Advanced Topics in Computational Geometry

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Take Home Exam
June 29–July 2, 2004

Answer four of the following problems. All have the same weight—25%. The exams should be returned to me on Friday, July 2nd, between 11:00 and 14:00 (Schreiber Bldg, Room 330), or sent by email as a .ps or .pdf file, no later than 14:00 Friday.
I will answer questions about the exam by email (or in person) during this period.
Good Luck!!

Problem 1

Let P be a set of m points in $\mathbb{R}^3$, and T a set of n (possibly intersecting) triangles in $\mathbb{R}^3$, all in general position. Give an algorithm, that uses cuttings based on vertical decomposition, that determines, for each point of P, the triangle lying directly below it (or reports that there is no such triangle). The running time of the algorithm should be $O^*(m^{2/3}n + m)$ (or better...) (where $O^*(\cdot)$ may include additional factors of the form m^ε and n^ε, for arbitrarily small $\varepsilon > 0$).

Problem 2

Let $P = \{p_1(t), \ldots, p_n(t)\}$ be a set of n points moving in the plane. Assume that for each $i = 1, \ldots, n$, each coordinate of $p_i(t)$ is given as a polynomial in t of degree at most k. Let $CH(t)$ denote the convex hull of P at time t. The combinatorial structure of $CH(t)$ is the circular list of indices $(i_1, \ldots, i_q)$ such that the vertices of $CH(t)$ are the points $p_{i_1}(t), \ldots, p_{i_q}(t)$, in this counterclockwise order along the hull.

(a) Show that the maximum possible number of changes in the combinatorial structure of $CH(t)$ over time is $O(n\lambda_{2k}(n))$. (**Hint:** Fix a point p_i, and consider the sequence whose elements are the points that appear, over time, as the next counterclockwise vertex of the hull after $p_i(t)$ (when p_i itself is a hull vertex).)

(b) Give a construction where the number of changes in the combinatorial structure of the convex hull is $\Omega(n^2)$. Try to make the degree k as small as you can.

Problem 3

(a) Let R be a set of n axis-parallel rectangles in the plane, and D a set of n disks in the plane. Consider the set of all the faces of the arrangement $\mathcal{A}(R \cup D)$ which contain a vertex of some rectangle on their boundary, and lie in the exterior of all the disks of D. Show that the combined complexity of all these faces is $O(n)$. (**Hint:** Use the combination lemma.)

(b) Give a tight upper bound on the combinatorial complexity of the Minkowski sum $\pi \oplus D$, where π is a polygonal path with n edges that does not cross itself, and D is a unit disk.

Problem 4

Let D be a set of n disks in the xy-plane. Lift each disk to a random height in the z-direction (e.g., enumerate the disks as $d_1, d_2, \ldots, d_n$, choose a random permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of $(1, 2, \ldots, n)$, and assign to disk d_i the height (z-coordinate) π_i.

We say that a vertex v of $\mathcal{A}(D)$, incident to the boundaries of two disks d_i, d_j, *survives* after the lifting if the z-vertical line passing through v meets the two lifted disks at two points w_i, w_j, so that the segment connecting them meets no other lifted disk.

(a) Show that the expected number of surviving vertices is $O(n \log n)$. (**Hint:** Express the probability of a vertex to survive in terms of the number of disks that contain it, and use Clarkson-Shor.)

(b) Does this result continue to hold if we replace D by a set of n axis-parallel rectangles? n axis-parallel squares?

Problem 5

(a) Let S be a set of n vertical line segments in the plane. Preprocess S into a data structure, so that, for a query point q, we can find in $O(\log n)$ time the segment that q sees with the largest angle (i.e., we simply seek the segment $ab \in S$ for which $\angle aqb$ is largest; the segments do not hide each other).

(b) Same as (a), but now we want to find the segment that forms with q the triangle with the largest area.

Problem 6

(a) Give an upper bound on the complexity of an arrangement of n triangles in $\mathbb{R}^3$ with the property that every vertical line intersects at most k triangles. How tight (in the worst case) is your bound? (**Hint:** Clarkson-Shor!)

(b) Let P be a set of n points in $\mathbb{R}^3$, and let γ be the circle $x^2 + y^2 = 1$, $z = 0$. Give an upper bound on the number of times where γ crosses between cells of the Voronoi diagram of P.