On the exponential type conjecture

1/31. Symplectic Zoominar.

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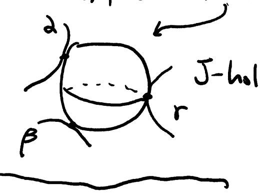
§1. The quantum connection

(X, ω) closed monotone symplectic manifold.

Genus 0 Gromov-Witten inv. $\rightsquigarrow$ quantum cohomology

$$QH^*(X;R) := (H^*(X;R), *)$$

$\langle \alpha * \beta, r \rangle := \#$



<u>Def</u>: The <u>quantum t-connection</u> on $H^*(X;R)[[t]]$ is given by

$$\nabla^{QH}_{d/dt} := \frac{d}{dt} + \frac{c_1 *}{t^2} + \frac{\mu}{t} \qquad \mu|_{H^q(X;R)} = \frac{q - \dim_{\mathbb{R}} X}{2}$$

This has a <u>quadratic singularity at $t=0$</u>, & a <u>regular singularity at $t=\infty$</u>
(mysterious!) (well-understood)

<u>Q</u>: can we say more about this?

<u>Conj</u> (Katzarkov-Kontsevich-Pantev 08, Galkin-Golyshev-Iritani 15)

Over $R=\mathbb{C}$, $\exists$ finite decomp $\nabla^{QH}_{d/dt} = \bigoplus_{\lambda} \nabla^{QH,\lambda}_{d/dt}$ s.t. for each λ. (The exponential type decomp)

$$\nabla^{QH,\lambda}_{d/dt} \sim \left(\frac{d}{dt} - \frac{\lambda}{t^2}\right) \otimes \boxed{\check{\nabla}^{QH,\lambda}_{d/dt}}$$

— regular singularity. (simple pole after gauge transf.)

<u>Thm CC.</u>) This conjecture holds for all closed monotone X.

Rmk: ① Exponential type singularities are in some sense the "simplest" irregular singularities. In particular, one can study their "fundamental solutions" via Fourier-Laplace transforms (which are regular singular!) & associated Stokes phenomenon.

... singularities ... In particular, we can study their fundamental solutions" via Fourier-Laplace transforms (which are regular singular!) & associated Stokes phenomenon.

② Pomerleano-Seidel ('23', '24') proved this conj. when X has a smooth symplectic divisor whose complement is Weinstein.

Strategy of proof: Mod p reduction. (3 steps)

§2. Mod p reduction & Katz's criterion

Step 1. $\exists$ domain R f.g. over $\mathbb{Z}$ and $\mathrm{Frac}(R) \supseteq \mathbb{Q}$, and ("global ring")

unique decomp. $\displaystyle\bigoplus_{\lambda \in \mathrm{Eig}(c_1\star)} H^*(X;R)[[t]]_\lambda$ s.t.

① it is preserved by ∇^{QH}

② $H^*(X;R)[[t]]_\lambda |_{t=0}$ = genearlized λ-eigenspace of $c_1\star \in \mathrm{End}(H^*(X;R))$.

$\rightsquigarrow$ $\nabla^{QH,\lambda}_{d/dt} := \nabla^{QH}_{d/dt} |_{H^*(X;R)[[t]]_\lambda} \sim \boxed{\frac{d}{dt} + \frac{\lambda + N_\lambda}{t^2} + G(t)}$ (N_λ nilpotent; $G(t)$ at most simple pole)

$\rightsquigarrow \tilde{\nabla}^{QH,\lambda}$ (the boxed operator without $\frac{\lambda}{t^2}$)

Want to show: $\tilde{\nabla}^{QH,\lambda}_{d/dt}$ is regular singular.

Katz's criterion: Let (∇, E) be a connection on variety X/R (global ring).

If for almost all $m \in \mathrm{mSpec}(R)$, $\nabla \otimes_R R/m$ has nilpotent p-curvature ($p = \mathrm{char}(R/m)$), then $\nabla \otimes_R \mathbb{C}$ has regular singularities.

p-curvature: (∇, E) a connection on variety X/k char p, and $D \in \mathrm{Der}_k \mathcal{O}_X$.

Then the p-curvature of ∇ along D is

$$\boxed{F^\nabla_D := (\nabla_D)^p - \nabla_{(D^p)}} \in \mathrm{End}_{\mathcal{O}_X}(E).$$

Step 2. Katz's criterion $\Rightarrow$ suffices to show that

$$F^{\tilde{\nabla}^{QH,\lambda}}_{d/dt} = F^{\nabla^{QH,\lambda}}_{d/dt} - \frac{\lambda^p}{t^{2p}}$$ is nilpotent.

§3. Quantum Steenrod operations (Fukaya, Wilkins, ...)

Over a char p field, there is a Frobenius linear algebra action

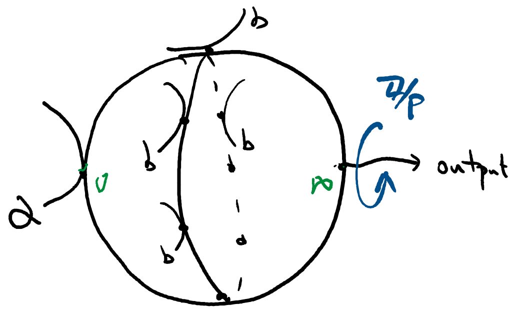
... Steenrod operations (Atiyah, Wilkins ....)

Over a char p field, there is a Frobenius linear algebra action

$$Q\Sigma : QH^*(X;k) \curvearrowright H^*(X;k)[[t]](\theta).$$
$$\quad\ \ \begin{matrix} \psi \\ b \end{matrix} \qquad\qquad\qquad \begin{matrix} \psi \\ a \end{matrix}$$

Defined via counting moduli space of J-hol

$Q\Sigma_b(a)$:



$Q\Sigma$ is called <u>quantum Steenrod operations</u> (action)

Step 3. 1) Lee: "$Q\Sigma$ should be related to the p-curvature of ∇^{QH}"

(he formulated & proved a precise version for symplectic resolutions)

Seidel in monotone case: $Q\Sigma_{c_1} = F^{QH}_{t^2 \partial_t}$.

(ultimately follows from Seidel-Wilkins: $[Q\Sigma, \nabla^{QH}]=0$).

2) Technical Lemma: $Q\Sigma$ is compatible with the decomp. $\nabla^{QH}_{\partial/\partial t} = \bigoplus_{\lambda \in eg(c_1*)} \nabla^{QH,\lambda}_{\partial/\partial t}$

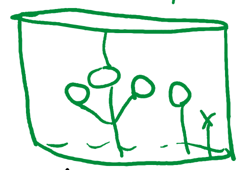
1) + 2) $\Rightarrow$ desired nilpotency result. $\square$

* §4. All of above (should) admit vast generalizations!

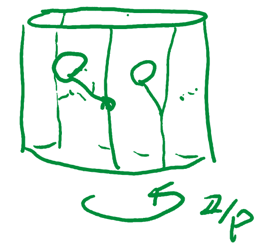
($\rightsquigarrow$ means letting $\mathcal{A} = \mathrm{Fuk}(X)$ recovers LHS from RHS, assuming $\mathrm{Fuk}(X)$ satisfies generation)

- $QH^*(X) \rightsquigarrow HH^*(\mathcal{A})$. (Seidel, FOOO, Ganatra, Ritter-Smith, ...)
- $(H^*(X;R)[[t]], \nabla^{QH}_{\partial/\partial t}) \rightsquigarrow (HP_*(\mathcal{A}), \nabla^{GGM}_{\partial/\partial t})$ (Ganatra-Perutz-Sheridan, Hugtenburg, Pomerleano-Seidel)
- $Q\Sigma: QH^*(X;\mathbb{F}_p) \curvearrowright H^*(X;\mathbb{F}_p)[[t]] \rightsquigarrow \bigcap^{\mathbb{Z}/p}: HH^*(\mathcal{A}) \curvearrowright HP_*(\mathcal{A})$. (C.)
  "$\mathbb{Z}/p$-equivariant cap product"
  Kontsevich-Soibelman
  $HH^*(\mathcal{A})^{\otimes n} \curvearrowright HH_*(\mathcal{A})$
  labeled by an operad
- Seidel-Wilkins $\rightsquigarrow$ $[\bigcap^{\mathbb{Z}/p}, \nabla^{GGM}]=0$ (C. in progress)
  $[\nabla^{QH}, Q\Sigma]=0$

- Seidel-Wilkins $\rightsquigarrow$ $[\cap^{\mathbb{Z}/p}, \nabla^{GGM}]=0$ (C. in progress). labeled by an spread
  $[\nabla^{QH}, QE]=0$



$\alpha \in H^*(X)$ on cylinder, $\cap^{\mathbb{Z}/p}$ arise from



Further Questions: i) Does the analogue of Lee's conj. hold? i.e.

$$\cap^{\mathbb{Z}/p} \sim \text{p-curvature of } \nabla^{GGM} ?$$

ii) KKP et al. also conjectured that $\nabla^{GGM}_{\partial/\partial t}$ for a general $\underline{\underline{\mathbb{Z}/2\text{-graded}}}$ uncurved A is regular singular. Can one prove this using reduction mod p?

(Not easy! Some issues: ① Hodge-to-de-Rham degeneration unknown in general (even if true, no canonical splitting); ② no analogue of "q" (Novikov variable) for general A.)

iii) When $A = D^b\mathrm{Coh}X$, what does $\cap^{\mathbb{Z}/p}$ recover? (conjectural answers & few examples in C. 24')

Thank you!

$$\nabla^{QH}_{q\partial_q} = q\frac{d}{dq} + \frac{1}{t}c_1 *.$$

1. fundamental solution at $q=0$ $\rightsquigarrow$ generating function for Grav. GW inv.

2. at $q=\infty$ ($\rightsquigarrow$ t-connection), analytic continue to 0, $\rightsquigarrow$ cohomology class. (Gamma conj., Dubrovin conj)