widetilde{X})$ as in $H^2(\widetilde{X}, M; \mathbb{Q})$, then we can write

$$c_1^{\mathbb{Q}}(\widetilde{X}) = \sum -a_i LD(E_i).$$

Then $\mathrm{md}(o) = \min\{a_i\}$ if $\min\{a_i\} \geq -1$, and $-\infty$ if $\min\{a_i\} < -1$.

A simple example

Let D be a Fano manifold and $-K_X = rL$ for $r > 0 \in \mathbb{Q}$. We consider

$$X = \mathrm{Spec} \oplus_{m=0}^{\infty} H^0(D, L^m)$$

i.e. X is obtained by contracting the zero section in L^{-1}. Then $\mathrm{md}(o) = r - 1$.

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The maximal index r is called the Fano index. In Brenton-Morrow's conjecture, because of the Kobayashi-Ochai Theorem, it suffices to prove the Fano index of D (the smooth divisor) is n.

The main technical result

Theorem (Li-Z. 24)

Let $o \in X$ be an isolated Fano cone singularity of dimension n. For any quasi-regular conic contact form η on the contact link M, we have the following formula for the minimal discrepancy:

$$2\mathrm{md}(o) = \inf_{\gamma} \mathrm{lSFT}_{\eta}(\gamma) > -2.$$

Here γ on the right ranges over all closed Reeb orbits of η. If moreover M admits a Liouville filling W such that $c_1^{\mathbb{Q}}(W) = 0$, then we have

$$2\mathrm{md}(o) = \inf\{d \mid SH_d^{+,S^1}(W;\mathbb{Q}) \neq 0\} + n - 3$$

where $SH_*^{+,S^1}(W;\mathbb{Q})$ denotes the $\mathbb{Q}$-coefficient S^1-equivariant positive symplectic homology of the Liouville filling W.

Conley-Zehnder index

In our case, $c_1^{\mathbb{Q}}(\xi) = 0$, we can trivialize $\det_{\mathbb{C}} \oplus^N \xi$. Hence we can define

$$\mu_{CZ}^{\mathbb{Q}}(\gamma) = \frac{1}{N}\mu_{CZ}(\oplus^N \rho(t))$$

where $\rho(t)$ is the linearized flow.

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where $\rho(t)$ is the linearized flow.
Let W be a symplectic filling of (M, ξ), we can view $c_1^{\mathbb{Q}}(W)$ as in $H^2(W, M; \mathbb{Q})$.
Let u be disk in W with boundary γ, then we have

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$\mu_{LCZ}^{\mathbb{Q}}$: lower semi-continuous extension of $\mu_{CZ}^{\mathbb{Q}}$.

$$\mathrm{lSFT}(\gamma) := \mu_{LCZ}(\gamma) + n - 3.$$

Why should they be related?

Theorem (McLean 16)

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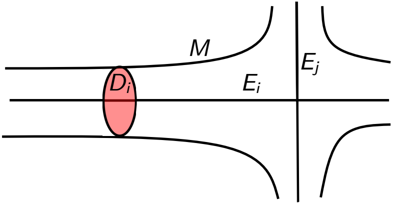

The moduli space of the above disk is 0, the difference in dimensions is precisely $2\langle c_1(\widetilde{X}), D_i\rangle$, which is $2a_i$

Computing md and lSFT

Algebraic side

$L^{-1} \to D$ is almost a resolution, except D has quotient singularities. Locally, L^{-1} is modeled on

$$\mathbb{C} \times \mathbb{C}^{n-1} / \frac{1}{m}(1, b_2, \ldots, b_n), 0 \leq b_i < m$$

$$\mathrm{md} = \min_{p, g \neq \mathrm{id}} \left\{ r, \frac{r w_1(g)}{m} + \sum_{i=2}^{n} \frac{w_i(g)}{m} \right\}$$

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Symplectic side

1. $2r$ is the Maslov index of the loop of the symplectic matrix from the linearized flow around a principle orbit.
2. Using the framing from the local model, one can compute a CZ index, whose multiple is principle $\Rightarrow$ discrepancy in framing $\Rightarrow$ CZ for all Reeb orbits.

Symplectic homology and its variants

Given a Liouville filling W of M, we can construct several Floer homology generated by Reeb orbits on M and cochain complex on W graded by μ_{CZ}.

1. $SH_*(W), SH_*^+(W)$ and $SH_*^{+,S^1}(W)$;
2. $SH_*^{+,S^1}(W)$'s chain complex is generated by Reeb orbits;
3. $\ldots \to H^{n-*}(W) \to SH_*(W) \to SH_*^+(W) \to \ldots$;
4. $SH_*(W)$ is a unital ring and $H^{n-*}(W) \to SH_*(W)$ is a ring map;
5. Gysin exact sequence, $\ldots \to SH_*^+(W) \to SH_*^{+,S^1}(W) \to SH_*^{+,S^1}(W) \to \ldots$;
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Example

$SH_*(\mathbb{C}^n) = 0$ and $SH_*^{+,S^1}(\mathbb{C}^n) = \oplus_{k\in\mathbb{N}}\mathbb{Q}[-2k-1-n]$.

A spectral sequence

We have a spectral sequence computing $SH_*^{+,S^1}(W)$ for Fano cone singularities with the first page:

$$E^1_{p,q} = \bigoplus_{p=N(\ell+\frac{k}{|G|})} H_{p+q-\mu_{\mathrm{LCZ}}(G,k,\ell)}(D^i_G;\mathbb{Q})$$

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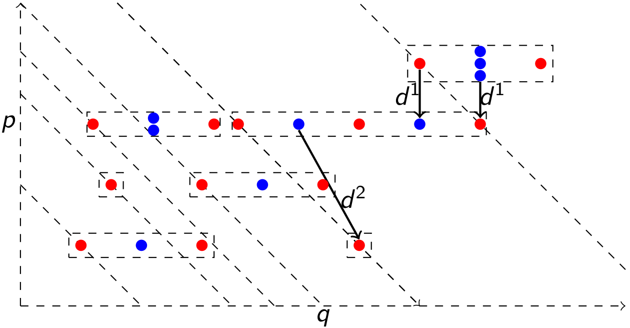

Figure: A schematic picture of the first page of the spectral sequence

Proof of the compactification results

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6. By our theorem, $\mathrm{md} = n - 1$.

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10. By Anderson's rigidity theorem, the original CY metric is flat.

Thank you!