Morse–Bott Floer homology and rectangular pegs

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History of rectangular peg problem

Toeplitz (1911): Does every Jordan curve (i.e. C^0 simple closed curve in $\mathbb{R}^2$) admits a square peg?

Schnirel'mann (1929): Every C^∞ simple closed curve in $\mathbb{R}^2$ admits a square peg.

Vaughan (1977): Every C^0 simple closed curve in $\mathbb{R}^2$ admits a rectangular peg. (Note: aspect angle unknown)
The proof uses non-existence of embedded Klein bottle in $\mathbb{R}^3$.

Greene-Lobb (2020): Every C^∞ simple closed curve in $\mathbb{R}^2$ admits a rectangular peg with any aspect angle $\phi \in (0, \frac{\pi}{2}]$.
The proof uses non-existence of embedded Lagrangian Klein bottle in $(\mathbb{R}^4, \omega_{\text{std}})$ (Shevchishin, Nemirovski 2007) or minimal Maslov number of embedded Lagrangian torus in $(\mathbb{R}^4, \omega_{\text{std}})$ is 2 (Polterovich, Viterbo 1990).

Results

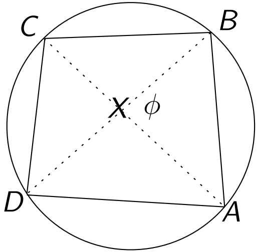
Theorem (G. 2024)

There is a generic class of C^∞ simple closed curves in the $\mathbb{R}^2$ in which each curve admits two geometrically distinct rectangular pegs with any aspect angle $\phi \in (0, \frac{\pi}{2})$.

Theorem (G. 2024)

There is a generic class of C^∞ simple closed curves in the $\mathbb{R}^2$ in which each curve admits two geometrically distinct cyclic quadrilateral pegs with any data $s, t \in (0, \frac{1}{2}]$ and $\phi \in (0, \pi)$ except $s = t = \frac{1}{2}$ and $\phi = \frac{\pi}{2}$.

Results



cyclic quadrilateral peg with data $s = \frac{|AX|}{|AC|}$, $t = \frac{|BX|}{|BD|}$, $\phi = \angle AXB$, and vertices (A, B, C, D) ordered counterclockwise.

Rectangular pegs and Clifford torus

Let $\gamma \in \mathrm{Emb}^{C^\infty}(S^1, \mathbb{C})$ which bounds area $\mathrm{Area}(\bar{\gamma}) = \pi$. It is in fact Hamiltonian diffeomorphic to $S^1(1)$ in $(\mathbb{C}, \omega_{\mathrm{std}})$.
There are two Lagrangian tori in $(\mathbb{C}^2, \omega_{\mathrm{std}})$:

$$L := F\left(\frac{1}{\sqrt{2}}(\gamma \times \gamma)\right), \qquad L_\phi := R_\phi F\left(\frac{1}{\sqrt{2}}(\gamma \times \gamma)\right)$$

where

$$F : (z_1, z_2) \mapsto \left(\frac{z_1 + z_2}{\sqrt{2}}, \frac{z_1 - z_2}{\sqrt{2}}\right)$$
$$R_\phi : (z_1, z_2) \mapsto (z_1, z_2 e^{i\phi})$$

are two Hamiltonian diffeomorphisms on $(\mathbb{C}^2, \omega_{\mathrm{std}})$. They are in fact Hamiltonian diffeomorphic to $T^2_{\mathrm{Cliff}} = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset \mathbb{C}^2$, which is a monotone Lagrangian torus with minimal Maslov number 2.

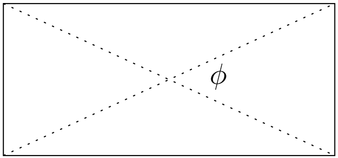
Rectangular pegs and Clifford torus

L and L_ϕ are ϱ-invariant where

$$\varrho : (z_1, z_2) \mapsto (z_1, -z_2)$$

is a symplectic involution on $(\mathbb{C}^2, \omega_{\text{std}})$ with $\text{Fix}(\varrho) = \mathbb{C} \times \{0\}$ that restricts to $\gamma \times \{0\}$ in which L and L_ϕ intersect cleanly.
An intersection point $p = (z_1, z_2) \in L \cap L_\phi$ determines a rectangular peg:

$C = z_1 - z_2$ $\qquad$ $z_1 + z_2 \cdot e^{i\phi} = B$



$D = z_1 - z_2 \cdot e^{i\phi}$ $\qquad$ $z_1 + z_2 = A$

Degenerate peg $A = B = C = D \Leftrightarrow p \in \gamma \times \{0\} \subset L \cap L_\phi$

Rectangular pegs and Clifford torus

Correspondence:

- When $\phi \in (0, \frac{\pi}{2})$, there is a bijection between sets

$$(L \cap L_\phi \setminus (\gamma \times \{0\}))/C_2 \xrightarrow{\cong} \mathrm{Rect}(\gamma, \phi)$$

  where C_2 is presented by $\langle \mathrm{id}, \varrho \mid \varrho^2 = \mathrm{id} \rangle$.
- When $\phi = \frac{\pi}{2}$, there is a bijection between sets

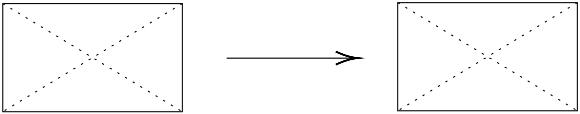
$$(L \cap L_\phi \setminus (\gamma \times \{0\}))/C_4 \xrightarrow{\cong} \mathrm{Rect}(\gamma, \phi)$$

  where C_4 is presented by $\langle \mathrm{id}, \varrho, \iota \mid \varrho^2 = \mathrm{id}, \iota^2 = \varrho \rangle$ and $\iota : (z_1, z_2) \mapsto (z_1, iz_2)$.

Rectangular pegs and Clifford torus

Action of ϱ on rectangular peg (A, B, C, D) with $\phi \in (0, \frac{\pi}{2})$:

$C = z_1 - z_2 \quad z_1 + z_2 e^{i\phi} = B \qquad C' = z_1 + z_2 \quad z_1 - z_2 e^{i\phi} = B'$



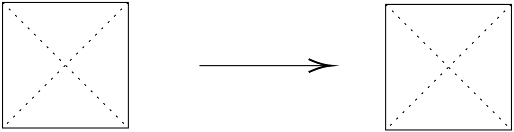
$D = z_1 - z_2 e^{i\phi} \quad z_1 + z_2 = A \qquad D' = z_1 + z_2 e^{i\phi} \, z_1 - z_2 = A'$

(A', B', C', D') geometrically coincides with (A, B, C, D) in $\mathbb{R}^2$.

Rectangular pegs and Clifford torus

Action of ι on rectangular peg (A, B, C, D) with $\phi = \frac{\pi}{2}$:

$C = z_1 - z_2 \quad z_1 + iz_2 = B \qquad C' = z_1 - iz_2 \quad z_1 - z_2 = B'$

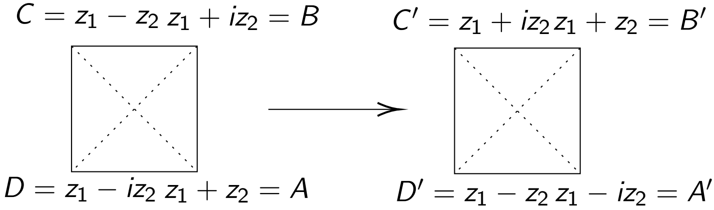


$D = z_1 - iz_2 \quad z_1 + z_2 = A \qquad D' = z_1 + z_2 \quad z_1 + iz_2 = A'$

(A', B', C', D') geometrically coincides with (A, B, C, D) in $\mathbb{R}^2$.

Rectangular pegs and Clifford torus

Action of $\varrho \circ \iota$ on rectangular peg (A, B, C, D) with $\phi = \frac{\pi}{2}$:



(A', B', C', D') geometrically coincides with (A, B, C, D) in $\mathbb{R}^2$.

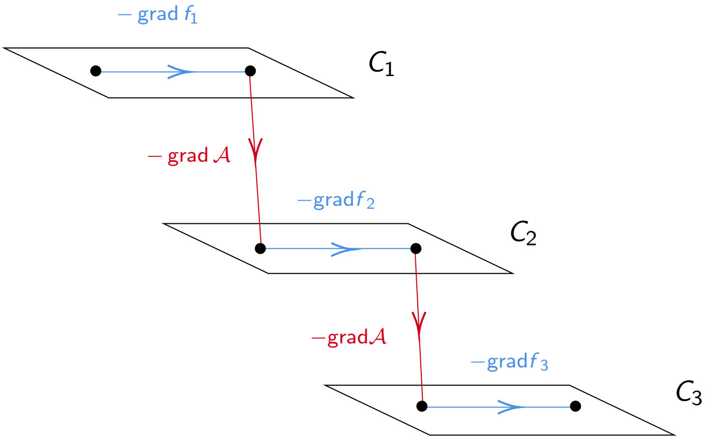
Morse–Bott Floer homology for monotone Lagrangians

Oh 1990s: For transversely intersecting monotone Lagrangians L_1 and L_2 with $\min\{N_{L_1}, N_{L_2}\} \geq 3$ or $L_2 = \varphi_H(L_1)$ with $N_{L_1} = N_{L_2} = 2$, Lagrangian Floer homology $HF_*(L_1, L_2)$ is well-defined.

Lagrangian Floer theory for clean intersection (i.e. Morse–Bott setting): Poźniak (1994), Fukaya-Oh-Ohta-Ono (2000s), Frauenfelder (2003), Biran-Cornea (2007), Seidel (2011), Sheridan (2011), Schmäschke (2016), etc.

For cleanly intersecting monotone Lagrangians L_1 and L_2 with $\min\{N_{L_1}, N_{L_2}\} \geq 3$ or $L_2 = \varphi_H(L_1)$ with $N_{L_1} = N_{L_2} = 2$, and $L_1 \cap L_2 = \bigsqcup_{i \in I} C_i$, the Morse–Bott Floer homology $HF_*(L_1, L_2, \{f_i\}, J, \{g_i\})$ is well-defined where ∂ is defined by counting cascade trajectories.

cascade trajectory:



Categorification of algebriac intersection number in Morse–Bott setting

Frauenfelder-G. (2024): Given two compact, oriented Lagrangians $L_1, L_2 \subset (M, \omega)$ intersecting cleanly with $L_1 \cap L_2 = \sqcup_{i \in I} C_i$ $(|I| < \infty)$:

- $L_1 \bullet L_2 = \sum_{i \in I} \mathrm{sign}(C_i)\chi(C_i)$
- when L_1 and L_2 are monotone as before:

$$\chi(HF_*(L_1, L_2, \{f_i\}; \Lambda)) = \pm \sum_{i \in I} \mathrm{sign}(C_i)\chi(C_i)$$

Indirect proof:

Assume $L \cap L_\phi \setminus (\gamma \times \{0\}) = \varnothing$. Then $L \cap L_\phi = \gamma \times \{0\}$, by choosing a perfect Morse function f on $\gamma \times \{0\}$ and lifts to all copies in the Nivikov covering $\widetilde{\mathcal{P}}(L, L_\phi)$, $HF_*(L, L_\phi, \{f\})$ is well-defined as $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}_2$-vector space.
Since L_ϕ can be Hamiltonian displaced away from L in $\mathbb{C}^2$, then by the Hamiltonian invariance:

$$HF_*(L, L_\phi) \cong \{0\}$$

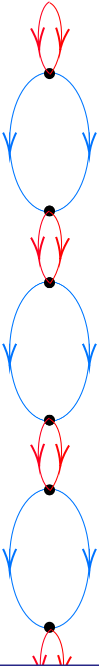
On the other hand, by choosing ϱ-invariant $J^\varrho \in \mathcal{J}^\varrho(\mathbb{C}^2, \omega_{\text{std}})$, we compute explicitly that $HF_*(L, L_\phi, \{f\}, J^\varrho) \ncong \{0\}$.
This will lead to a contradiction, and therefore $L \cap L_\phi \setminus (\gamma \times \{0\}) \neq \varnothing$.

On the existence of rectangular peg

- Equivariant transversality (Khovanov-Seidel 2002): J^{ϱ}-holomorphic strip u_0 with $\mathrm{im}(u_0) \subset \mathrm{Fix}(\varrho)$ is the only obstruction for $\widetilde{\mathcal{M}}(J^{\varrho})$ to be transversely cutout.
  However, such u_0 in here has $\partial\, \mathrm{im}(u_0) = \gamma \times \{0\} \subset \mathrm{Fix}(\varrho)$, and the Maslov index $\mu(u_0) = 4$. Thus for a generic choice of $J^{\varrho} \in \mathcal{J}^{\varrho}(\mathbb{C}, \omega_{\mathrm{std}})$, $\widetilde{\mathcal{M}}_{\mathrm{cas}}(J^{\varrho})_{[1]}$ and $\widetilde{\mathcal{M}}_{\mathrm{cas}}(J^{\varrho})_{[2]}$ are transversely cutout as they do not contain such u_0, and $HF_*(L, L_\phi, \{f\}, J^{\varrho})$ is still well-defined.
- When a J^{ϱ}-holomorphic strip u has $\mathrm{im}(u) \not\subset \mathrm{Fix}(\varrho)$, then there is another one $u^{\varrho} := \varrho \circ u$ with $\mathrm{im}(u^{\varrho}) \not\subset \mathrm{Fix}(\varrho)$. These cancel in pairs in $CF_*(L, L_\phi, \{f\}, J^{\varrho})$.

Thus we have:

$$HF_*(L, L_\phi, \{f\}, J^{\varrho}) \cong H_*(S^1; \mathbb{F}_2)$$



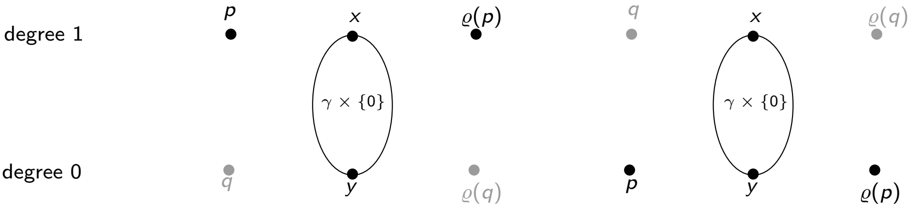
On the generic doubling of rectangular pegs

Geomtric transversality: There is a generic class of C^∞ simple closed curves γ's such that L and L_ϕ intersect transversely away from $\gamma \times \{0\}$.

Indirect proof: Assume there is only one pair $p, \varrho(p) \in L \cap L_\phi \setminus (\gamma \times \{0\})$.

$$\begin{aligned}\chi(HF_*(L, L_\phi, \{f\}; \Lambda)) &:= \sum_{i=0,1} (-1)^i \operatorname{rank}_\Lambda HF_i(L, L_\phi, \{f\}; \Lambda) \\ &= \sum_{i=0,1} (-1)^i \operatorname{rank}_\Lambda CF_i(L, L_\phi, \{f\}; \Lambda) \\ &= \begin{cases} 2 & \text{if } |p| = |\varrho(p)| = 0 \\ -2 & \text{if } |p| = |\varrho(p)| = 1 \end{cases}\end{aligned}$$

and this contradicts to $HF_*(L, L_\phi; \Lambda) \cong \{0\}$. Thus, there must be another distinct pair $q, \varrho(q)$ of intersection points in $L \cap L_\phi \setminus (\gamma \times \{0\})$.


degree 1
degree 0
p
x
ϱ(p)
q
x
ϱ(q)
γ × {0}
γ × {0}
q
y
ϱ(q)
p
y
ϱ(p)

On the generic doubling of rectangular pegs

This can also be proved by computing the algebraic intersection number: asumme there is only one pair $p, \varrho(p) \in L \cap L_\phi \setminus (\gamma \times \{0\})$, then

$$L \bullet L_\phi = \mathrm{sign}(p) + \mathrm{sign}(\varrho(p)) \pm \chi(\gamma \times \{0\}) = \pm 2$$

which contradicts to $L \bullet L_\phi = 0$.

This approach can be applied to show the generic doubling result of the cyclic quadrilateral pegs based on the existence result by Greene-Lobb (2020) (in the case other than rectangles, the Lagrangian Floer theory is not well established for the Lagrangian tori that are not monotone).

Towards C^0 rectangular peg problem

Greene-Lobb (2024): partial case of rectangular peg on some specific rectifiable Jordan curves.
In their paper, the spectral invariant from an alternative version of Floer homology (construction inspired by Heegaard Floer homology) for the Lagrangian tori is involved.

Using the Morse–Bott Floer homology here, one can directly apply Leclercq's work on Lagrangian spectral invariant for the study.

Asano-Ike (2024): rectangular peg problem is solved for all rectifiable Jordan curves.
Their proof applies microlocal sheaf theory.

Thank You!