On smooth critical subsolutions of the Hamilton-Jacobi Equation

The Hamilton-Jacobi Equation is a PDE that was set up by Hamilton and Jacobi to find trajectories of Classical Mechanical Systems. Since its discovery it has had many applications in Dynamical Systems, PDE, Calculus of variations, Control Theory, and it is has had far reaching generalizations in the field it generated Symplectic Geometry .

This Hamilton-Jacobi Equation is the PDE

$$H(x, d_x u) = c,$$

where $H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, (x, p) \mapsto H(x, p)$ is a given function called the Hamiltonian, c is a constant, and $u : \mathbf{R}^n \to \mathbf{R}$ is the sought solution, which enter the equation only through $d_x u$ its the differential or gradient at the point $x \in \mathbf{R}^n$.

Although the results are true for general manifolds, we will only deal with the case of a Hamiltonian $H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, (x, p) \mapsto H(x, p)$. Since we will need compactness in the space variable x, we will assume that H is $\mathbf{Z}^n$ periodic in the space variable. We are interested in Hamiltonians that come from classical calculus of variations (Tonelli's theory), therefore we will assume some regularity of H, and convexity and superlinearity of H in p (all these conditions will be explained in detail during the lecture). We will also require that the solutions $u : \mathbf{R}^n \to \mathbf{R}$ of the Hamilton-Jacobi Equation

$$H(x, d_x u) = c,$$

to be periodic in $\mathbf{Z}^n$ in the $x \in \mathbf{R}^n$ variable.

A simple argument (explained during the lecture) shows that there is at most one constant c for which this equation can have a C^1 solution. This value $c[0]$ is called the Mañé critical value. Although, generally the Hamilton-Jacobi does not have a C^1 solution, it always has a viscosity solution (in the sense of Crandall and Lions).

A critical subsolution is a function $u : \mathbf{R}^n \to \mathbf{R}$, still assumed $\mathbf{Z}^n$-periodic, such that

$$H(x, d_x u) \leq c[0].$$

One can show that there is a Lipschitz function u which satisfies the inequality above almost everywhere. A few years ago wuth Antonio Siconolfi we showed that there are C^1 solutions. This work was explained in a colloquium at TAU in December 2005. Since then Patrick Bernard showed that there always exists such a C^1 solution with a Lipschitz derivative. An example shows that one cannot do better. We will explain the very clever proof of Patrick Bernard which uses directly the Lax-Oleinik semi-group.

Time permitting, we will also discuss work that was done to find conditions under which there are smoother subsolutions.

During the lectures we will introduce whatever is necessary to the understanding of the subject. No previous knowledge of the subject is assumed.

Hamilton-Jacobi and Denjoy-Schwartz:
why dynamics matter in the regularity of smooth subsolutions

In this follow-up lecture of the colloquium, we will explain that, in dimension n=2, the classical work of Denjoy on C^2 diffeomorphisms of the circle (or rather its generalization by Schwartz to flows on surfaces) puts strong restrictions on the existence of smoother critical subsolutions of the Hamilton-Jacobi equation. For this we will have to introduce Mather's α function (called the homogenized Hamiltonian in PDE) and Schwartzman asymptotic cycles. We will also develop the results obtained by Patrick Bernard (and even some generalization) for existence of smoother critical subsolutions.