Errata for

Theory and Applications of Stochastic Processes

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- p.3, line 7, add before "T = absolute ..." the words R is the universal gas constant,"

- p.9 Equation (1.36) should be

$$\begin{aligned}&\lim_{\gamma\to\infty}\langle\gamma\boldsymbol{v}(t_1)\cdot\gamma\boldsymbol{v}(t_2)\rangle\\ =&\lim_{\gamma\to\infty}\frac{\gamma^2}{m^2}\int_0^{t_1}\int_0^{t_2}e^{-\gamma(t_1-s_1)}e^{-\gamma(t_2-s_2)}\langle\boldsymbol{\Xi}(s_1)\cdot\boldsymbol{\Xi}(s_2)\rangle\,ds_1\,ds_2\\ =&\lim_{\gamma\to\infty}\frac{\gamma^2 q}{m^2}\int_0^{t_1}e^{-\gamma(t_1-s_1)}e^{-\gamma(t_2-s_1)}\,ds_1\\ =&\lim_{\gamma\to\infty}\frac{\gamma q}{m^2}\left[e^{-\gamma(t_2-t_1)}-e^{-\gamma(t_2+t_1)}\right]=0\end{aligned}$$

  Equation (1.37) should be

$$\langle\gamma\boldsymbol{v}(t_1)\cdot\gamma\boldsymbol{v}(t_2)\rangle=\frac{\gamma q}{m^2}e^{-\gamma|t_2-t_1|}(1+o(1))\ \text{ for }\gamma(t_1\wedge t_2)\gg 1.$$

- p.16, line 6 below (1.70). Change "wise-band" to "wideband".

- p.17, 5 lines above (1.74), replace $I_j=\sum_i q_i\boldsymbol{v}_i\cdot u_j(\boldsymbol{x}_i)$ with $I_j=\sum_i q_i\boldsymbol{v}_i\cdot\nabla u_j(\boldsymbol{x}_i)$

- p.32, Example 2.4, line 3, change definition to $\tau_y(\omega)=\inf\{t\geq 0\,|\,\omega(t)\geq y\}$. Change displayed equation and the following text to

$$\{\omega\in\Omega\,|\,\tau_y(\omega)\leq t\}=\bigcap_{n=1}^{\infty}\bigcup_{r\in\mathbb{Q}_t}\left\{\omega\in\Omega\,|\,\omega(r)\geq y-\frac{1}{n}\right\},$$

  which is a countable intersection of countable unions of the cylinders $C(r,(y-1/n,\infty))$ for rational $r\leq t$ and $n=1,2,\ldots$ and is thus in $\mathcal{F}_t$.

- p.32, In Example 2.6 change $A \in \Omega$ to $A \subset \Omega$.
- p.34, line 3, change definition to $\tau_y(\omega) = \inf\{t \geq 0 \,|\, x(t,\omega) \geq y\}$. Change displayed equation to

$$\{\omega \in \Omega \,|\, \tau_y(\omega) \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{r \in \mathbb{Q}_t} \left\{\omega \in \Omega \,|\, x(r,\omega) \geq y - \frac{1}{n}\right\},$$

- p.40, Third equation from top, change on the left hand side big right brace to regular right brace.
- p.49 Section 2.3.2, line 2 of third paragraph: replace $T_1 \setminus T_1$ with $T_2 \setminus T_1$
- p.86 line 2, Change $|f_{xx}(x,t)| \leq A(t)e^{\alpha(t)|x|}$ to $|f_{x^i x^j}(\boldsymbol{x},t)| \leq A(t)e^{\alpha(t)|\boldsymbol{x}|}$.
- p.94 A line above (4.8). Change "domain $D \subset \mathbb{R}^n$" to "interval $D \subset \mathbb{R}$"
- p.95 lines 1 and 4, change $\omega \in D$ to $\omega \in \Omega$.
- p.97 Change superscript kj to ij in second line of (4.15).
- p.102 line 4 from bottom of page, replace $H[0,T]$ with $H_2[0,T]$.
- p.104 line 2 from bottom, change $\mathbb{M}_{n,m}$ to $\mathbb{M}_{d,m}$.
- p.105 line 8 from top, replace $H[0,T]$ with $H_2[0,T]$.
- p.121, line 2, eq. (4.118), add $d\boldsymbol{y}$ in the integral on the LHS.
- p.144, line 2 should be

$$-u = \frac{v - \eta - [-\gamma\eta + f(x - \eta\Delta t)]\Delta t}{\sqrt{\varepsilon\gamma\Delta t}},$$

  Change (5.22) to

$$p_N(\boldsymbol{y}, t+\Delta t) = \int\limits_{\mathbb{R}^d} \frac{p_N(\boldsymbol{x},t)\,d\boldsymbol{x}}{(4\pi\Delta t)^{d/2}\sqrt{\det \boldsymbol{\sigma}(\boldsymbol{x},t)}} \exp\left\{-\frac{\mathcal{B}(\boldsymbol{x},\boldsymbol{y},t)}{4\Delta t}\right\}.$$

- p.147 Replace all text from p.147, Definition 5.2.1, to p.151, up to eq.(5.49) with the following text:

Definition 5.2.1 (Unidirectional flux). *The* unidirectional probability current (flux) density *at a point x_1 is the probability of trajectories that propagate from the ray $x < x_1$ into the ray $x > x_1$ in unit time. It is given by*

$$J_{LR}(x_1, t) = \lim_{\Delta t \to 0} J_{LR}(x_1, t, \Delta t), \tag{5.34}$$

where

$$\begin{aligned} &J_{LR}(x_1, t, \Delta t) \\ &= \frac{1}{\Delta t} \int_{x_1}^{\infty} dx \int_{-\infty}^{x_1} \frac{dy}{\sqrt{4\pi \Delta t \sigma(y,t)}} \exp\left\{ -\frac{[x - y - a(y,t)\Delta t]^2}{4\sigma(y,t)\Delta t} \right\} p_N(y,t). \end{aligned} \tag{5.35}$$

Remark 5.2.2. Note that the dependence of p_N on the initial point has been suppressed in (5.35).

Theorem 5.2.3 (Unidirectional and net fluxes in one dimension) *The discrete probability flux densities at a point x_1 are given by*

$$J_{LR,RL}(x_1, t, \Delta t) = \sqrt{\frac{\sigma(x_1,t)}{\pi \Delta t}} p_N(x_1, t) \pm \frac{1}{2} J(x_1, t) + O(\sqrt{\Delta t}), \tag{5.36}$$

where the net flux is

$$\begin{aligned} J(x_1, t) &= \lim_{\Delta t \to 0} [J_{LR}(x_1, t, \Delta t) - J_{RL}(x_1, t, \Delta t)] \\ &= \left\{ -\frac{\partial\left[\sigma(x,t) p\left(x,t\right)\right]}{\partial x} + a(x,t) p\left(x,t\right) \right\}_{x = x_1}. \end{aligned} \tag{5.37}$$

Remark 5.2.3. It is clear from (5.36) that the probability flux densities in Definition 5.2.1 are infinite, but the net flux is finite.

Proof: The integral (5.35) can be calculated by the Laplace method [20] at the saddle point $x = y = x_1$. First, we change variables in (5.35) to $x = x_1 + \xi\sqrt{\Delta t}$ and $y = x_1 - \eta\sqrt{\Delta t}$ to obtain

$$J_{LR}(x_1,t,\Delta t) = \int_0^\infty d\xi \int_0^\infty \frac{d\eta}{\sqrt{4\pi\Delta t\sigma(x_1-\eta\sqrt{\Delta t},t)}}$$
$$\times \exp\left\{-\frac{\left[\xi+\eta-a(x_1-\eta\sqrt{\Delta t},t)\sqrt{\Delta t}\right]^2}{4\sigma(x_1-\eta\sqrt{\Delta t},t)}\right\} p_N\left(x_1-\eta\sqrt{\Delta t},t\right),$$

and changing the variable in the inner integral to $\eta = \zeta - \xi$, we get

$$J_{LR}(x_1,t,\Delta t) = \int_0^\infty d\xi \int_\xi^\infty \frac{d\zeta}{\sqrt{4\pi\Delta t\sigma(x_1-(\zeta-\xi)\sqrt{\Delta t},t)}} \tag{5.38}$$
$$\times \exp\left\{-\frac{\left[\zeta-a(x_1-(\zeta-\xi)\sqrt{\Delta t},t)\sqrt{\Delta t}\right]^2}{4\sigma(x_1-(\zeta-\xi)\sqrt{\Delta t},t)}\right\} p_N\left(x_1-(\zeta-\xi)\sqrt{\Delta t},t\right).$$

Next, we expand the exponent in powers of $\sqrt{\Delta t}$ to obtain

$$\frac{\left[\zeta-a(x_1-(\zeta-\xi)\sqrt{\Delta t},t)\sqrt{\Delta t}\right]^2}{4\sigma(x_1-(\zeta-\xi)\sqrt{\Delta t},t)} \tag{5.39}$$
$$= \frac{\zeta^2}{4\sigma(x_1,t)} + \left[\frac{\zeta^2(\zeta-\xi)\sigma_x(x_1,t)}{4\sigma^2(x_1,t)} - \frac{\zeta a(x_1,t)}{2\sigma(x_1,t)}\right]\sqrt{\Delta t} + O(\Delta t),$$

the pre-exponential factor,

$$\frac{1}{\sqrt{\sigma\left(x_1-(\zeta-\xi)\sqrt{\Delta t},t\right)}} = \frac{\left[1+\frac{\sigma_x(x_1,t)}{2\sigma(x_1,t)}(\zeta-\xi)\sqrt{\Delta t}+O(\Delta t)\right]}{\sqrt{\sigma(x_1,t)}},$$

and the pdf

$$p_N\left(x_1-(\zeta-\xi)\sqrt{\Delta t},t\right)$$
$$= p_N(x_1,t) - \frac{\partial p_N(x_1,t)}{\partial x}(\zeta-\xi)\sqrt{\Delta t} + O(\Delta t). \tag{5.40}$$

Using the expansions (5.39)–(5.40) in (5.38), we obtain

$$
\begin{aligned}
&J_{LR}(x_1, t, \Delta t)(5.41)\\
&= \int_0^\infty d\xi \int_\xi^\infty \frac{d\zeta}{\sqrt{4\pi\Delta t\sigma(x_1,t)}} \exp\left\{-\frac{\zeta^2}{4\sigma(x_1,t)}\right\} \left\{ p_N(x_1,t) \right.\\
&- \sqrt{\Delta t}\left[\left(\frac{\zeta^2\,(\zeta-\xi)\,\sigma_x(x_1,t)}{4\sigma^2(x_1,t)} - \frac{\zeta a(x_1,t)}{2\sigma(x_1,t)} - \frac{\sigma_x(x_1,t)\,(\zeta-\xi)}{2\sigma(x_1,t)}\right) p_N(x_1,t)\right.\\
&\left.\left. + \ (\zeta-\xi)\, p_{N,x}(x_1,t)\right] + O(\Delta t)\right\}.
\end{aligned}
$$

Similarly, $J_{RL}(x_1,t) = \lim_{\Delta t\to 0} J_{RL}(x_1,t,\Delta t)$, where

$$
\begin{aligned}
&J_{RL}(x_1, t, \Delta t) \qquad (5.43)\\
&= \frac{1}{\Delta t}\int_{-\infty}^{x_1} dx \int_{x_1}^{\infty} \frac{dy}{\sqrt{4\pi\Delta t\sigma(y,t)}} \exp\left\{-\frac{[x-y-a(y,t)\Delta t]^2}{4\sigma(y,t)\Delta t}\right\} p_N(y,t).
\end{aligned}
$$

The change of variables $x = x_1 - \xi\sqrt{\Delta t},\ y = x_1 + \eta\sqrt{\Delta t}$ in (5.43) gives

$$
\begin{aligned}
&J_{RL}(x_1, t, \Delta t)\\
&= \int_0^\infty d\xi \int_\xi^\infty \frac{d\zeta}{\sqrt{4\pi\Delta t\sigma(x_1,t)}} \exp\left\{-\frac{\zeta^2}{4\sigma(x_1,t)}\right\} \left\{ p_N(x_1,t) \right.\\
&+ \sqrt{\Delta t}\left[\left(\frac{\zeta^2\,(\zeta-\xi)\,\sigma_x(x_1,t)}{\sigma^2(x_1,t)} - \frac{\zeta a(x_1,t)}{2\sigma(x_1,t)} - \frac{\sigma_x(x_1,t)\,(\zeta-\xi)}{2\sigma(x_1,t)}\right) p_N(x_1,t)\right.\\
&\left.\left. - \ (\zeta-\xi)\, p_{N,x}(x_1,t)\right] + O(\Delta t)\right\}. \qquad (5.44)
\end{aligned}
$$

Both $J_{LR}(x_1,t)$ and $J_{RL}(x_1,t)$ are infinite, because $p_N(x_1,t) > 0$. Using the identities of Exercise 5.13 below, we find that the net flux density is, however, finite and is given by

$$
\begin{aligned}
J_{\text{net}}(x_1,t) =& \lim_{\Delta t\to 0}\left\{J_{LR}(x_1,t,\Delta t) - J_{RL}(x_1,t,\Delta t)\right\}\\
=& -2\int_0^\infty d\xi \int_\xi^\infty \frac{d\zeta}{\sqrt{4\pi\Delta t\sigma(x_1,t)}}\exp\left\{-\frac{\zeta^2}{4\sigma(x_1,t)}\right\}\\
&\times\left[\left(\frac{\zeta^2\left(\zeta-\xi\right)\sigma_x(x_1,t)}{4\sigma^2(x_1,t)} - \frac{\zeta a(x_1,t)}{2\sigma(x_1,t)} - \frac{\sigma_x(x_1,t)\left(\zeta-\xi\right)}{2\sigma(x_1,t)}\right)\right.\\
&\left.\times\, p_N(x_1,t) + \left(\zeta-\xi\right)p_{N,x}(x_1,t)\right]\\
=& \left\{-\frac{\partial\left[\sigma(x,t)p\left(x,t\right)\right]}{\partial x} + a(x,t)p\left(x,t\right)\right\}_{x=x_1}, \qquad (5.45)
\end{aligned}
$$

as asserted. □

Exercise 5.13 (Identities). Prove that the following identities are obtained by changing the order of integration,

$$
\begin{aligned}
\int_0^\infty d\xi\int_\xi^\infty \frac{\zeta^2\left(\zeta-\xi\right)d\zeta}{\sqrt{4\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} &= \int_0^\infty \frac{\zeta^4\,d\zeta}{4\sqrt{\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} = 3\sigma^2\\
\int_0^\infty d\xi\int_\xi^\infty \frac{\zeta\,d\zeta}{\sqrt{4\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} &= \int_0^\infty \frac{\zeta^2\,d\zeta}{\sqrt{4\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} = \sigma\\
\int_0^\infty d\xi\int_\xi^\infty \frac{\left(\zeta-\xi\right)d\zeta}{\sqrt{4\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} &= \frac{1}{4}\int_0^\infty \frac{\zeta^2\,d\zeta}{\sqrt{\pi\sigma}}\exp\left\{-\frac{\zeta^2}{4\sigma}\right\} = \frac{\sigma}{2}.
\end{aligned}
$$

□

Equation (5.37) is the classical expression for the probability (or heat) current in diffusion theory [82]. The FPE (4.121) can be written in terms of the flux density function $J(x,t)$ in the conservation law form

$$
\frac{\partial p\left(x,t\right)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}. \qquad (5.46)
$$

In $\mathbb{R}^d$ the probability flux density is the probability density of trajectories that propagate per unit time from a domain D across its boundary, ∂D, into the complementary part of space, D^c. It is given by $J_{\text{out}}(\partial D,t)=\lim_{\Delta t\to 0}J_{\text{out}}(\partial D,t,\Delta t)$, where

$$\begin{aligned}&J_{\text{out}}(\partial D,t,\Delta t) &(5.47)\\ &=\frac{1}{\Delta t}\int\limits_{D^c}d\boldsymbol{x}\int\limits_{D}\frac{p_N(\boldsymbol{y},t)\,d\boldsymbol{y}}{(4\pi\Delta t)^{d/2}\sqrt{\det\boldsymbol{\sigma}(\boldsymbol{y},t)}}\\ &\quad\times\ \exp\left\{-\frac{(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{a}(\boldsymbol{y},t)\Delta t)^T\boldsymbol{\sigma}^{-1}(\boldsymbol{y},t)(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{a}(\boldsymbol{y},t)\Delta t)}{4\Delta t}\right\}.\end{aligned}$$

Similarly, the probability flux density into the domain is defined as the limit of

$$\begin{aligned}&J_{\text{in}}(\partial D,t,\Delta t) &(5.48)\\ &=\frac{1}{\Delta t}\int\limits_{D}d\boldsymbol{x}\int\limits_{D^c}\frac{p_N(\boldsymbol{y},t)\,d\boldsymbol{y}}{(4\pi\Delta t)^{d/2}\sqrt{\det\boldsymbol{\sigma}(\boldsymbol{y},t)}}\\ &\quad\times\ \exp\left\{-\frac{(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{a}(\boldsymbol{y},t)\Delta t)^T\boldsymbol{\sigma}^{-1}(\boldsymbol{y},t)(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{a}(\boldsymbol{y},t)\Delta t)}{4\Delta t}\right\}.\end{aligned}$$

The net flux from the domain is defined as the limit

$$J_{\text{net}}(\partial D,t)=\lim_{\Delta t\to 0}J_{\text{net}}(\partial D,t,\Delta t),$$

where

$$J_{\text{net}}(\partial D,t,\Delta t)=J_{\text{out}}(\partial D,t)-J_{\text{in}}(\partial D,t,\Delta t).$$

Theorem 5.2.4 (Unidirectional and net fluxes in $\mathbb{R}^d$). *The discrete probability flux density densities at a boundary point $\boldsymbol{x}_B$ are given by*

$$\begin{aligned}\boldsymbol{J}_{out,in}(\boldsymbol{x}_B,t)\cdot\boldsymbol{n}(\boldsymbol{x}_B)=&\sqrt{\frac{\sigma_n(\boldsymbol{x}_B,t)}{\pi\Delta t}}p\left(\boldsymbol{x}_B,t\right)\pm\frac{1}{2}\boldsymbol{J}_{net}(\boldsymbol{x}_B,t)\cdot\boldsymbol{n}(\boldsymbol{x}_B)\\ &+O(\sqrt{\Delta t}), \qquad (5.49)\end{aligned}$$

- p.149 Line 6 from bottom of page. Change (5.129) to (4.121)

- p.153 Line 5 from top of page and lines 6 and 13 of proof. Change (5.129), (5.130) to (4.121), (4.122).
- p.154 Line 1. Change "As in Chapter 5" to "As in (5.11)"
- p.155, change right parenthesis to a right brace in the middle term of eq. (5.69)
- p.156 Replace (5.68) with

$$\begin{aligned} &p_{\Delta t}(\boldsymbol{y}, t+\Delta t\, \boldsymbol{x}_0, s) \\ &= \int_D \frac{p_{\Delta t}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, s)\, d\boldsymbol{x}}{(4\pi\Delta t)^{d/2}\sqrt{\det \boldsymbol{\sigma}(\boldsymbol{x}, t)}} \exp\left\{-\frac{\mathcal{B}(\boldsymbol{x}, \boldsymbol{y}, t)}{4\Delta t}\right\}, \end{aligned} \tag{5.68}$$

- p.172, Line 4 of Girsanov's theorem, replace $a(x(t,\omega))/b^2(x(t,\omega))$ with $a(x(t,\omega),t)/b^2(x(t,\omega),t)$
- p.186, change eq.(6.34) to

$$\bar{\tau}(x) = \frac{\displaystyle\int_{-1}^{x} \exp\left\{\frac{U(y)}{\varepsilon}\right\} dy \int_{x}^{1}\int_{-1}^{1} \exp\left\{\frac{U(y)-U(z)}{\varepsilon}\right\} dz\, dy}{\displaystyle\varepsilon \int_{-1}^{1} \exp\left\{\frac{U(y)}{\varepsilon}\right\} dy} - \frac{\displaystyle\int_{x}^{1} \exp\left\{\frac{U(y)}{\varepsilon}\right\} dy \int_{-1}^{x}\int_{-1}^{1} \exp\left\{\frac{U(y)-U(z)}{\varepsilon}\right\} dz\, dy}{\displaystyle\varepsilon \int_{-1}^{1} \exp\left\{\frac{U(y)}{\varepsilon}\right\} dy}$$

- p.186, Line 2 below (6.32), remove coma from Andronov,
- p.209, End of equation in Example 7.2 should be $= \dfrac{2}{\pi\varepsilon} \neq 0$,
- p.213, line 1, remove period after "time step"

- p.213, the last line, change E to $\mathbb{E}$
- p.215, line 3 below eq.(7.27), change Takacs to Tàkacs
- p.221, line 9 below Definition 7.1.5, insert “in” before “numerous”
- p.221, the first line of Theorem 7.1.4, replace “time after” with “after time”
- p.222, the first line below Definition 7.2.1, replace “because” by “since”
- p.223, line 1. Change to ”If $F_\tau(t) = 1 - e^{-\lambda t}$ ...” Line 8 from bottom, equation should be

$$\hat{f}_\tau(s) = \frac{\lambda}{\lambda + s} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n s^n}{\lambda^n}, \quad 0 \leq s < \lambda.$$

- p.233, next to the last line, change “Eqation” to “Equation”
- p.237, in eq. (7.90), replace “for $x > 0$” with “for $t > 0$”
- p.237, one line below eq.(7.91), delete > 0
- p.246, second line of Exercise 7.12, replace βx with $\beta \dot{x}$
- p.247, add left parenthesis before β in eq. (7.114)
- p.248, one line below (eq. (7.120), add a right parenthesis to equation number (7.120
- p.249, two lines above eq.(7.123), replace “to” after $n_\varepsilon(x)$ by “and”
- p.250, change comma to a full stop et the end of eq. (7.133)
- p.257, line 3 of Section 8.1, replace “equation” by “equations”
- p.260, one line below eq. (8.7), delete “white noise”
- p.268, at end of eq.(8.37), change $v > 0$ to $v < 0$
- p.288, caption of fig. 8.5, change $\gamma \to \infty$ to $\gamma \to 0$
- p.289, third line in the second paragraph after Exercise 8.19(iv), change (8.75) to “in Figure 8.5.”

- p.300, line 5 from the bottom of the page, insert ε between $\lim\limits_{\varepsilon\to 0}$ and log
- p.304 Two lines above (9.10), put a full stop before "We...". Line 4 from bottom. In the proof of Lemma 9.1.1 remove emphasis from (9.10).
- p.305 Change the lower limit 0 of integration in equation (9.15) to $\tilde{x}$. Replace the text from line 5 to the last line before Exercise 9.1 with the following text: Given the solution $\psi(y)$ of the eikonal equation (9.10), the solution of the transport equation (9.11) is

$$K_0(y) = \frac{\tilde{K}_0}{\sqrt{\int_{\mathbb{R}} \xi w(\xi \,|\, y) e^{\xi\psi'(y)}\, d\xi}} \times \exp\left\{ -\frac{1}{2} \int\limits_{\tilde{x}}^{y} \frac{\int\limits_{\mathbb{R}} \xi \dfrac{\partial w(\xi \,|\, \eta)}{\partial \eta} e^{\xi\psi'(\eta)}\, d\xi}{\int\limits_{\mathbb{R}} \xi w(\xi \,|\, \eta) e^{\xi\psi'(\eta)}\, d\xi}\, d\eta \right\},$$

- p.308 Replace eq.(9.30) by

$$\int\limits_{\mathbb{R}} [n_D(x+\varepsilon\xi) w(\xi \,|\, x)\, d\xi - n_D(x) = -1$$

- p.309 Replace eq.(9.34) by

$$C_1(\varepsilon) \sim \frac{\tilde{K}_0}{K_0(B)} \frac{e^{\psi(B)/\varepsilon}}{\int\limits_{-\infty}^{0} e^{\eta\psi'(B)} \int\limits_{-\infty}^{\eta} w(\xi \,|\, B) U_0(\eta-\xi)\, d\xi\, d\eta}.$$

  Replace eq.(9.39) by

$$C_2(\varepsilon) \sim \sqrt{\frac{\pi}{\varepsilon}} \frac{\tilde{K}_0}{K_0(B)} e^{\psi(B)/\varepsilon} \sqrt{\left|\frac{m_1'(B)}{m_2(B)}\right|} \frac{1}{m_2(B)}.$$

  The function $\psi(y)$ is the solution of the eikonal equation (9.10), $K_0(y)$ is given in (9.15), and $\tilde{K}_0$ is given in (9.16).

- p.312 Replace eq.(9.59) by

$$C(\varepsilon) \sim \frac{\tilde{K}_0}{K_0(B)} \frac{e^{\psi(B)/\varepsilon}}{\int\limits_{-\infty}^{0} e^{\eta\psi'(B)} \int\limits_{-\infty}^{\eta} w(\xi \,|\, B) U_0(\eta - \xi)\, d\xi\, d\eta},$$

- p.313 Replace eq.(9.69) by

$$C_2(\varepsilon) \sim \sqrt{\frac{\pi}{\varepsilon}} \frac{\tilde{K}_0}{K_0(B)} e^{\psi(B)/\varepsilon} \sqrt{\left|\frac{m_1'(B)}{m_2(B)}\right|} \frac{1}{m_2(B)}.$$

- p.324, Exercise 9.8, remove "the"
- p.345, two lines above (10.19), insert "using" before "Taylor's"
- p.348, two lines above Exercise 10.7, replace frac12 with $\frac{1}{2}$.
- p.353, The first line of Section 10.1.4, change $U(\alpha) < U(\beta) \max_{[\alpha,\beta]} U(y)$ to $U(\alpha) < U(\beta) = \max_{[\alpha,\beta]} U(y)$ (insert =)
- p.359, title of subsection 10.1.6, change "The MFPT" to "MFPT and"
- p.361, line 3 below eq. (10.57), change "eq. (10.57)" to "eq. (10.52)"
- p.362, One line above Section 10.2.2, replace $q_\varepsilon(\boldsymbol{y} \,|\, \boldsymbol{x})$ with $K_0(\boldsymbol{y} \,|\, \boldsymbol{x})$
- p.365, second line of eq.(10.77), the first term should be $\dfrac{\partial a^i(\boldsymbol{y})}{\partial y^i}$.
- p.367, l.9, replace (10.59) with (10.57).
- p.369, line below (10.93), change $\alpha, \beta > 0$ to $\alpha > 0, \beta \geq 0$.
- p.369, replace eq. (10.95) by

$$\boldsymbol{a}(\rho, s) = \left[a^0(s)\rho\nabla\rho + B(s)\nabla s\right][1 + o(1)]. \qquad (10.95)$$

- p.373, equation below (10.117), insert = between $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau$ and $\lim_{\varepsilon\to 0} \varepsilon \left[\sup_{\boldsymbol{y}\in\partial D} \log \boldsymbol{J}(\boldsymbol{y} \,|\, \boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{y}) - \sup_{\boldsymbol{y}\in D} \log p_\varepsilon(\boldsymbol{y} \,|\, \boldsymbol{x})\right]$.

- p.376, equation (10.125) should be

$$B(s)\frac{d}{ds}K_0(0,s) = -\left[a^0(s) + B'(s) + \sigma(s)d\phi(s)\right]K_0(0,s)$$

- p.377, lines 3,4 from bottom of page, replace $\gamma(s_i)$ and $\gamma'(s_i)$ by $\xi(s_i)$ and $\xi'(s_i)$, respectively.

- p.378, line 3 of Section 10.2.9, replace $\boldsymbol{y}_k \in D$ with $\boldsymbol{y}_k \in \partial D$

- p.378, line 2 above (10.130). Replace (10.104) with (10.102).

- p.379, line 3 below (10.136), the broken formula should be $U(\boldsymbol{x}_S) = \min_{\boldsymbol{x}\in\partial D} U(\boldsymbol{x})$

- p.380, line 2 from bottom of page, replace "andthere" with "and there"

- p.384, in eq. (10.142), replace 1 by $\text{sgn}(x - x_C)$

- p.385, one line below (10.147), add at the beginning of the line "and changing (y,η) to $-(y,\eta)$"

- p.386, Replace the first 7 lines of the proof of Theorem 10.3.2 by:
  The mean first passage time from $(x,y) \in D$ to Γ is the solution of the Andronov–Vitt–Pontryagin boundary value problem

$$\begin{aligned}\varepsilon\gamma\bar{\tau}_{yy} + y\bar{\tau}_x - [\gamma y + U'(x)]\bar{\tau}_y =& -1 \text{ for } (x,y) \in D \\ \bar{\tau} =& \,0 \text{ for } (x,y) \in \Gamma,\end{aligned} \tag{1}$$

where D is the domain enclosed by Γ. Scaling $\bar{\tau}(x,y) = C_\varepsilon q(x,y)$, where $\max_D q(x,y) = 1$, we see from the previous section that $C_\varepsilon \to \infty$ as $\varepsilon \to 0$. It follows that for fixed $(x,y) \in D$ and $\varepsilon \to 0$

$$\varepsilon\gamma q_{yy} + yq_x - [\gamma y + U'(x)]q_y \sim 0, \quad q \to 1 \tag{2}$$

with the boundary condition $q = 0$ for $(x,y) \in \Gamma$. The characteristics of the reduced eq.(2) converge to the stable equilibrium point $(x_A, 0)$, so there is no internal layer there. Therefore q is asymptotically 1 in D and has a boundary layer near Γ and so is $\bar{\tau}$. The value of the constant

C_ε is recovered by multiplying eq.(1) by $e^{-E/\varepsilon}$ and integrating over D. The resulting Lagrange identity gives

$$\bar{\tau} \sim C_\varepsilon = \frac{\displaystyle\int_D e^{-E/\varepsilon}\,dx\,dy}{-\gamma\varepsilon \displaystyle\oint_\Gamma e^{-E/\varepsilon} q_y(x,y)\nu_2(x,y)\,dx} \tag{3}$$

for fixed $(\boldsymbol{x}, y) \in D$ and $\varepsilon \to 0$.

- p.387, equation below (10.158), replace in the denominator $\rho^2(x_C, 0)$ by $\rho_y^2(x_C, 0)$

- p.387, replace equations (10.159), (10.160) with

$$\begin{aligned} b_0(x) = {} & \nu_1(x', y_\Gamma(x'))\nu_2(x', y_\Gamma(x')) - \gamma\nu_2(x', y_\Gamma(x')) \\ & - U''(x')\nu_1(x', y_\Gamma(x'))\nu_2(x', y_\Gamma(x')) \to \frac{\omega_C^2}{\lambda} \text{ as } x' \to x_C \end{aligned}$$

$$\rho_y^2(x, y_\Gamma(x)) = \frac{y_\Gamma^2(x')}{y_\Gamma^2(x') + [\gamma y_\Gamma(x') + U'(x')]^2} \to \frac{1}{1+\lambda^2}.$$

- p.389, Theorem 10.3.3, change $\omega_C^2 + \lambda^2$ to $\gamma\lambda$ in formulas

- p.392, one line above Corollary 10.3.2, insert“the” after “Scaling”.

- p.394, line 1 of *Proof*: hyphenate $\sqrt{\varepsilon}$-neighborhood

- p.394, line 3 of *Proof*: change powers to power

- p.395, one line below (10.198), change (8.92) to (10.197)

- p.397, change in eq. (10.205) $+$ into $-$.

- p.397, change $\omega_C^2 + \lambda^2$ to $\gamma\lambda$ in formulas below eq. (10.205)

- p.397, replace the line below (10.205) with “so, using the identity $\lambda^2 - \omega_C^2 = \gamma\lambda$, we find that the most likely exit point is”

- p.405, one line below eq. (11.27), replace “(11.26)” by Lp

- p.446, Reference [73], delete the word ”values” and the first hyphen.

See errata at http://www.math.tau.ac.il/∼schuss/