

**The Algebraic Revolution in
Combinatorial and Computational Geometry:
State of the Art**

Micha Sharir

Tel Aviv University

Historical Review: To get us to the present

Combinatorial Geometry owes its roots to
(many, but especially to) **Paul Erdős** (1913–1996)



[Erdős, 80th birthday]:

My most striking contribution to geometry is, no doubt, my problem on the number of **distinct distances**. This can be found in many of my papers on combinatorial and geometric problems.

One of the two problems posed in [Erdős, 1946]

Both have kept many good people sleepless for many years

Distinct distances: Estimate the smallest possible number $D(n)$ of distinct distances determined by any set of n points in the plane

Repeated distances: Estimate the maximum possible number of pairs, among n points in the plane, at distance exactly 1

Erdős's distinct distances problem:

Estimate the smallest possible number $D(n)$ of distinct distances determined by any set of n points in the plane

[Erdős, 1946] conjectured:

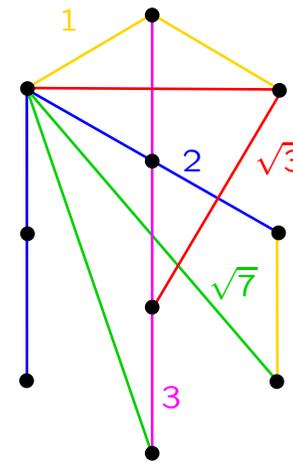
$$D(n) = \Omega(n/\sqrt{\log n})$$

(Cannot be improved:
Tight for the integer lattice)

A hard nut; Slow steady progress

Best bound before the
“algebraic revolution”:

$$\Omega(n^{0.8641}) \quad [\text{Katz-Tardos 04}]$$



$$5 \ll \binom{10}{2} = 45$$

**The founding father of the revolution:
György Elekes (passed away in September 2008)**



Elekes's insights

Circa 2000, Elekes was studying

Erdős's distinct distances problem

Found an ingenious transformation of this problem to an Incidence problem between points and curves (lines) in 3D

For the transformation to work, Elekes needed

A couple of deep conjectures on the new setup

(If proven, they yield the almost tight lower bound $\Omega(n/\log n)$)

Nobody managed to prove his conjectures; he passed away in 2008, three months before the revolution began

The first breakthrough

[Larry Guth and Nets Hawk Katz, 08]:

Algebraic Methods in Discrete Analogs of the Kakeya Problem

Showed: The number of **joints** in a set of n lines in 3D is $O(n^{3/2})$

A **joint** in a set L of n lines in \mathbb{R}^3 :

Point incident to (at least) three **non-coplanar lines** of L

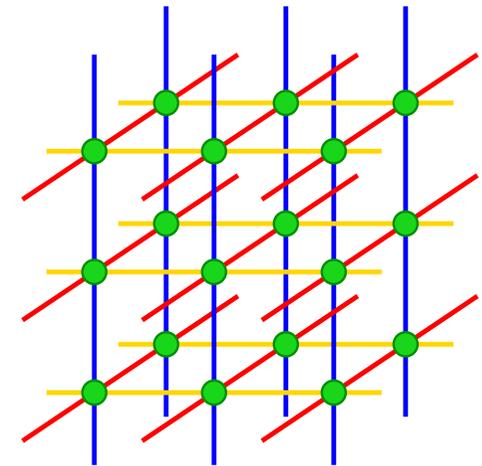
Proof uses simple **algebraic tools**:

Low-degree polynomials vanishing

On many points in \mathbb{R}^d

And some elementary tricks in

Algebraic Geometry



The joints problem

The bound $O(n^{3/2})$ conjectured in [Chazelle et al., 1992]

Worst-case tight: $\sqrt{n} \times \sqrt{n} \times \sqrt{n}$ lattice; $3n$ lines and $n^{3/2}$ joints

In retrospect, a “trivial” problem

In general, in d dimensions

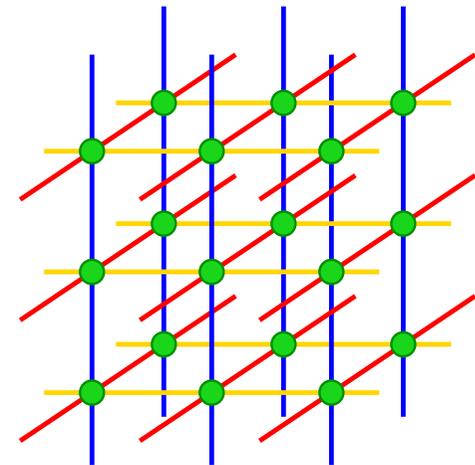
Joint = point incident to at least d lines, not all on a hyperplane

Max number of joints is $\Theta(n^{d/(d-1)})$

[Kaplan, S., Shustin, 10],

[Quilodrán, 10]

(Similar, and very simple proofs)



From joints to distinct distances

The new algebraic potential (and Elekes's passing away)

Triggered me to air out Elekes's ideas in 2010

Guth and Katz picked them up,

Used more advanced algebraic methods

And obtained their second (main) breakthrough:

- [Guth, Katz, 10]: The number of distinct distances in a set of n points in the plane is $\Omega(n/\log n)$

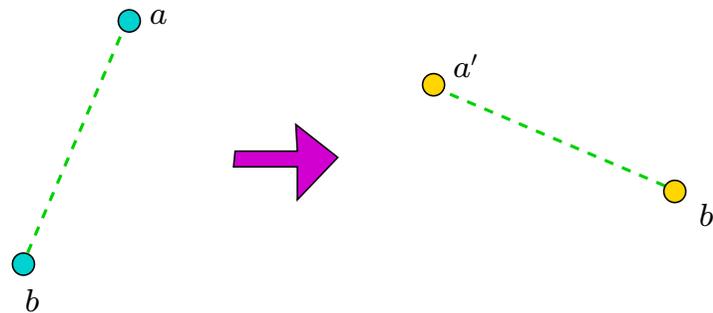
Settled Elekes's conjectures (in a more general setup)

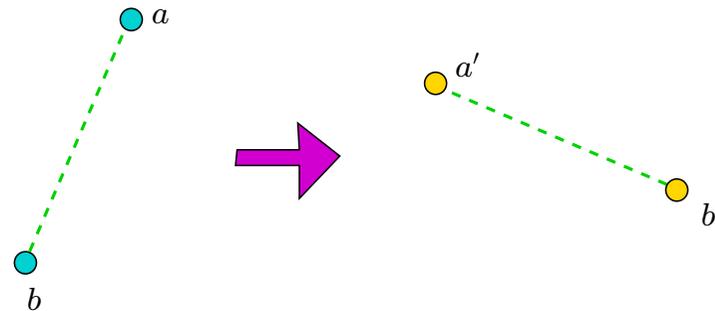
And solved (almost) completely the distinct distances problem

End of prehistory; the dawn of a new era

Erdős's distinct distances problem Elekes's transformation: Some hints

- Consider the 3D parametric space of rigid motions (“rotations”) of \mathbb{R}^2
- There is a rotation mapping a to a' and b to b'
 $\Leftrightarrow \text{dist}(a, b) = \text{dist}(a', b')$





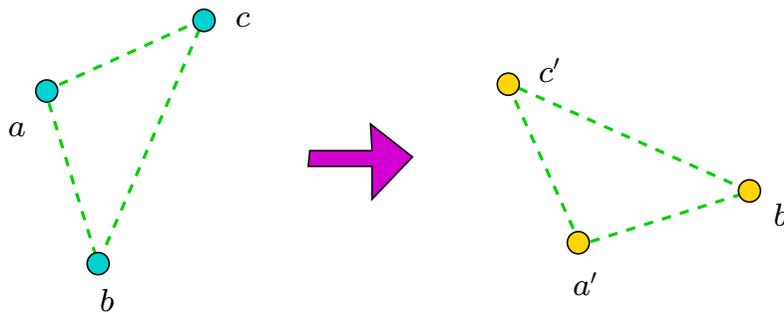
- Elekes assigns each pair $a, a' \in S$ to the locus $h_{a,a'}$ of all rotations that map a to a' (with suitable parameterization, $h_{a,a'}$ is a line in 3D)
- So if $dist(a, b) = dist(a', b')$ then $h_{a,a'}$ and $h_{b,b'}$ meet at a common point (rotation)

- After some simple (but ingenious) algebra, **Elekes's main conjecture** was:

Number of rotations that map $\geq k$ points of S
to $\geq k$ other points of S

= Number of points (in 3D) incident to $\geq k$ lines $h_{a,a'}$

= $O\left((\text{Num of lines})^{3/2}/k^2\right) = O\left(n^3/k^2\right)$



Summary

- Both problems (joints and distinct distances) reduce to Incidence problems of points and lines in three dimensions
- Both problems solved by Guth and Katz using new algebraic machinery
- Both are hard problems, resisting decades of “conventional” geometric and combinatorial attacks
- New algebraic machinery picked up, extended and adapted
Yielding solutions to many old and new difficult problems:
Some highlighted in this survey

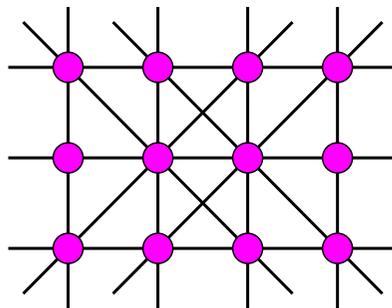
A few words about incidences

Incidences between points and lines in the plane

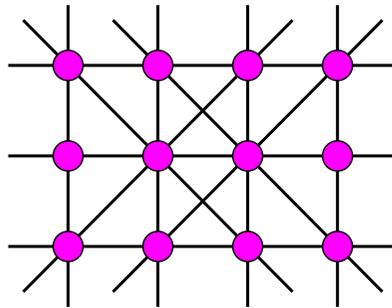
P : Set of m distinct points in the plane

L : Set of n distinct lines

$I(P, L) =$ Number of incidences between P and L
 $= |\{(p, \ell) \in P \times L \mid p \in \ell\}|$



Incidences between points and lines in the plane



$$I(m, n) = \max \{I(P, L) \mid |P| = m, |L| = n\}$$

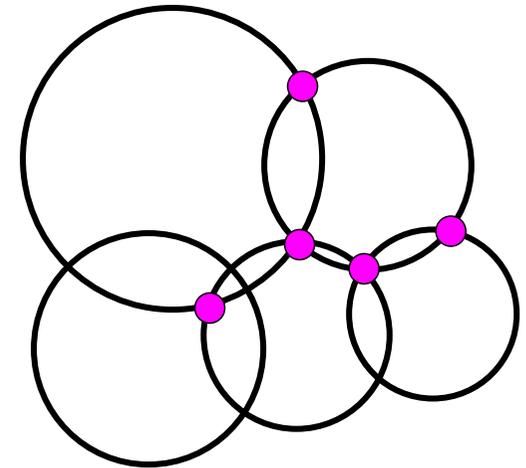
$$I(m, n) = \Theta(m^{2/3}n^{2/3} + m + n) \quad [\text{Szemerédi–Trotter 83}]$$

Why incidences?

- Because it's there—another Erdős-like cornerstone in geometry
- Simple question; Unexpected bounds; Nontrivial analysis
- Arising in / related to many topics:
Repeated and distinct distances and other configurations
Range searching in computational geometry
The [Kakeya problem](#) in harmonic analysis
- Triggered development of sophisticated tools
([space decomposition](#)) with many other applications

Many extensions

- Incidences between **points and curves** in the plane
- Incidences with **lines, curves, flats, surfaces**, in **higher dimensions**
- In most cases, no known sharp bounds
Point-line incidences is the exception...



Incidences in the new era

The present high profile of incidence geometry:

Due to Guth and Katz's works: Both study

Incidences between points and lines in three dimensions

Interesting because they both are "truly 3-dimensional":

Controlling coplanar lines

(If all points and lines lie in a common plane,

Cannot beat the planar Szemerédi-Trotter bound)

Old-new Machinery from Algebraic Geometry and Co.

- Low-degree polynomial vanishing on a given set of points
- Polynomial ham sandwich cuts
- Polynomial partitioning
- Miscellany (Thom-Milnor, Bézout, Harnack, Warren, and co.)
- Miscellany of newer results on the algebra of polynomials
- And just plain good old stuff from the time when Algebraic geometry was algebraic geometry (Monge, Cayley–Salmon, Severi; 19th century)

Point-line Incidences in \mathbb{R}^3

Elekes's conjecture: Follows from the point-line incidence bound:

Theorem: ([Guth-Katz 10])

For a set P of m points

And a set L of n lines in \mathbb{R}^3 , such that no plane contains more than $O(n^{1/2})$ lines of L (“truly 3-dimensional”)

(Holds in the Elekes setup)

$$\max I(P, L) = \Theta(m^{1/2}n^{3/4} + m + n)$$

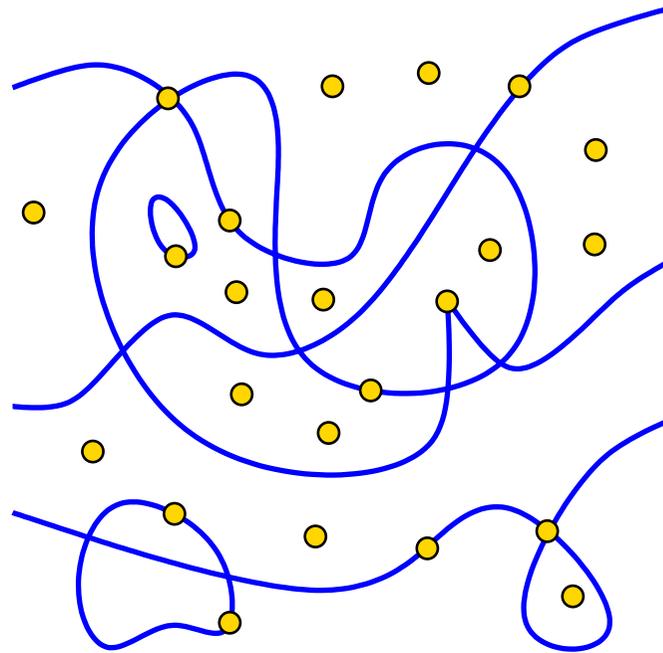
Proof uses **polynomial partitions**

Polynomial partitioning of a point set

[Guth-Katz 10]: A set S of n points in \mathbb{R}^d can be partitioned into $O(t)$ subsets, each consisting of at most n/t points, By a polynomial p of degree $D = O(t^{1/d})$, Each subset in a distinct connected component of $\mathbb{R}^d \setminus Z(f)$

Proof based on the polynomial Ham Sandwich theorem of [Stone, Tukey, 1942]

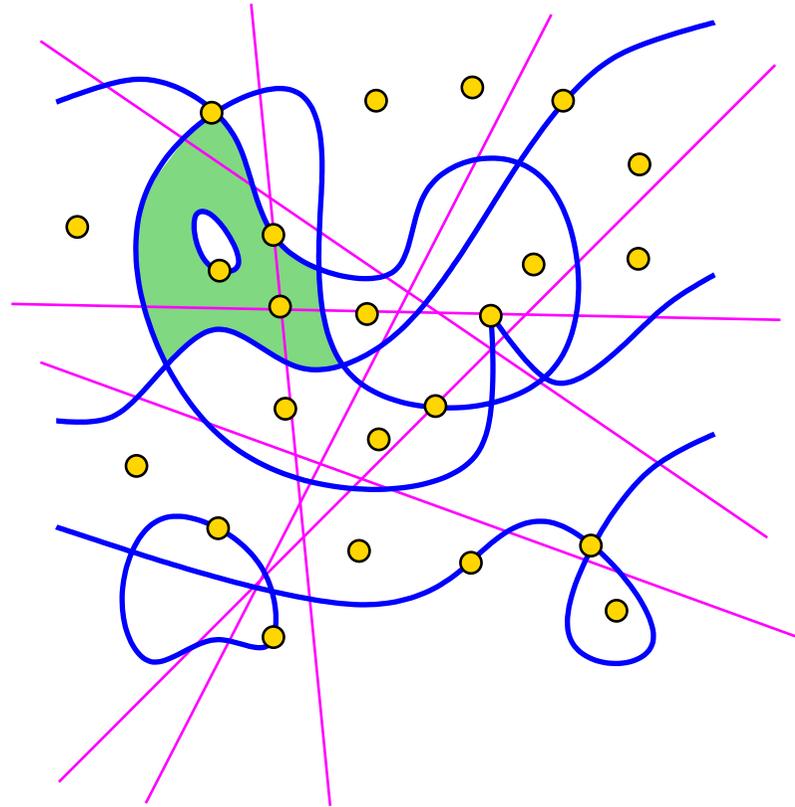
Polynomial partitioning



Polynomial partitioning: Restatement and extension

[Guth-Katz 10]: For a set S of n points in \mathbb{R}^d , and degree D
Can construct a polynomial p of degree D
Such that each of the $O(D^d)$ connected components of $\mathbb{R}^d \setminus Z(f)$
contains at most $O(n/D^d)$ points of S

[Guth 15]: For a set S of n k -dimensional constant-degree
algebraic varieties in \mathbb{R}^d , and degree D
Can construct a polynomial p of degree D
Such that each of the $O(D^d)$ connected components of $\mathbb{R}^d \setminus Z(f)$
is intersected by at most $O(n/D^{d-k})$ varieties of S



Polynomial partitioning

- A new kind of space decomposition
Excellent for [Divide-and-Conquer](#)
- Competes (very favorably) with [cuttings](#), [simplicial partitioning](#)
(Conventional decomposition techniques from the 1990's)
- Many advantages (and some challenges)
- A major new tool to take home

Incidences via polynomial partitioning

In five easy steps (for Guth-Katz's m points / n lines in \mathbb{R}^3):

- Partition \mathbb{R}^3 by a polynomial f of degree D :
 $O(D^3)$ cells, $O(m/D^3)$ points and $O(n/D^2)$ lines in each cell
- Use a trivial bound in each cell:
 $O(\text{Points}^2 + \text{Lines}) = O((m/D^3)^2 + n/D^2)$
- Sum up: $O(D^3) \cdot (m^2/D^6 + n/D^2) = O(m^2/D^3 + nD)$
- Choose the right value: $D = m^{1/2}/n^{1/4}$, substitute
- Et voilà: $O(m^{1/2}n^{3/4})$ incidences

But...

For here lies the point: [Hamlet]

What about the points that lie on the surface $Z(f)$?

Method has no control over their number

Here is where all the fun (and hard work) is:

Incidences between points and lines on a 2D variety in \mathbb{R}^3

Need advanced algebraic geometry tools:

Can a surface of degree D contain many lines?!

Ruled surfaces

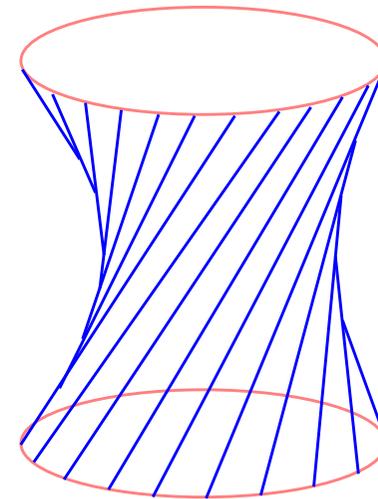
Can a surface of degree D contain many lines?!

Yes, but only if it is **ruled by lines**

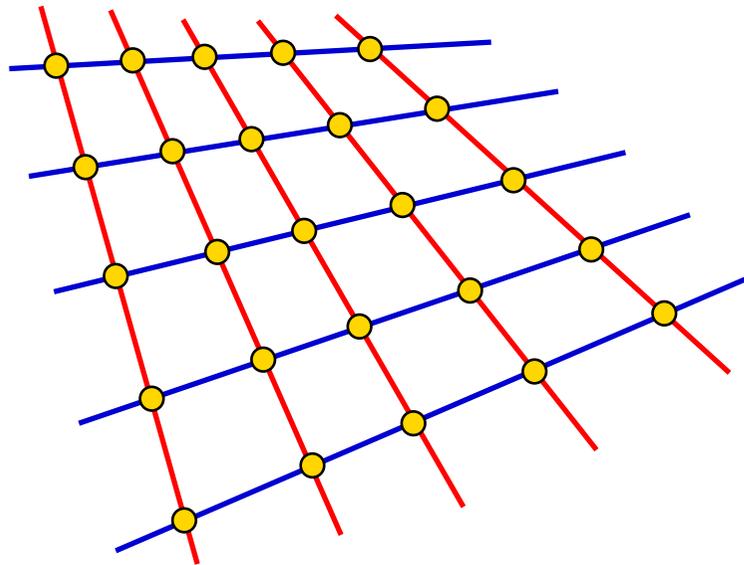
Hyperboloid of one sheet

(Doubly ruled)

$$z^2 = x^2 + y^2 - 1$$



Ruled and non-ruled surfaces



Hyperbolic paraboloid
(Doubly ruled)

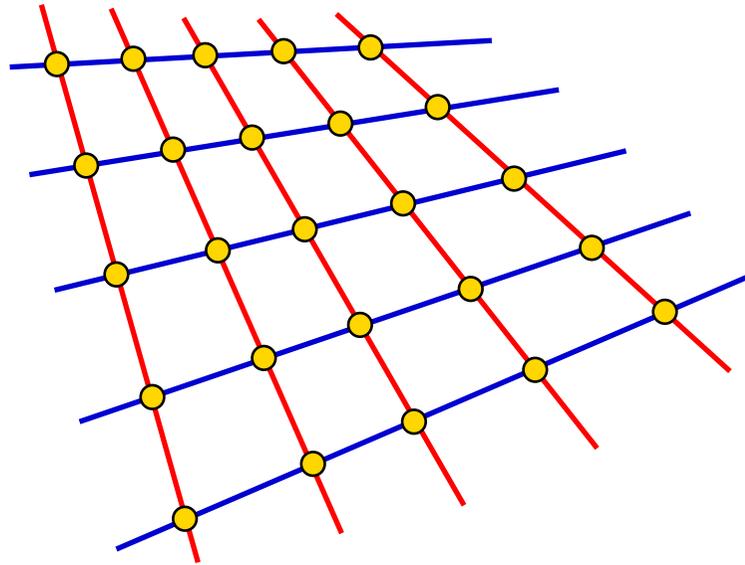
$$z = xy$$

A non-ruled surface of degree D can contain at most

$D(11D - 24)$ lines

[Monge, Cayley–Salmon, 19th century]

Ruled and non-ruled surfaces



If $Z(f)$ not ruled: Contains only “few” lines; “easy” to handle

If $Z(f)$ is (singly) ruled: “Generator” lines meet one another only at singular points; again “easy” to handle

(The only doubly ruled surfaces are these two quadrics)

Point-line incidences in \mathbb{R}^3

Finally, if $Z(f)$ contains planes

Apply the Szemerédi-Trotter bound in each plane

(No 3D tricks left...)

But we assume that no plane contains more than $n^{1/2}$ lines:

The incidence count on these planes is not too large

And we are done: $O(m^{1/2}n^{3/4} + m + n)$ incidences

A new algebraic era in combinatorial geometry:

Polynomial partitioning and other algebraic geometry tools
Gave the whole area a huge push
Many new results, new deep techniques, and a lot of excitement

Opening up the door to questions about
Incidences between **lines**, or **curves**, or **surfaces**,
in **three** or **higher dimensions**

And many other “non-incidence” problems
Some using incidences in the background, some don’t

Unapproachable, “not-in-our-lifetime” problems before the
revolution

Now falling down, one after the other, rather “easily”

The new algebraic era

Ending of fairy tales in French:

Et ils véchurent heureux et eurent beaucoup d'enfants

(And they lived happily and had lots of children)

Hopefully, this is not the end of the fairy tale yet...

But they already have lots of children

Many children

- New proofs of old results (simpler, different)
[Kaplan, Matoušek, S., 11]
- Unit distances in three dimensions
[Zahl 13], [Kaplan, Matoušek, Safernová, S. 12]
- Point-circle incidences in three dimensions
[S., Sheffer, Zahl 13], [S., Solomon 17]
- Complex Szemerédi-Trotter incidence bound and related bounds
[Solymosi, Tao 12], [Zahl 15], [Sheffer, Szabó, Zahl 15]
- Range searching with semi-algebraic ranges
An algorithmic application; [Agarwal, Matoušek, S., 13]

And more children

- Incidences between points and lines in four dimensions
[S., Solomon, 16]
- Incidences between points and curves in higher dimensions
[S., Sheffer, Solomon, 15], [S., Solomon, 17]
- Incidences in general and semi-algebraic extensions
[Fox, Pach, Sheffer, Suk, Zahl, 14]
- Algebraic curves, rich points, and doubly-ruled surfaces
[Guth, Zahl, 15]

And more

- Distinct distances between two lines
[S., Sheffer, Solymosi, 13]
- Distinct distances: Other special configurations
[S., Solymosi, 16], [Pach, de Zeeuw, 17],
[Charalambides, 14], [Raz, 17], [S., Solomon, 17]
- Arithmetic combinatorics:
Sums vs. products and related problems
[Iosevich, Roche-Newton, Rudnev, Shkredov]

And more

- Polynomials vanishing on grids:
The Elekes–Rónyai–Szabó problems revisited
[Raz, S., Solymosi, 16], [Raz, S., de Zeeuw, 16,17]
- Triple intersections of three families of unit circles
[Raz, S., Solymosi, 15]
- Unit-area triangles in the plane
[Raz, S., 15]
- Lines in space and rigidity of planar structures
[Raz, 16]

And more

- Almost tight bounds for eliminating depth cycles for lines in three dimensions
[Aronov, S., 16]
- Eliminating depth cycles among triangles in three dimensions
[Aronov, Miller, S., 17]
- New bounds on curve tangencies and orthogonalities
[Ellenberg, Solymosi, Zahl, 16]
- Cutting algebraic curves into pseudo-segments and applications
[S., Zahl, 17]

Distinct distances between two lines

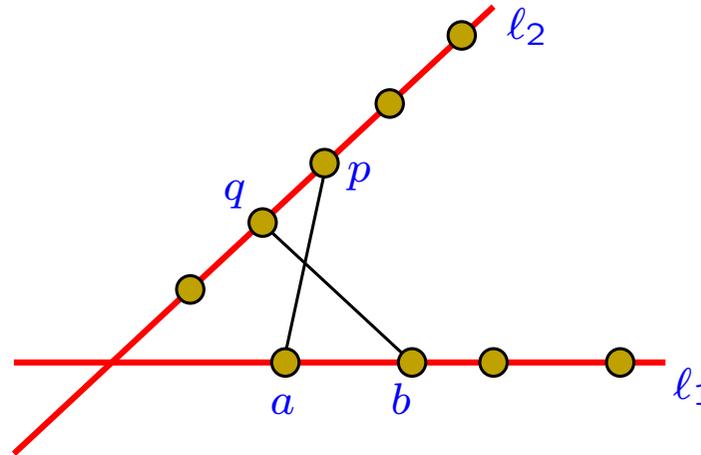
[S., Sheffer, Solymosi, 13]

ℓ_1, ℓ_2 : Two lines in \mathbb{R}^2 , non-parallel, non-orthogonal

P_1, P_2 : Two n -point sets, $P_1 \subset \ell_1, P_2 \subset \ell_2$

$D(P_1, P_2)$: Set of distinct distances between P_1 and P_2

Theorem: $|D(P_1, P_2)| = \Omega(n^{4/3})$



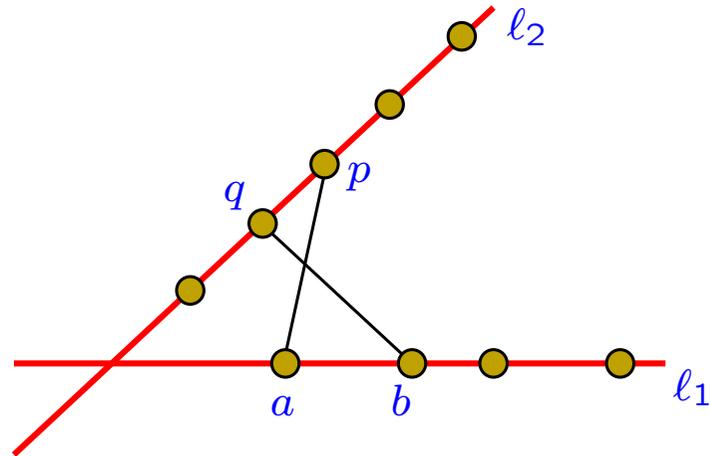
$D(P_1, P_2)$ can be $\Theta(n)$ when

l_1, l_2 are parallel:

(Take $P_1 = P_2 = \{1, 2, \dots, n\}$)

or orthogonal:

(Take $P_1 = P_2 = \{1, \sqrt{2}, \dots, \sqrt{n}\}$)



- A superlinear bound conjectured by [Purdy]
- And proved by [Elekes, Rónyai, 00]
- And improved to $\Omega(n^{5/4})$ by [Elekes, 1999]

Distinct distances between two lines

In [S., Sheffer, Solymosi]: Ad-hoc proof; reduces to
Incidences between points and hyperbolas in the plane

But also a special case of old-new algebraic theory of
[Elekes, Rónyai, Szabó 00, 12]

Enhanced by [Raz, S., Solymosi 16], [Raz, S., de Zeeuw 16]

The Elekes–Rónyai–Szabó Theory

A, B, C : Three sets, each of n real numbers

$F(x, y, z)$: A real trivariate polynomial (constant degree)

How many zeroes does F have on $A \times B \times C$?

Focus only on “bivariate case”: $F(x, y, z) = z - f(x, y)$

The bivariate case $F = z - f(x, y)$

$$Z(F) = \{(a, b, c) \in A \times B \times C \mid c = f(a, b)\}$$

$$|Z(F)| = O(n^2)$$

And the bound is worst-case tight:

$$A = B = C = \{1, 2, \dots, n\} \text{ and } z = x + y$$

$$A = B = C = \{1, 2, 4, \dots, 2^n\} \text{ and } z = xy$$

The bivariate case

The amazing thing ([Elekes-Rónyai, 2000]):

For a quadratic number of zeros,

$$z = x + y \text{ (and } A = B = C = \{1, 2, \dots, n\})$$

$$z = xy \text{ (and } A = B = C = \{1, 2, 4, \dots, 2^n\})$$

Are essentially **the only two possibilities!**

The bivariate case

Theorem ([Elekes-Rónyai],

Strengthened by [Raz, S., Solymosi 16]):

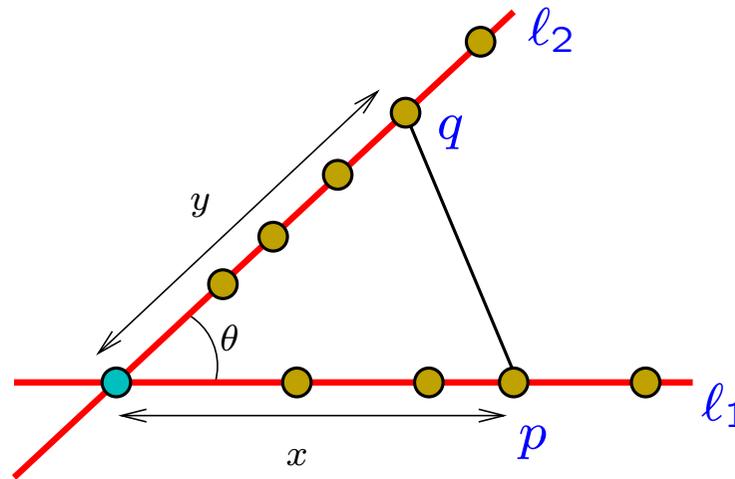
If $z - f(x, y)$ vanishes on $\Omega(n^2)$ points of some $A \times B \times C$, with $|A| = |B| = |C| = n$, then f must have the **special form**

$$f(x, y) = p(q(x) + r(y)) \quad \text{or} \quad f(x, y) = p(q(x) \cdot r(y))$$

for suitable **polynomials** p, q, r

If f does not have the special form, then the number of zeros is always $O(n^{11/6})$ [Raz, S., Solymosi 16]

Distinct distances on two lines: What's the connection?



$$z = f(x, y) = \|p(x) - q(y)\|^2 = x^2 + y^2 - 2xy \cos \theta$$

$$A = P_1, B = P_2$$

$C =$ Set of (squared) distinct distances between P_1 and P_2

$$z = f(x, y) = \|p(x) - q(y)\|^2 = x^2 + y^2 - 2xy \cos \theta$$

$$A = P_1, B = P_2$$

$C =$ Set of (squared) distinct distances between P_1 and P_2

How many zeros does $z - f(x, y)$ have on $A \times B \times C$?

Answer: $|P_1| \cdot |P_2| = n^2$

Does f have the special form?

No (when $\theta \neq 0, \pi/2$)

Yes (when $\theta = 0, \pi/2$: parallel / orthogonal lines)

Here A, B, C have different sizes

Using unbalanced version of [Elekes, Rónyai, Raz, S., Solymosi]:

$$\begin{aligned}n^2 &= \text{Num. of zeros} = O(|P_1|^{2/3}|P_2|^{2/3}|C|^{1/2}) \\ &= O(n^{4/3}|C|^{1/2})\end{aligned}$$

Hence $|C| = \text{number of distinct distances} = \Omega(n^{4/3})$

Cycles, Tangencies, and Lenses

Eliminating depth cycles among lines in \mathbb{R}^3

[Aronov, S. 16]

Eliminating depth cycles among triangles in \mathbb{R}^3

[Aronov, Miller, S. 17]

Tangencies between algebraic curves in the plane

[Ellenberg, Solymosi, Zahl 16]

Cutting lenses and new incidence bounds for curves in the plane

[S., Zahl 16]

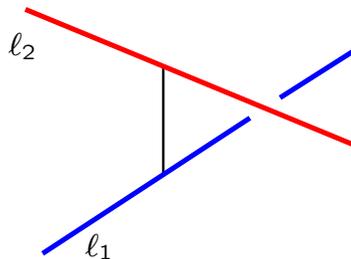
Eliminating depth cycles for lines

L : Set of n lines in \mathbb{R}^3

Non-vertical, in general position

Depth (above/below) relation:

$l_1 < l_2$: On the z -vertical line passing through both l_1, l_2
 l_1 passes below l_2



Depth cycles

\prec is a total relation, but can contain cycles:

Goal: Eliminate all cycles

\equiv Cut the lines into a small number of pieces (segments, rays, lines)

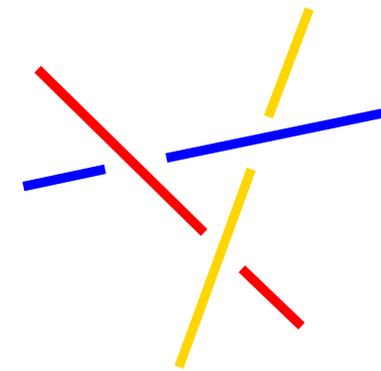
Such that the (now partial) depth relation among the pieces is acyclic

\equiv **Depth order**

• Hard open problem (much harder than joints), for ≥ 35 years

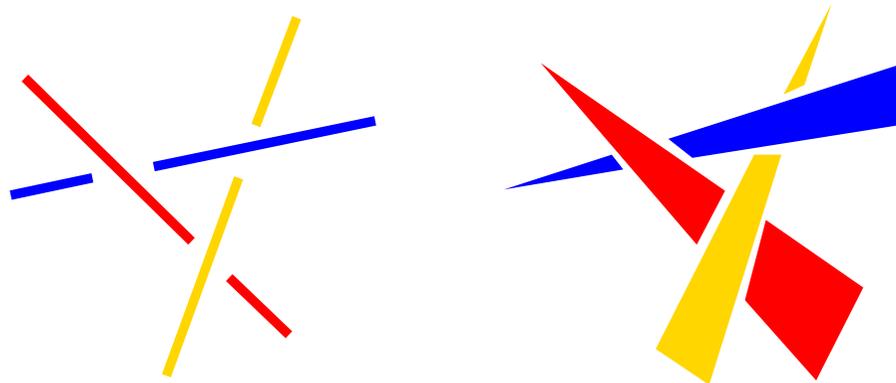
Miserable prehistory (skipped here):

Very weak bounds, only for special configurations



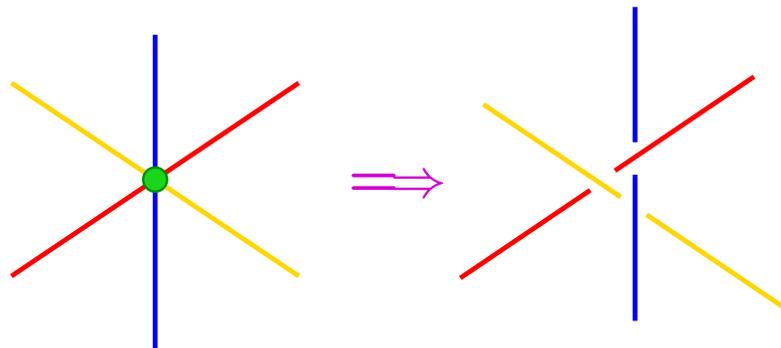
Motivation: Painter's Algorithm in computer graphics

- Draw objects in scene in back-to-front order
- Nearer objects painted over farther ones
- Works only if **no cycles** in depth relation



Generalization of the joints problem

Small perturbation of the lines turns a joint into a cycle
So $\Omega(n^{3/2})$ cuts needed in the worst case



Eliminating cycles for lines

All cycles in a set of n lines in \mathbb{R}^3 can be eliminated with $O(n^{3/2}\text{polylog}(n))$ cuts

Almost tight! [Aronov, S. 16]

Also works for line segments (trivial)

And for constant-degree algebraic arcs

Relatively easy proof, using polynomial partitioning

In a somewhat unorthodox way

How does it work?

Recall the variant of polynomial partitioning in [Guth, 15]:

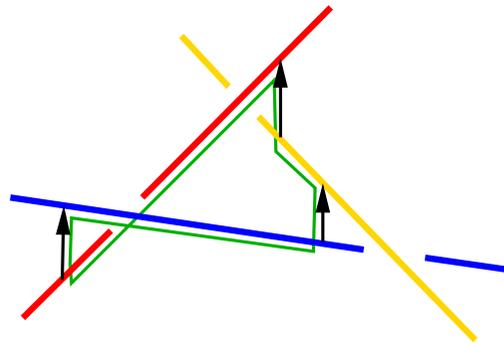
Given a set L of n lines in \mathbb{R}^3 , and degree D , there exists a polynomial f of degree D such that each of the $O(D^3)$ cells of $\mathbb{R}^3 \setminus Z(f)$ is crossed by at most $O(n/D^2)$ lines of L

Cutting cycles I

L : input set of n lines in \mathbb{R}^3

C : a k -cycle (of k lines)

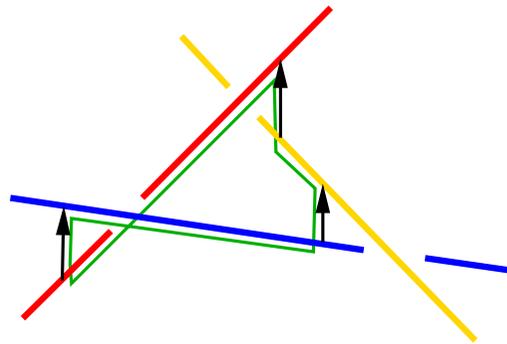
$l_1 \prec l_2 \prec \dots \prec l_k \prec l_1$



$\pi(C)$: The green polygonal loop representing C

Eliminate $C \equiv$ Cut one of the “line portions” of $\pi(C)$

Cutting cycles II

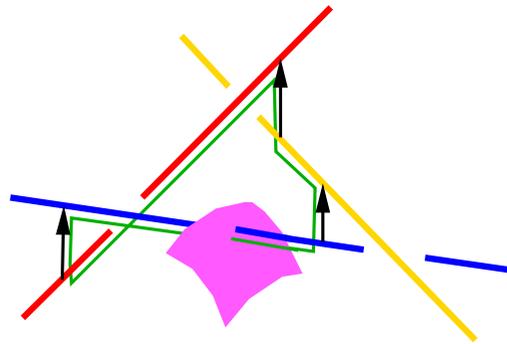


If $Z(f)$ does not cross $\pi(C)$:

$\pi(C)$ fully contained in a cell of $\mathbb{R}^3 \setminus Z(f)$:

Will be handled **recursively**

Cutting cycles III

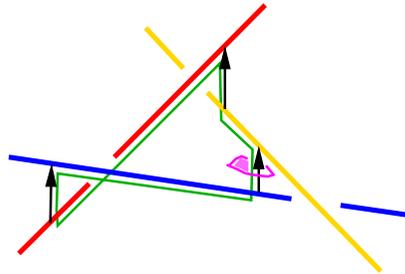


Cut each line at its intersections with $Z(f)$

If $Z(f)$ crosses a line-segment of $\pi(C)$:

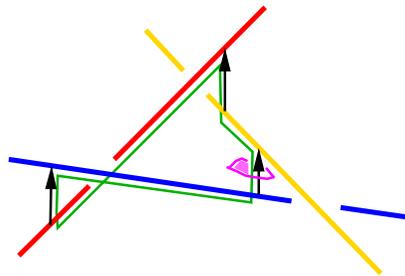
C is eliminated

Cutting cycles IV



If $Z(f)$ crosses a vertical jump segment of $\pi(C)$:
The **level** of $\pi(C)$ in $Z(f)$ **goes up**
(Level of $q \approx$ Number of layers of $Z(f)$ below q)

Cutting cycles IV, cont.



$\pi(C)$ is a closed loop:

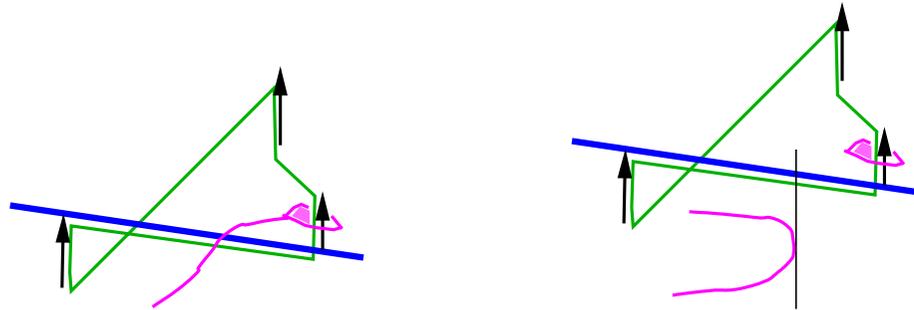
What goes up must come down... How?

Not at vertical jumps! they always go up!

Either $Z(f)$ also crosses a line-segment of $\pi(C)$ (C is cut)

Or the level changes “abruptly” below a line segment

Cutting cycles IV, cont.



First change occurs $\leq nD$ times

Second change occurs $O(D^2)$ times per line (Bézout, Harnack)

Cut each line also over each such change

(Within its own “vertical curtain”)

In total, $O(nD^2)$ non-recursive cuts

Cutting cycles V

Recurrence:

$$K(n) = O(D^3)K(n/D^2) + O(nD^2)$$

For $D = n^{1/4}$, solves to $K(n) = O(n^{3/2} \text{polylog} n)$

Cycles and lenses

- Same approach also eliminates all cycles for constant-degree algebraic arcs

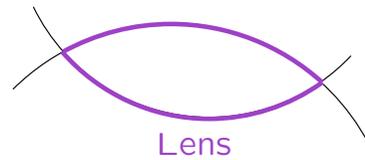
Leads to:

- [S., Zahl 16]:
 n constant-degree algebraic arcs in the plane can be cut into $O(n^{3/2}\text{polylog}(n))$ pseudo-segments
(Each pair intersect at most once)
- Was known before only for circles and pseudo-circles
Open for > 10 years
Crucial for improved incidence bounds in the plane

Lenses and cycles

To cut the curves into pseudo-segments

Need to cut every lens



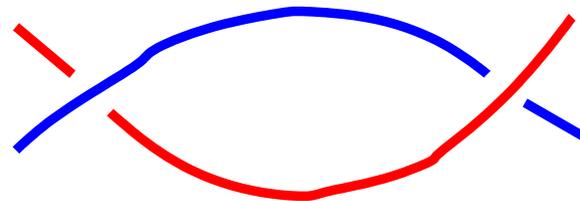
The new idea [Ellenberg, Solymosi, Zahl, 16]:

(Simplified version:)

Map a plane curve $y = f(x)$ to
a space curve $\{(x, f(x), f'(x)) \mid x \in \mathbb{R}\}$

z -coordinate = slope

A lens becomes a 2-cycle!



Cycles, lenses, and incidences

- Eliminate all cycles of the lifted curves \Rightarrow
Cut all the lenses of the original curves \Rightarrow
Turn them into pseudo-segments with $O(n^{3/2} \text{polylog}(n))$ cuts
- Previously known only for circles or pseudo-circles
($O(n^{3/2} \log n)$ cuts [Marcus, Tardos 06])
- Impossible for arbitrary 3-intersecting curves:
 $\Theta(n^2)$ cuts might be needed

Cycles, lenses, and incidences

- Leads to improved incidence bounds for Points and curves in the plane [S., Zahl 16]
- For pseudo-segments, the Szemerédi-Trotter bound applies (By [Székely 97], using the Crossing Lemma)
- Assume the curves come from an s -dimensional family
Add some other divide-and-conquer tricks:

$$I(P, C) = O\left(m^{2/3}n^{2/3} + m^{\frac{2s}{5s-4}}n^{\frac{5s-6+\varepsilon}{5s-4}} + m + n\right),$$

for any $\varepsilon > 0$.

- (For circles, $s = 3$; \approx reconstructs known bound [Agarwal et al. 04], [Marcus, Tardos 06]:

$$I(P, C) = O\left(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} \log^{2/11}(m^3/n) + m + n\right)$$

The last slide (that everybody is too exhausted to read)

- A mix of

Algebra, Algebraic Geometry, Differential Geometry, Topology
In the service of Combinatorial (and Computational) Geometry

- Dramatic push of the area
Many hard problems solved

- And still many deep challenges ahead

Most ubiquitous: **Distinct distances in three dimensions**

Elekes's transformation leads to difficult incidence questions

Involving points and 2D or 3D surfaces in higher dimensions

(5 to 7)

And the really last slide (that everybody is so happy to read)

Thank You