

The Simplex Algorithm in Dimension Three¹

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Abstract

We investigate the worst-case behavior of the simplex algorithm on linear programs with 3 variables, that is, on 3-dimensional simple polytopes. Among the pivot rules that we consider, the “random edge” rule yields the best asymptotic behavior as well as the most complicated analysis. All other rules turn out to be much easier to study, but also produce worse results: Most of them show essentially worst-possible behavior; this includes both Kalai’s “random-facet” rule, which is known to be subexponential without dimension restriction, as well as Zadeh’s deterministic history-dependent rule, for which no non-polynomial instances in general dimensions have been found so far.

1 Introduction

The simplex algorithm is a fascinating method for at least three reasons: For computational purposes it is still the most efficient general tool for solving linear programs, from a complexity point of view it is the most promising candidate for a strongly polynomial time linear programming algorithm, and last but not least, geometers are pleased by its inherent use of the structure of convex polytopes.

The essence of the method can be described geometrically: Given a convex polytope P by means of inequalities, a linear functional φ “in general position,” and some vertex v_{start} ,

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the simplex algorithm chooses an edge to a neighboring vertex along which φ decreases strictly. Iterating this yields a φ -monotone edge-path. Such a path can never get stuck, and will end at the unique φ -minimal (“optimal”) vertex of P .

Besides implementational challenges, a crucial question with respect to efficiency asks for a suitable *pivot rule* that prescribes how to proceed with the monotone path at any vertex. Since Dantzig invented the simplex algorithm in the late 1940’s [4], a great variety of pivot rules have been proposed. Most of them (including Dantzig’s original “largest coefficient rule”) have subsequently been shown to lead to exponentially long paths in the worst case. (See [1] for a survey.) Prominent exceptions are Zadeh’s history-dependent “least entered” rule, and several randomized pivot rules. Particularly remarkable is the “random facet” rule proposed by Kalai [10]; its expected path length for all instances is bounded subexponentially in the number of facets. See also Matoušek et al. [15].

In this paper, we analyze the worst-case behavior of the simplex method on 3-dimensional simple polytopes for some well-known pivot rules. At first glance, the 3-dimensional case may seem trivial, since by Euler’s formula a 3-polytope with n facets has at most $2n - 4$ vertices (with equality if and only if the polytope is simple), and there are examples where $n - 3$ steps are needed for any monotone path to the optimum (see, e.g., Figure 1). Therefore, for *any* pivot rule the simplex algorithm is linear, with at least $n - 3$ and at most $2n - 5$ steps in the worst case. However, no pivot rule is known that would work with at most $n - 3$ steps.

In order to summarize our results, we define the following measure of quality. Fix a pivot rule $\mathcal{R}$. For every 3-dimensional simple polytope $P \subset \mathbb{R}^3$ and for every linear functional $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ *in general position* with respect to P (i.e., no two vertices of P have the same φ -value), denote by $\lambda_{\mathcal{R}}(P, v_{\text{start}})$ the path length (expected path length, if $\mathcal{R}$ is randomized) produced by the simplex algorithm with the pivot rule $\mathcal{R}$, when started at vertex v_{start} . The *linearity coefficient* of $\mathcal{R}$ is

$$\Lambda(\mathcal{R}) := \limsup_{n(P) \rightarrow \infty} \left\{ \frac{\lambda_{\mathcal{R}}(P, v_{\text{start}})}{n(P)} : P, \varphi, v_{\text{start}} \text{ as above} \right\},$$

where $n(P)$ is the number of facets of P . With the usual simplifications for a geometric analysis (compare [14], [20, Lect. 3], [1]), we may restrict our attention to *simple* 3-dimensional polytopes P (where each vertex is contained in precisely 3 facets). So we only consider 3-dimensional polytopes P , with $n = n(P)$ facets, $3n - 6$ edges, and $2n - 4$ vertices. By the discussion above, the linearity coefficient satisfies $1 \leq \Lambda(\mathcal{R}) \leq 2$ for every pivot rule $\mathcal{R}$.

The most remarkable aspect of the picture that we obtain, in Section 3, is that the “random edge” rule (“RE” for short) performs quite well (as it is conjectured for general dimensions), but it is quite tedious to analyze (as it has already been observed for general dimensions). Our main result is in the following bounds for the random edge rule:

$$1.3473 \leq \Lambda(\text{RE}) \leq 1.4943.$$

Thus we manage to separate $\Lambda(\text{RE})$ from the rather easily achieved lower bound of $\frac{4}{3}$, as well as from the already non-trivial upper bound of $\frac{3}{2}$, which we had obtained in an earlier phase of our work, using a simpler and considerably shorter proof.

On the other hand, in Section 4 we prove that the linearity coefficient for the “greatest decrease” pivot rule is $\Lambda(\text{GD}) = \frac{3}{2}$, while many other well-known rules have linearity coefficient $\Lambda = 2$, among them the largest coefficient, least index, steepest decrease, and the shadow vertex rules, as well as Zadeh’s history-dependent least entered rule (not known to be super-polynomial in general), and Kalai’s random facet rule (known to be sub-exponential in general).

2 Basics

Klee [13] proved in 1965 that the “monotone Hirsch conjecture” is true for 3-dimensional polytopes, that is, whenever the graph of a 3-dimensional polytope P with n facets is oriented by means of a linear functional in general position there is a monotone path of length at most $n - 3$ from any vertex to the sink $v_{\min}$. (See Klee & Kleinschmidt [14] for a survey of the Hirsch conjecture and its ramifications.) Unfortunately, Klee’s proof is not based on a pivot rule.

Theorem 2.1 (Klee [13]). *For any simple 3-polytope $P \subset \mathbb{R}^3$, a linear functional $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ in general position for P , and any vertex v_{start} of P , there is a φ -monotone path from v_{start} to the φ -minimal vertex $v_{\min}$ of P that does not revisit any facet.*

In particular, there is a φ -monotone path from v_{start} to $v_{\min}$ of length at most $n - 3$.

It is not too hard to come up with examples showing that the bound provided by Theorem 2.1 is best possible. One of the constructions will be important for our treatment later on, so we describe it below in Figure 1.

A particularly useful tool for constructing LP-oriented 3-polytopes is the following result due to Mihalisin and Klee. It is stated in a slightly weaker version in their paper, but their proof actually shows the following.

Theorem 2.2 (Mihalisin & Klee [17]). *Let $G = (V, E)$ be a planar 3-connected graph, $f : V \rightarrow \mathbb{R}$ any injective function, and denote by $\vec{G}$ the acyclic oriented graph obtained from G by directing each edge to its endnode with the smaller f -value. Then the following are equivalent:*

1. *There exist a polytope $P \subset \mathbb{R}^3$ and a linear functional $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ in general position for P , such that G is isomorphic to the graph of P and, for every $v \in V$, $f(v)$ agrees with the φ -value of the vertex of P corresponding to v .*
2. *Both (a) and (b) hold:*
 - (a) *$\vec{G}$ has a unique sink in every facet (induced non-separating cycle) of G , and*
 - (b) *there are 3 node-disjoint monotone paths joining the (unique) source to the (unique) sink of $\vec{G}$.*

Here the fact that the source and the sink of $\vec{G}$ are unique (referred to in condition (b)) follows from (a); compare Joswig et al. [8]. Equipped with Theorem 2.2, one readily verifies that the family of directed graphs indicated in Figure 1 ($n \geq 4$) demonstrates that Klee’s bound of $n - 3$ on the length of a shortest monotone path cannot be improved.

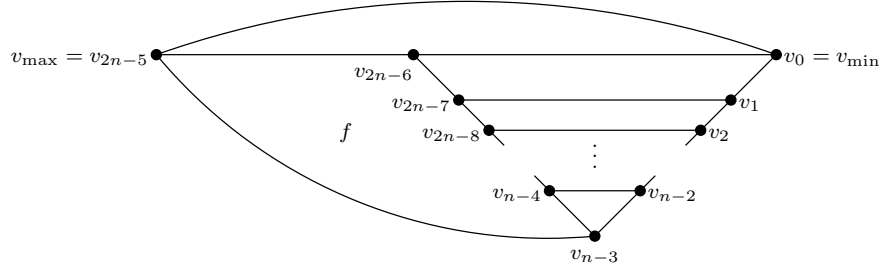


Figure 1: A worst case example for Klee's theorem, starting at v_{n-3} (and for Bland's rule, starting at v_{2n-6} ; see Section 4.1). All edges are oriented from left to right.

3 The Random Edge Rule

At any non-optimal vertex, the *random edge* pivot rule takes a step to one of its improving neighbors, chosen uniformly at random. Thus the expected number $\mathbf{E}(v)$ of steps that the random edge rule would take from a given vertex v to the optimal one $v_{\min}$ may be computed recursively as

$$\mathbf{E}(v) = 1 + \frac{1}{|\delta^+(v)|} \sum_{u: (v,u) \in \delta^+(v)} \mathbf{E}(u), \quad (1)$$

where $\delta^+(v)$ denotes the set of edges that leave v (that is, lead to *better* vertices), so that $|\delta^+(v)|$ is the number of neighbors of v whose φ -value is smaller than that of v .

Despite its simplicity and its (deceptively) simple recursion, this rule has by now resisted several attempts to analyze its worst-case behavior, with a few exceptions for special cases, namely linear assignment problems (Tovey [19]), the Klee-Minty cubes (Kelly [12], Gärtner et al. [6]), and d -dimensional linear programs with at most $d + 2$ inequalities (Gärtner et al. [7]). All known results leave open the possibility that the expected number of steps taken by the random edge rule could be bounded by a polynomial, perhaps even by $O(n^2)$ or $O(dn)$, where n is the number of facets.

3.1 Lower Bounds

The lower bound calculations appear to be much simpler if we do not use the recursion given above, but instead use a “flow model.” For this, fix a starting vertex v_{start} , and denote by $p(v)$ the probability that the vertex v will be visited by a random edge path from v_{start} to $v_{\min}$, and similarly by $p(e)$ the probability that a directed edge e will be traversed. Then the probability that a vertex v is visited is the sum of the probabilities that the edges leading into v are traversed,

$$p(v) = \sum_{e \in \delta^-(v)} p(e)$$

if v is not the starting vertex. (Here $\delta^-(v)$ denotes the set of edges that enter v .) Furthermore, by definition of the random edge rule we have

$$p(e) = \frac{1}{|\delta^+(v)|} p(v) \quad \text{for all } e \in \delta^+(v) \quad (2)$$

We take the maximal vertex of the configuration at v_{2k-7} as the starting vertex v_{start} . Using equation (3) we obtain for the expected cost $\mathbf{E}(v_{\text{start}})$:

$$\mathbf{E}(v_{\text{start}}) = (k - 2) \frac{1897}{128} - 1.$$

With $n = k + 10(k - 2)$ this yields

$$\mathbf{E}(v_{\text{start}}) = \frac{n-2}{11} \cdot \frac{1897}{128} - 1 = \frac{1897}{1408}n - \frac{5202}{1408},$$

which proves the lower bound. $\square$

The configuration depicted in Figure 3 was found by complete enumeration of the “abstract objective functions” (that is, acyclic orientations satisfying condition (a) of Theorem 2.2; see Kalai [9]) on 3-polytopes with $n \leq 12$ facets. In particular, our proof of Theorem 3.1 includes a worst-possible example for $n = 12$. We refer to Mechtel [16] for more details of the search procedure, as well as for a detailed analysis of properties of worst-case examples for the random edge rule.

3.2 Upper Bounds

Theorem 3.2. *The linearity coefficient of the random edge rule satisfies*

$$\Lambda(\text{RE}) \leq \frac{130}{87} < 1.4943.$$

For the proof consider any linear program on a simple 3-polytope with n facets, with a linear objective function φ in general position. We will refer to the φ -value of a vertex as its “height.” A *1-vertex* will denote a vertex with exactly one neighbor that is smaller with respect to φ . Similarly, a *2-vertex* has exactly 2 smaller neighbors. Consequently, the random edge rule proceeds deterministically from any 1-vertex to its unique improving neighbor, and it proceeds from any 2-vertex to each of its two improving neighbors with probability $\frac{1}{2}$.

Basic counting yields that our LP has exactly $n - 3$ 1-vertices and $n - 3$ 2-vertices in addition to the unique maximal vertex v_{max} and the unique minimal vertex v_{min} , which have 3 and 0 smaller neighbors, respectively. For the following, we also assume that the vertices are sorted and labelled $v_{2n-5}, \dots, v_1, v_0$ in decreasing order of their objective function values, with $v_{\text{max}} = v_{2n-5}$ and $v_{\text{min}} = v_0$.

For any vertex v , let $N_1(v)$ (resp., $N_2(v)$) denote the number of 1-vertices (resp., 2-vertices) that are not higher than v (including v itself). Put $N(v) = N_1(v) + N_2(v)$. For all vertices v other than the maximal one this is the number of vertices lower than v , that is, $N(v_i) = i$ for all $i \neq 2n - 5$.

The core of our upper bound on the linearity coefficient of the random edge rule is the following result.

Lemma 3.3. *For each vertex v other than the maximal vertex v_{2n-5} ,*

$$\mathbf{E}(v) \leq \frac{46}{87}N_1(v) + \frac{42}{87}N_2(v).$$

Proof of Theorem 3.2. Lemma 3.3 implies that $\mathbf{E}(v) \leq \frac{130(n-3)}{87} + \frac{15}{29} = \frac{130}{87} \cdot n - \frac{115}{29}$ for all v . Here, the “ $\frac{15}{29}$ ” comes from the fact that Lemma 3.3 applies only for $v \neq v_{2n-5}$; thus we bound $\mathbf{E}(v_{2n-5})$ by $1 + \frac{1}{3} \sum_{i=1}^3 \mathbf{E}(w_i)$, where w_1, w_2, w_3 are the neighbors of v_{2n-5} , and we exploit the fact that $N(w_1) + N(w_2) + N(w_3) \leq 6n - 21$. $\square$

Proof of Lemma 3.3. We will establish the lemma by deriving the generic inequality

$$\mathbf{E}(v) \leq \alpha N_1(v) + \beta N(v), \quad (4)$$

where α and β are constants whose values will be fixed later. The proof will proceed by induction and will be subdivided into 24 distinct cases. Each case yields a linear inequality on α and β that, when satisfied, justifies the induction step and implies that (4) holds in that case. Since our case analysis is complete, we have a proof of (4) for any pair (α, β) that satisfies all the 24 inequalities. It is easy to see (and will be argued in more detail below) that for each such pair we get an upper bound of $\alpha + 2\beta$ for the linearity coefficient $\Lambda(\text{RE})$. Thus we minimize $\alpha + 2\beta$ subject to the linear constraints posed by the various cases; this leads to an LP in two variables with 24 constraints whose optimal solution is $(\alpha, \beta) = (\frac{46}{87}, \frac{42}{87})$, of value $\frac{130}{87}$.

Inequality (4), with those values of α and β , is proved by induction on $N(v)$. The base case $N(v) = 0$ is obvious, since v is the optimum in this case, and $\mathbf{E}(v) = 0$. Suppose now that the inequality of the lemma holds for all vertices lower than some vertex v .

By an appropriate unwinding of the recursion (1), we express $\mathbf{E}(v)$ in terms of the expected costs $\mathbf{E}(w)$ of certain vertices w that are reachable from v via a few downward edges. The general form of such a recursive expression will be

$$\mathbf{E}(v) = c + \sum_{i=1}^k \lambda_i \mathbf{E}(w_i),$$

where $\lambda_i > 0$ for $i = 1, \dots, k$, and $\sum_i \lambda_i = 1$.

Since we assume by induction that $\mathbf{E}(w_i) \leq \alpha N_1(w_i) + \beta N(w_i)$, for each i , it suffices to show that

$$\sum_{i=1}^k \alpha \lambda_i (N_1(v) - N_1(w_i)) + \sum_{i=1}^k \beta \lambda_i (N(v) - N(w_i)) \geq c.$$

Write

$$\Delta_1(w_i) := N_1(v) - N_1(w_i), \quad \Delta(w_i) := N(v) - N(w_i),$$

for $i = 1, \dots, k$. (These terms are defined with respect to the vertex v that is currently considered.) Here $\Delta(w_i)$ is the *distance* between v and w_i , that is, one plus the number of vertices between v and w_i in the numbering of the vertices $(v_{2n-5}, \dots, v_0)$ detailed above. Clearly $\Delta(w_i) \geq \Delta_1(w_i)$.

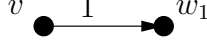
We thus need to show that for each vertex v ,

$$\alpha \sum_{i=1}^k \lambda_i \Delta_1(w_i) + \beta \sum_{i=1}^k \lambda_i \Delta(w_i) \geq c. \quad (5)$$

At this point we start our case analysis.

Case 1: v is a 1-vertex.

Let w_1 denote the target of the unique downward edge emanating from v as in the following figure, where (here and in all subsequent figures) each edge is labelled by the probability of reaching it from v .

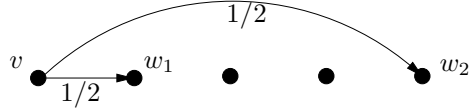


In this case, $\mathbf{E}(v) = 1 + \mathbf{E}(w_1)$. In the setup presented above, we have $\lambda_1 = 1$, $c = 1$, $\Delta_1(w_1) \geq 1$, and $\Delta(w_1) \geq 1$, thus (5) is implied by

$$\alpha + \beta \geq 1. \quad (6)$$

Case 2: v is a 2-vertex.

Let w_1 and w_2 denote the targets of the two downward edges emanating from v , where w_2 is lower than w_1 .



We have

$$\mathbf{E}(v) = 1 + \frac{1}{2}\mathbf{E}(w_1) + \frac{1}{2}\mathbf{E}(w_2),$$

hence we need to require that

$$\frac{\alpha}{2}\Delta_1(w_1) + \frac{\alpha}{2}\Delta_1(w_2) + \frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1.$$

Note that $\Delta(w_2) > \Delta(w_1) \geq 1$.

Case 2.a: $\Delta(w_2) \geq 4$ (as in the preceding figure).

Ignoring the effect of the $\Delta_1(w_j)$'s, it suffices to require that

$$\frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1,$$

which will follow if

$$\beta \geq \frac{2}{5}. \quad (7)$$

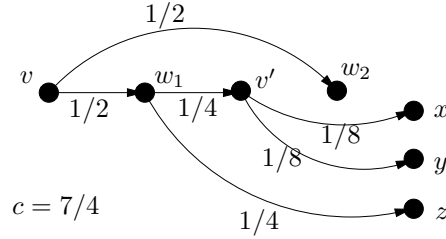
Case 2.b.i: $\Delta(w_2) = 3$ and one of the two vertices above w_2 and below v is a 1-vertex.

In this case $\Delta_1(w_2) \geq 1$ and $\Delta(w_1) + \Delta(w_2) \geq 4$, so (5) is implied by

$$\frac{1}{2}\alpha + 2\beta \geq 1. \quad (8)$$

Case 2.b.ii: $\Delta(w_2) = 3$ and the two vertices between v and w_2 are 2-vertices. Denote the second intermediate vertex as v' . We may assume that v' is reachable from v (that is, from w_1), otherwise we can ignore it and reduce the situation to Case 2.c treated below (by choosing another ordering of the vertices producing the same oriented graph). Three subcases can arise.

First, assume that none of the three edges that emanate from w_1 and v' further down reaches w_2 . Denote by x, y the two downward neighbors of v' and by z the downward neighbor of w_1 other than v' . The vertices x, y, z need not be distinct (except that $x \neq y$), but none of them coincides with w_2 .



We have here $c = 7/4$.

To make the analysis simpler to follow visually, we present it in a table. Each row denotes one of the target vertices w_2, x, y, z , ‘multiplied’ by the probability of reaching it from v . The left (resp., right) column denotes a lower bound on the corresponding quantities $\Delta_1(\cdot)$ (resp., $\Delta(\cdot)$). To obtain an inequality that implies (2), one has to multiply each entry in the left (resp., right) column by the row probability times α (resp., times β), and require that the sum of all these terms be $\geq c$.

	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	0	3
$1/8x$	0	4
$1/8y$	0	5
$1/4z$	0	4

Note the following: (a) We do not assume that the rows represent distinct vertices (in fact, $x = z$ is implicit in the table); this does not cause any problem in applying the rule for deriving an inequality from the table. (b) We have to squeeze the vertices so as to make the resulting inequality as sharp (and difficult to satisfy) as possible; thus we made one of x, y the farthest vertex, because making z the farthest vertex would have made the inequality easier to satisfy.

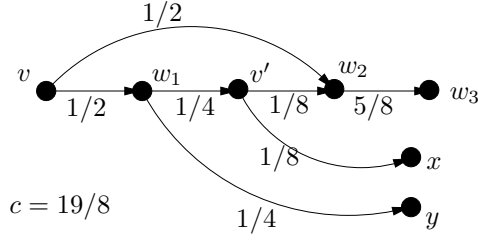
We thus obtain

$$\left(\frac{3}{2} + \frac{4}{8} + \frac{5}{8} + \frac{4}{4}\right)\beta \geq \frac{7}{4},$$

or

$$\beta \geq \frac{14}{29}. \tag{9}$$

Next, assume that w_2 is connected to v' . In this case w_2 is a 1-vertex, and we extend the configuration to include its unique downward neighbor w_3 .



Let x denote the other downward neighbor of v' and let y denote the other downward neighbor of w_1 . In the following table, the ‘worst’ case is to make w_3 and y coincide, and make x the farthest vertex.

	$\alpha\Delta_1$	$\beta\Delta$
$5/8w_3$	1	4
$1/8x$	1	5
$1/4y$	1	4

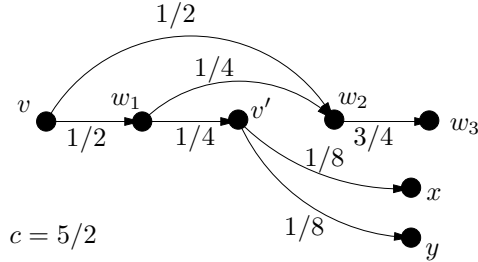
We then obtain

$$\alpha + \left(\frac{20}{8} + \frac{5}{8} + \frac{4}{4} \right) \beta \geq \frac{19}{8},$$

or

$$\alpha + \frac{33}{8}\beta \geq \frac{19}{8}. \quad (10)$$

Finally, assume that w_2 is connected to w_1 . Here too w_2 is a 1-vertex, and we extend the configuration to include its unique downward neighbor w_3 .



Denoting by x, y the two downward neighbors of v' , our table and resulting inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$3/4w_3$	1	4
$1/8x$	1	4
$1/8y$	1	5

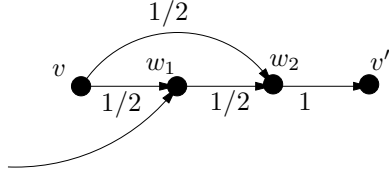
$$\alpha + \frac{33}{8}\beta \geq \frac{5}{2}, \quad (11)$$

which, by the way, is stronger than (10).

Case 2.c: $\Delta(w_2) = 2$. Hence, the only remaining case is that w_1 and w_2 are the two vertices immediately following v .

Case 2.c.i: w_1 is a 1-vertex (whose other upward neighbor lies above v). Its unique downward edge ends at some vertex which is either w_2 or lies below w_2 .

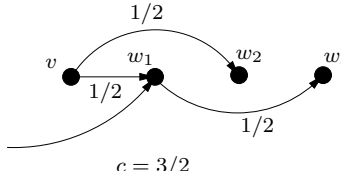
Assume first that this vertex coincides with w_2 , which makes w_2 a 1-vertex, whose unique downward neighbor is denoted as v' . The local structure, table, and inequality are



	$\alpha\Delta_1$	$\beta\Delta$
v'	2	3

$$2\alpha + 3\beta \geq \frac{5}{2}. \quad (12)$$

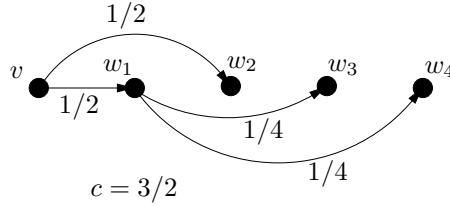
Suppose next that the downward neighbor w_3 of w_1 lies below w_2 . We get



	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	1	2
$1/2w_3$	1	3

$$\alpha + \frac{5}{2}\beta \geq \frac{3}{2}. \quad (13)$$

Case 2.c.ii: w_1 is a 2-vertex, both of whose downward neighbors lie strictly below w_2 . Denote these neighbors as w_3, w_4 , with w_3 lying above w_4 .



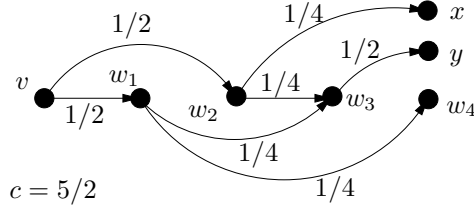
We may assume that $\Delta(w_3) = 3$ (i.e., there is no vertex between w_2 and w_3), since the case $\Delta(w_3) \geq 4$ is already covered by (9).

Case 2.c.ii.1: w_2 is a 1-vertex. Then the table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	0	2
$1/4w_3$	1	3
$1/4w_4$	1	4

$$\frac{1}{2}\alpha + \frac{11}{4}\beta \geq \frac{3}{2}. \quad (14)$$

Case 2.c.ii.2: w_2 is a 2-vertex but w_3 is a 1-vertex. Then w_3 (which satisfies $\Delta(w_3) = 3$) is connected either to w_2 or to a vertex above v . In the former case, let x denote the other downward neighbor of w_2 , and let y denote the unique downward neighbor of w_3 . The local structure looks like this (with x, y, w_4 not necessarily distinct, but they all are below w_3 due to $\Delta(w_3) = 3$):

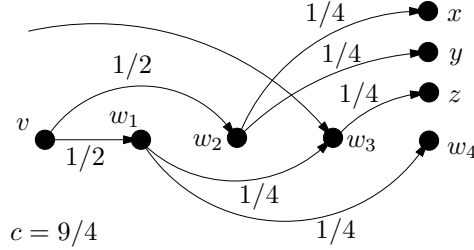


The (worst) table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	4
$1/2y$	1	4
$1/4w_4$	1	5

$$\alpha + \frac{17}{4}\beta \geq \frac{5}{2}. \quad (15)$$

The next case is where the other upward neighbor of w_3 lies above v . Let x, y denote the two downward neighbors of w_2 , and let z denote the unique downward neighbor of w_3 . (Again, x, y, z, w_4 need not be distinct, but they all are below w_3 due to $\Delta(w_3) = 3$.) The local structure is:

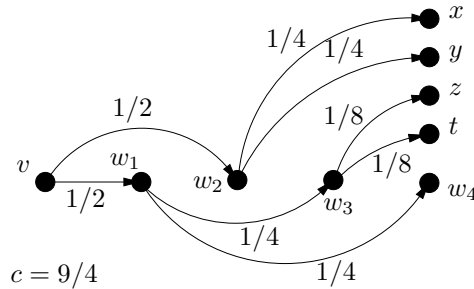


The (worst) table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	4
$1/4y$	1	4
$1/4z$	1	5
$1/4w_4$	1	5

$$\alpha + \frac{9}{2}\beta \geq \frac{9}{4}. \quad (16)$$

Case 2.c.ii.3: Both w_2 and w_3 are 2-vertices. We have to consider the following type of configuration (where x, y, z, t, w_4 need not all be distinct, but $x \neq y$ and $z \neq t$, and we may assume $x \neq t, y \neq z$; also, because $\Delta(w_3) = 3$, both x and y are lower than w_3):



Intuitively, a worst table is obtained by ‘squeezing’ x, y, z, t , and w_4 as much to the left as possible, placing two of them at distance 4 from v , two at distance 5, and one at distance

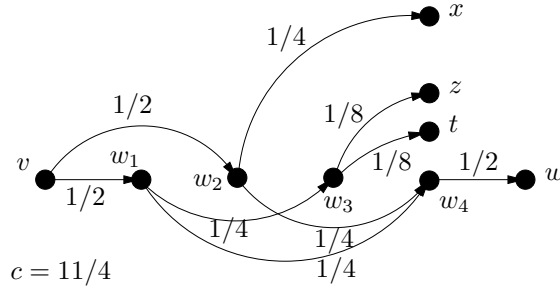
6. However, squeezing them this way will make some pairs of them coincide and form 1-vertices, which will affect the resulting tables and inequalities.

Suppose first that among the three ‘heavier’ targets x, y, w_4 , at most one lies at distance 4 from v . The worst table and the associated inequality are (recall that $x \neq y$):

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	0	4
$1/4y$	0	5
$1/8z$	0	4
$1/8t$	0	6
$1/4w_4$	0	5

$$\frac{19}{4}\beta \geq \frac{9}{4}. \quad (17)$$

Suppose then that among $\{w_4, x, y\}$, two are at distance 4 from v , say w_4 and y . Then $w_4 = y$ is a 1-vertex, and we denote by w its unique downward neighbor. The local structure is:



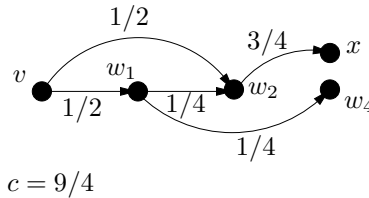
Two equally worst tables, and the resulting common inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	5
$1/8z$	1	6
$1/8t$	1	7
$1/2w$	1	5

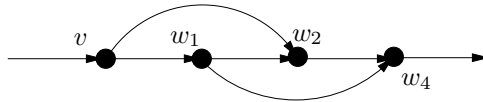
	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	6
$1/8z$	1	6
$1/8t$	1	5
$1/2w$	1	5

$$\alpha + \frac{43}{8}\beta \geq \frac{11}{4}. \quad (18)$$

Case 2.c.iii: w_1 is a 2-vertex that reaches w_2 ; that is, one of its downward neighbors, say w_3 , coincides with w_2 . Then w_2 is a 1-vertex, and we denote by x its unique downward neighbor.



A crucial observation is that x cannot be equal to w_4 . Indeed, if they were equal, then w_4 would be a 1-vertex.



In this case, cutting the edge graph G of P at the downward edge emanating from w_4 and at the edge entering v would have disconnected G , contradicting the fact that G is 3-connected.

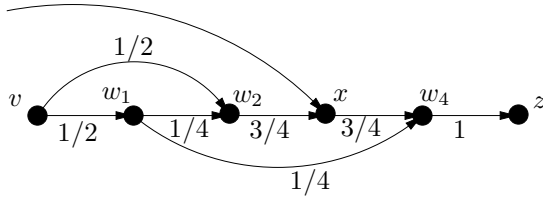
We first dispose of the case where x lies lower than w_4 . The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/4x$	1	4
$1/4w_4$	1	3

$$\alpha + \frac{15}{4}\beta \geq \frac{9}{4}. \quad (19)$$

In what follows we thus assume that x lies above w_4 .

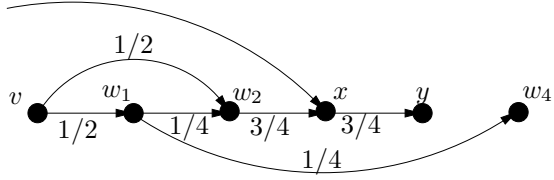
Case 2.c.iii.1: x is a 1-vertex that precedes w_4 . Suppose first that w_4 is the unique downward neighbor of x . Then w_4 is a 1-vertex, and we denote its unique downward neighbor by z . The local structure, table and inequality are:



	$\alpha\Delta_1$	$\beta\Delta$
z	3	5

$$3\alpha + 5\beta \geq 4. \quad (20)$$

Suppose next that the unique downward neighbor y of x is not w_4 . The local structure, table and inequality look like this (y is drawn above w_4 because this yields a sharper inequality):

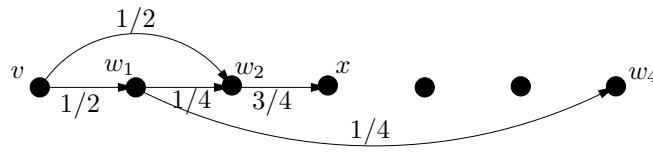


	$\alpha\Delta_1$	$\beta\Delta$
$3/4y$	2	4
$1/4w_4$	2	5

$$2\alpha + \frac{17}{4}\beta \geq 3. \quad (21)$$

Case 2.c.iii.2: x is a 2-vertex that precedes w_4 . This subcase splits into several subcases, where we assume, respectively, that $\Delta(w_4) \geq 6$, $\Delta(w_4) = 4$, and $\Delta(w_4) = 5$.

Case 2.c.iii.2(a). Suppose first that $\Delta(w_4) \geq 6$. The configuration looks like this:



$$c = 9/4$$

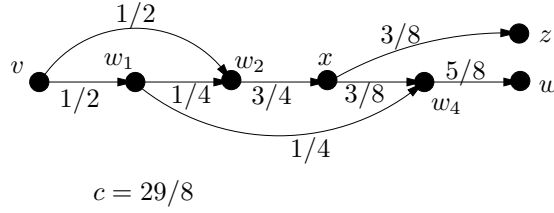
The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/4x$	1	3
$1/4w_4$	1	6

$$\alpha + \frac{15}{4}\beta \geq \frac{9}{4}. \quad (22)$$

Note that this is the same inequality as (19).

Case 2.c.iii.2(b). Suppose next that $\Delta(w_4) = 4$, and that one of the downward neighbors of x is w_4 . Let z denote the other downward neighbor. w_4 is a 1-vertex, and we denote by w its unique downward neighbor.



The 3-connectivity of the edge graph of P implies, as above, that $w \neq z$. Since we assume that $\Delta(w_4) = 4$, z also lies below w_4 , and the table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$5/8w$	2	5
$3/8z$	2	6

$$2\alpha + \frac{43}{8}\beta \geq \frac{29}{8}. \quad (23)$$

Suppose next that $\Delta(w_4) = 4$ and w_4 is not a downward neighbor of x . Denote those two neighbors as w and z , both of which lie lower than w_4 , by assumption, and are clearly distinct. The configuration, table and inequality look like this:

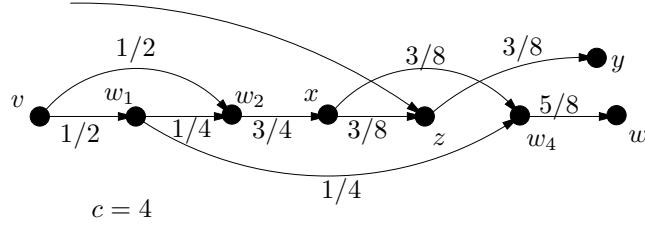
$c = 3$

	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	1	4
$3/8w$	1	5
$3/8z$	1	6

$$\alpha + \frac{41}{8}\beta \geq 3. \quad (24)$$

Case 2.c.iii.2(c). It remains to consider the case $\Delta(w_4) = 5$. Let z denote the unique vertex lying between x and w_4 . We may assume that z is connected to x , for otherwise z is not reachable from v , and we might as well reduce this case to the case $\Delta(w_4) = 4$ just treated.

Consider first the subcase where the other downward neighbor of x is w_4 itself. Then w_4 is a 1-vertex, and we denote by w its unique downward neighbor. This subcase splits further into two subcases: First, assume that z is a 1-vertex, and let y denote its unique downward neighbor. Clearly, y must lie below w_4 (it may coincide with or precede w). The configuration looks like this:

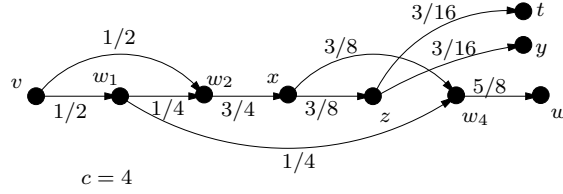


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/8y$	3	6
$5/8w$	3	6

$$3\alpha + 6\beta \geq 4. \quad (25)$$

In the other subcase, z is a 2-vertex; we denote its two downward neighbors as y and t . The vertices w, y, t all lie below w_4 and may appear there in any order (except that $w \neq t$). The configuration looks like this:

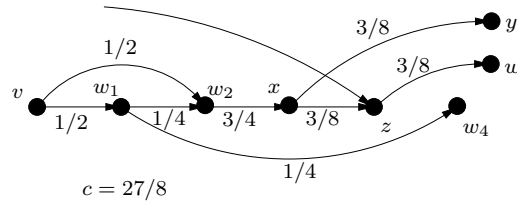


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/16y$	2	6
$3/16t$	2	7
$5/8w$	2	6

$$2\alpha + \frac{99}{16}\beta \geq 4. \quad (26)$$

Consider next the subcase where w_4 is not a downward neighbor of x . Denote the other downward neighbor of x as y , which lies strictly below w_4 . This subcase splits into three subcases. First, assume that z is a 1-vertex, and denote its unique downward neighbor as w . The configuration looks like this:

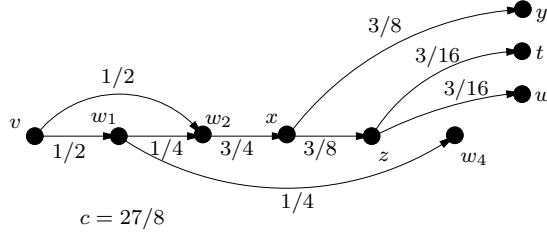


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	2	5
$3/8y$	2	6
$3/8w$	2	5

$$2\alpha + \frac{43}{8}\beta \geq \frac{27}{8}. \quad (27)$$

Second, assume that z is a 2-vertex, so that none of its two downward neighbors is w_4 . Denote these neighbors as w and t . All three vertices y, t, w lie strictly below w_4 , and $w \neq t$. The configuration looks like this:

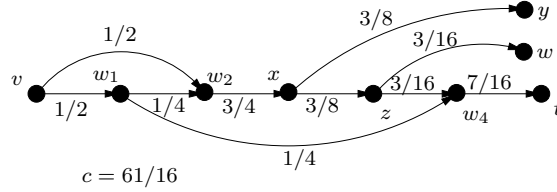


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	1	5
$3/8y$	1	6
$3/16w$	1	6
$3/16t$	1	7

$$\alpha + \frac{95}{16}\beta \geq \frac{27}{8}. \quad (28)$$

Finally, assume that z is a 2-vertex, so that one of its two downward neighbors is w_4 . Denote the other neighbor as w . In this case w_4 is a 1-vertex, and we denote its unique downward neighbor as t . All three vertices y, t, w lie strictly below w_4 . The configuration looks like this:



The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/8y$	2	6
$3/16w$	2	7
$7/16t$	2	6

$$2\alpha + \frac{99}{16}\beta \geq \frac{61}{16}, \quad (29)$$

which, by the way, is weaker than (26).

Assuming that we have α and β that satisfy all the inequalities (6)–(29), each of the induction steps is justified, and the inequality (4) follows. As already noted, since we always have $N_1(v), N_2(v) \leq n - 3$, we obtain

$$\mathbf{E}(v) \leq \alpha N_1(v) + \beta(N_1(v) + N_2(v)) = (\alpha + \beta)N_1(v) + \beta N_2(v) \leq (\alpha + 2\beta)(n - 3)$$

for $v \neq v_{2n-5}$. Hence we choose (α, β) to minimize $\alpha + 2\beta$, subject to all the inequalities that we have derived. This yields the parameters that appear in the statement of Lemma 3.3. $\square$

Discussion. (1) The analysis has used (twice) the fact that G is a 3-connected graph. Without this assumption, the linearity coefficient becomes $13/8$: A lower bound construction can be derived from the figure shown in Case 2.c.iii, and an upper bound can be obtained along the same lines of the preceding proof, using a much shorter case analysis. It is interesting that the proof did not use at all the planarity of the polytope graph G .

(2) In an earlier phase of our work, we obtained the upper bound of $3/2$ on the linearity coefficient, using a similar but considerably shorter case analysis. For a long time, it seemed that this approach cannot yield a smaller bound. The reason for presenting here the lengthier case analysis is to demonstrate that the problem is probably far from admitting a clean and simple solution (at least using this approach). Of course, it would be interesting to find an alternative simpler way of attacking the problem.

(3) We note that Lemma 3.3 is not the end of the story. The solution $(\alpha, \beta) = (\frac{46}{87}, \frac{42}{87})$ satisfies (9) and (20) with equality. If we examine the configuration corresponding to (9) and expand it further, we can replace (9) by better inequalities, which result in a (slightly) improved bound on the linearity coefficient, at the costs of lengthening further our case analysis. We have no idea whether this iterative refinement process ever converges to some critical configuration, whose further expansion does not improve the bound, and which is then likely to yield a tight bound on the linearity coefficient.

4 Other Pivot Rules

4.1 Bland's Rule

For Bland's *least index* pivot rule [2] the facets (inequalities) are numbered. At every non-minimal vertex the rule then dictates to choose the edge that leaves the facet with the smallest number. (A special feature of Bland's rule is that it does not admit cycling even on degenerate programs/non-simple polytopes, when our geometric description of the rule is, however, not applicable.)

Proposition 4.1. *The linearity coefficient of Bland's rule is 2.*

Proof. Figure 1 illustrates a family of 3-dimensional LPs on which Bland's rule, started at $v_{\text{start}} = v_{2n-6}$, visits all but one of the vertices. (The directed graph that is depicted is readily verified to satisfy the conditions of Theorem 2.2.) Specifically, choose an initial numbering of the facets where the largest index is assigned to facet f . When starting at the vertex $v_{\text{start}} = v_{2n-6}$ the simplex algorithm with Bland's rule visits the $2n - 5$ vertices $v_{2n-6}, \dots, v_0$. $\square$

4.2 Dantzig's Rule

Dantzig's rule is the original rule proposed by Dantzig when he invented the simplex algorithm. In his setting of a maximization problem formulated in the language of simplex tableaus, the rule requires to pivot into the basis the variable that has the largest reduced cost coefficient (if no variable has positive reduced cost, the current tableau is optimal).

By suitably scaling the inequalities of the LP, Dantzig's rule follows the same path as Bland's rule; see Amenta & Ziegler [1, Observation 2.6]. Thus Dantzig's rule cannot be faster than Bland's rule and we obtain with Proposition 4.1

Proposition 4.2. *The linearity coefficient of Dantzig's rule is 2.*

4.3 Greatest Decrease Rule

The *greatest decrease* rule moves from any non-optimal vertex to the neighbor with the smallest objective function value. We assume that the objective function is generic, so the vertex is unique. However, the greatest decrease rule may compare non-adjacent neighbors, so the information given by the directed graph is not sufficient to implement it.

Proposition 4.3. *The linearity coefficient of the greatest decrease rule is $\frac{3}{2}$.*

Proof. First we show that $\Lambda(\text{GD}) \geq \frac{3}{2}$. Figure 4 indicates a family of 3-dimensional LPs. By Theorem 2.2, there is a realization of these LPs with the objective function linear ordering on the vertices given by the left-to-right ordering in our figure. Started at $v_{\text{start}} = v_{2n-6}$ the greatest decrease rule visits all 1-vertices, the global sink, and half of the 2-vertices. Thus it needs $\frac{3}{2}(n-3)$ pivot steps to reach $v_{\text{min}} = v_0$.

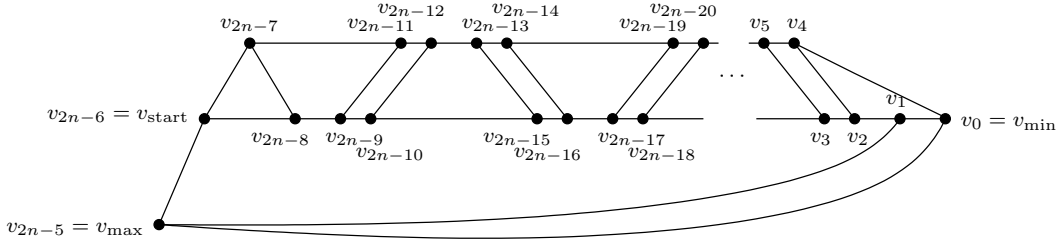


Figure 4: Lower bound for the greatest decrease rule. All edges are oriented from left to right.

For the proof of $\Lambda(\text{GD}) \leq \frac{3}{2}$, we consider an arbitrary instance $(n, P, \varphi, v_{\text{start}})$ as above. Denote by n_1 and n_2 the number of visited 1-vertices resp. 2-vertices. Thus there are $n-3-n_1$ and $n-3-n_2$ unvisited 1-vertices and 2-vertices, respectively. For every visited 2-vertex v only one of the two direct successors v' and v'' is visited. Assuming that $\varphi(v') > \varphi(v'')$, the greatest decrease rule will go directly from v to v'' and thus skip v' , whose objective function value satisfies $\varphi(v) > \varphi(v') > \varphi(v'')$. Thus there is an unvisited vertex uniquely associated with every visited 2-vertex. Thus $n_2 \leq 2n-6-n_1-n_2$, which is equivalent to $n_1+2n_2 \leq 2n-6$. We get

$$n_1 + n_2 = \frac{1}{2}n_1 + \frac{1}{2}(n_1 + 2n_2) \leq \frac{1}{2}(n-3) + \frac{1}{2}(2n-6) \leq \frac{3}{2}(n-3).$$

This yields $\Lambda(\text{GD}) \leq \frac{3}{2}$ and completes the proof. $\square$

4.4 Steepest Decrease Rule

At any non-minimal vertex v the *steepest decrease* pivot rule moves to the neighbor w with vw being the steepest decreasing edge, that is, such that $\frac{\langle c, w-v \rangle}{\|w-v\| \|c\|}$ is minimal (where $\langle c, x \rangle$ is the objective function).

Proposition 4.4. *The linearity coefficient of the steepest decrease rule is 2.*

Proof. Figure 5 depicts a planar projection onto the (x_1, x_2) -plane of an LP that is easily constructed either “by hand” or as a deformed product (see Amenta & Ziegler [1]). If the polytope is scaled to be very flat in the x_3 -direction, then steepest decrease tells the simplex algorithm to use the edge that in the projection has the smallest slope (in absolute value). Thus starting at $v_{\text{start}} = v_{2n-5}$, the steepest decrease rule visits *all* the vertices. $\square$

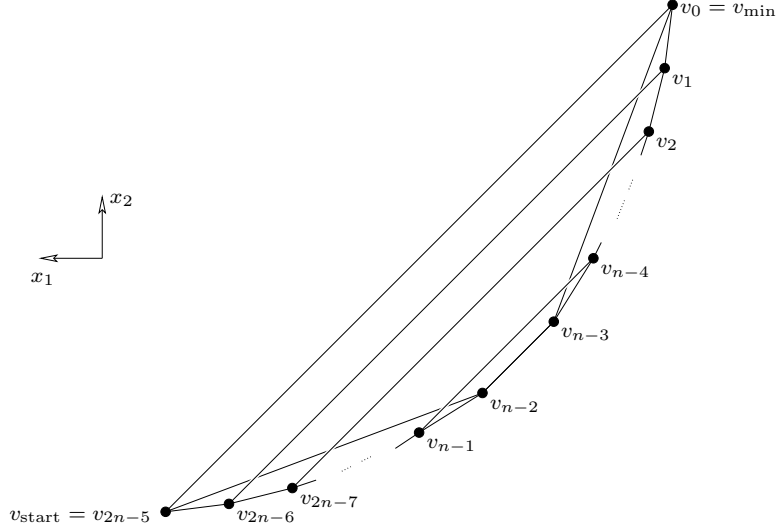


Figure 5: Lower bound for the steepest decrease and shadow vertex rules. Planar projection of the polytope: The objective function is x_1 ; it directs all edges from left to right.

4.5 Shadow Vertex Rule

The *shadow vertex* pivot rule chooses a sequence of edges that lie on the boundary of the 2-dimensional projection of the polytope given by $x \mapsto (c^\top x, d^\top x)$, where $c^\top x$ is the given objective function, and $d^\top x$ is an objective function that is constructed to be optimal at the starting vertex v_{start} . The vertices that are visited on the path from v_{start} and v_{min} are then optimal for objective functions that interpolate between $d^\top x$ and $c^\top x$. (This pivot rule is known to be polynomial on “random linear programs” in specific models; compare Borgwardt [3], Ziegler [21], and Spielman & Teng [18].)

Proposition 4.5. *The linearity coefficient of the shadow vertex rule is 2.*

Proof. We reuse the linear programs of Proposition 4.4/Figure 5. Here $v_{2n-5} = v_{\text{max}}$ is optimal for the starting objective function $d^\top x = x_2$, while v_0 is optimal for $c^\top x = x_1$. On the way from v_{2n-5} to $v_{\text{min}} = v_0$ the shadow vertex rule visits *all* the vertices. $\square$

4.6 Random Facet

The *random facet* pivot rule, due to Kalai [11, p. 228], is as follows:

- (RF) At any non-optimal vertex v choose one facet f containing v uniformly at random and solve the problem restricted to f by applying (RF) recursively.
 The recursion will eventually restrict to a one-dimensional subproblem (that is, an edge), which is solved by following the edge.

The one-dimensional base case singled out here is only implicit in Kalai's work. This is probably the reason why there are different versions of this rule in the literature which unfortunately were not distinguished. They all differ in the way how 1-vertices are treated. Since the (unique) out-edge of a 1-vertex is always taken with probability one (regardless of which facets we restrict to) we could use the following alternative formulations of the random facet rule:

- (RF1) At each non-optimal vertex v follow the (unique) outgoing edge if v is a 1-vertex. Otherwise choose one facet f uniformly at random containing v and solve the problem restricted to f by applying (RF1) recursively.
- (RF2) At any non-optimal vertex v choose one facet f containing v uniformly at random and solve the problem restricted to f by applying (RF2) recursively. The minimal vertex $\text{opt}(f)$ of f is a 1-vertex and we follow the (unique) outgoing edge of the vertex $\text{opt}(f)$.

The variant (RF1) appears in Gärtner, Henk & Ziegler [6, p. 350], while the version (RF2) is from Gärtner [5], who, however, formulated this variant of the random facet rule for combinatorial cubes, where the formulations above are equivalent.

Note that (RF) uses randomness at any vertex and (RF1) would follow a path of 1-vertices deterministically while (RF2) takes at most one deterministic step in a row. This results in distinct pivot rules, with different worst case examples.

Proposition 4.6. *The linearity coefficient of the random facet rule (RF) is 2.*

Proof. Figure 6 depicts a family of LPs with $2n - 4 = 2a + 2b + 2$ vertices and $n = a + b + 3$ facets. At each of the b 1-vertices $v_{\text{start}} = v_{2n-7}, v_{2n-9}, \dots, v_{2n-5-2b}$, the random facet rule chooses with probability $\frac{1}{2}$ to “stick” to facet f .

Choosing $a = k^2$ and $b = k$ we obtain a family of LPs with $n = k^2 + k + 3$ facets. Then (RF) sticks to facet f with probability $p \geq 1 - (\frac{1}{2})^k$. Thus the expected number of visited vertices is at least

$$(1 - (\frac{1}{2})^k) (2a + b) \geq 2k^2 - \frac{2k^2}{2^k}.$$

Since there are $n = k^2 + k + 3$ facets the linearity coefficient is 2. $\square$

If we use the version (RF1) of the random facet rule, then the simplex algorithm follows the path of 1-vertices $v_{\text{start}} = v_{2n-7}, v_{2n-9}, \dots, v_{2n-5-2b}$ deterministically. We can cut off each of these vertices. This yields the graphs depicted in Figure 7. At each source of the new facets $\Delta_1, \dots, \Delta_b$ the facet f is chosen with probability $\frac{1}{3}$. Thus the linearity coefficient remains 2, only the rate of convergence decreases. The same works for (RF2) as well.

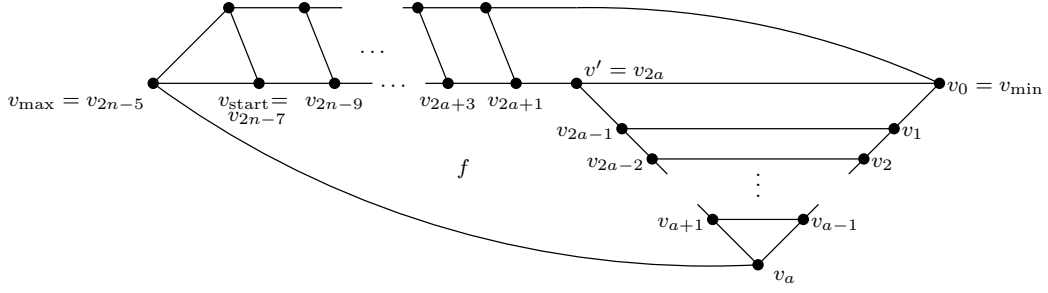


Figure 6: Lower bounds for the random facet rule (RF). All edges are oriented from left to right.

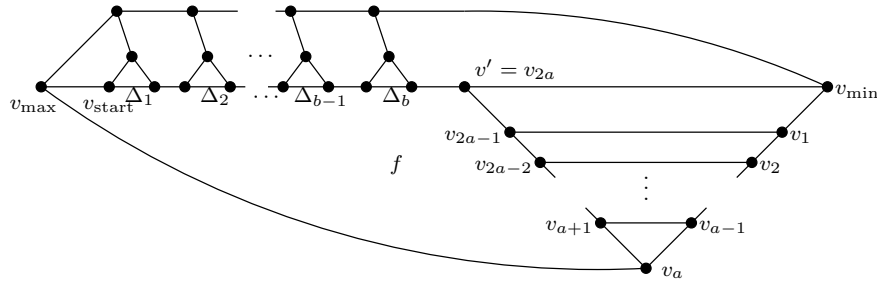


Figure 7: Lower bound for the (RF1) and (RF2) variants of the random facet rule, and for the least entered rule with random edge as the tie-breaking rule. All edges are oriented from left to right.

4.7 Least Entered Rule

At any non-optimal vertex, the *least entered* pivot rule chooses the decreasing edge that leaves the facet that has been left least often in the previous moves. In case of ties a tie-breaking rule is used to determine the decreasing edge to be taken. Any other pivot rule can be used as a tie-breaking rule.

The least entered rule was first formulated by Norman Zadeh around 1980 (see [14] and [21]). It has still not been determined whether Zadeh's rule is polynomial if the dimension is part of the input. Zadeh has offered \$1000 for solving this problem.

Proposition 4.7. *The linearity coefficient of the least entered rule with greatest decrease as tie-breaking rule is 2.*

Proof. Figure 8 describes a family of 3-dimensional LPs, where the left-to-right ordering of the vertices suggested by the figure can be realized, according to the Mihalisin–Klee Theorem 2.2. Starting at $v_{\text{start}} = v_{2n-6}$, the greatest decrease rule decides to leave the facet f . Following two 1-vertices the facet f is entered again. All upcoming facets have not been visited before. Thus the least entered rule “sticks” to the facet f and $2n - 7$ vertices (that is, all but 3 vertices) are visited. $\square$

Proposition 4.8. *The linearity coefficient of the least entered rule with random edge as the tie-breaking rule is 2.*

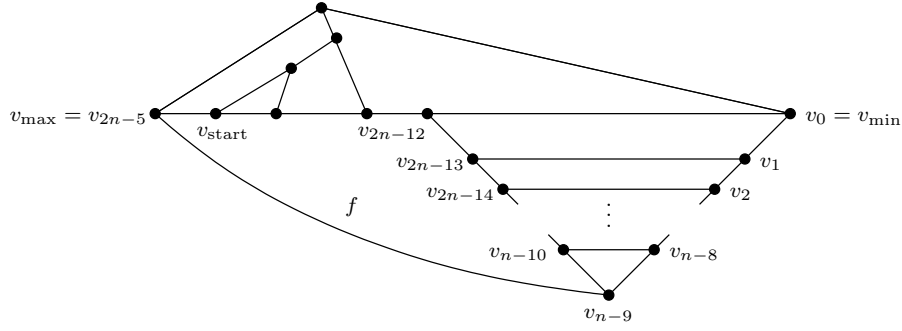


Figure 8: Lower bound for the least entered rule with greatest decrease as the tie-breaking rule. All edges are oriented from left to right.

Proof. Figure 7 describes LPs with $2n - 4 = 2a + 4b + 2$ vertices and $n = a + 2b + 3$ facets. At the sources of the facets Δ_i the random edge rule leaves the facet f with probability $\frac{1}{2}$. As soon as f is left once, it will be revisited and the least entered rule will “stick” to the facet f . (Thus the only way not to “stick” to f is that the random edge rule chooses to continue along f until it reaches the vertex v' .) When the least entered rule “sticks” to the facet f all of the $2a$ vertices $v_{2a-1}, v_{2a-2}, \dots, v_1, v_0$ are visited.

Now the analysis is exactly the same as in the proof of Proposition 4.6. Thus choosing $a = k^2$ and $b = k$ yields that the linearity coefficient is 2. $\square$

Acknowledgements. We are grateful to Emo Welzl and Günter Rote for inspiring discussions and helpful comments.

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