Theorem 6.14 (Dvoretzky-Rogers). If $B_2^n$ is the maximal volume ellipsoid of a convex body $K \in K_n^0$, there exists an orthonormal sequence $z_1, \ldots, z_n$ in $\mathbb{R}^n$ such that

$$\left(\frac{n-i+1}{n}\right)^{1/2} \leq \|z_i\| \leq |z_i| = 1$$

for all $i = 1, \ldots, n$.

Proof. We define the $z_i$ inductively. Choose $z_1$ to be any contact point of $K$ and $B_2^n$, and assume that $z_1, \ldots, z_k$ have been chosen for some $k < n$.

We set $F_k = \text{span}\{z_1, \ldots, z_k\}$. Then, $\text{tr}(P_{F_k^\perp}) = n - k$, and applying Lemma 6.13 we may find a contact point $y_{k+1}$ of $K$ and $B_2^n$ such that

$$|P_{F_k^\perp} y_{k+1}|^2 = \langle y_{k+1}, P_{F_k^\perp} y_{k+1} \rangle \geq \frac{n-k}{n},$$

and the inductive step is complete. □

Corollary 6.15. Assume that $B_2^n$ is the maximal volume ellipsoid of a convex body $K \in K_n^0$. If $k = \lfloor n/2 \rfloor + 1$, we can find orthonormal vectors $z_1, \ldots, z_k$ such that

$$\frac{1}{\sqrt{2}} \leq \|z_j\| \leq 1$$

for all $j = 1, \ldots, k$.

Remark 6.16. It is useful to note that one also has a version of Corollary 6.15 that provides a bound for the norms of all of the orthonormal vectors $z_i$, I left it for you to prove in the exercise sheet.

6.2 Another proof for John’s theorem

As explained in class, in a maximization problem we can many times deduce the existence of some good structure (special contact points, an isotropic measure) simply by using the fact that a certain differential vanishes. Let us have a second look at John’s theorem with these “eyeglasses” on.

I am doing the centrally symmetric case for simplicity.

Lemma 6.17. Let $K = -K$ be a convex body and assume $B_2^n \subset K$ is the maximal volume ellipsoid. Then for any $T \in SL_n$ there exists a point $u \in S^{n-1}$ such that

$$\|Tu\|_K \geq 1.$$
**Lemma 6.18.** Let $K = -K$ be a convex body and assume $B^n_2 \subset K$ is the maximal volume ellipsoid. Then for any $S \in L(\mathbb{R}^n, \mathbb{R}^n)$ there exists a contact point $u \in S^{n-1} \cap \partial K$ such that

$$\|Su\|_K \geq \frac{1}{n} \text{tr} S.$$ 

**Proof.** Define $T_\varepsilon = \frac{1}{\det(I + \varepsilon S)^{1/n}} (I + \varepsilon S)$ so that $T_\varepsilon \in SL_n$. By Lemma 6.17 we may find for every $\varepsilon$ a point $u_\varepsilon \in S^{n-1}$ such that

$$\|T_\varepsilon u_\varepsilon\|_K \geq 1$$

that is

$$\|u_\varepsilon + \varepsilon Su_\varepsilon\|_K \geq \det(I + \varepsilon S)^{1/n}.$$ 

Using that $B^n_2 \subset K$ we know $\|u_\varepsilon\| \leq |u_\varepsilon| = 1$ so that for $\varepsilon \geq 0$ we have

$$1 + \varepsilon \|Su_\varepsilon\|_K \geq \|u_\varepsilon\| + \varepsilon \|Su_\varepsilon\|_K \geq \det(I + \varepsilon S)^{1/n} = 1 + \varepsilon \frac{1}{n} \text{tr} S + o(\varepsilon).$$

Therefore

$$\|Su_\varepsilon\|_K \geq \frac{1}{n} \text{tr} S + o(1).$$

Take now a sequence $u_\varepsilon \to u$ as $\varepsilon \to 0$ then $u \in S^{n-1}$ and $\|Su\|_K \geq \frac{1}{n} \text{tr} S$. Clearly as $1 \leq \|T_\varepsilon u_\varepsilon\|_K = \|u_\varepsilon + O(\varepsilon)\|_K$ we have that in the limit also $\|u\|_K \geq 1$ but since $B^n_2 \subset K$ we actually have $\|u\|_K = 1$, so that $u$ is a contact point. 

Finally, we can show

**Lemma 6.19.** Let $K = -K$ be a convex body and assume $B^n_2 \subset K$ is the maximal volume ellipsoid. Then for any $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ there exists a contact point $u \in S^{n-1} \cap \partial K$ such that

$$\langle Tu, u \rangle \geq \frac{1}{n} \text{tr} T.$$ 

**Proof.** Given $T$ define $S_\varepsilon = (I + \varepsilon T)$ and by Lemma 6.18 find for every $\varepsilon$ (now it may be also negative, by the way) a contact point $u_\varepsilon \in S^{n-1} \cap \partial K$ such that

$$\|u_\varepsilon + \varepsilon Tu_\varepsilon\|_K = \|S_\varepsilon u_\varepsilon\|_K \geq \frac{1}{n} \text{tr} S_\varepsilon = 1 + \varepsilon \frac{1}{n} \text{tr} T.$$ 

The left hand side is equal to

$$\|u_\varepsilon\|_K + \varepsilon \langle \nabla \|\cdot\|_K, Tu_\varepsilon \rangle + o(\varepsilon) = 1 + \varepsilon \langle \nabla \|\cdot\|_K, Tu_\varepsilon \rangle + o(\varepsilon).$$

The gradient of the norm is a functional $w \in \partial K^o$ which is the supporting functional at $u$, that is, it satisfies $\langle u, w \rangle = 1$. However, since $u$ is a contact point of $K$ and $B^n_2$, the normal at $u$ is $u$ itself. We thus get for every $\varepsilon$ a contact point $u_\varepsilon$ for which

$$1 + \varepsilon \langle u_\varepsilon, Tu_\varepsilon \rangle + o(\varepsilon) \geq 1 + \varepsilon \frac{1}{n} \text{tr} T,$$

which can be rewritten (for $\varepsilon > 0$, say) as

$$\langle u_\varepsilon, Tu_\varepsilon \rangle + O(\varepsilon) \geq \frac{1}{n} \text{tr} T.$$ 

Taking a converging subsequence of contact points as $\varepsilon \to 0^+$, we end up with a contact point
Proof of John’s theorem. To complete the proof of John’s theorem in this vein is actually quite simple. Consider as in the original proof, the convex hull of contact points
\[ C = \{ u \otimes u : |u| = \|u\|_K = 1 \} \]
and assume it may be separated from \( \frac{1}{n} I \), using some linear functional given by \( T \in L(\mathbb{R}^n, \mathbb{R}^n) \). This means that
\[ \langle T, \frac{1}{n} I \rangle > c > \langle T, u \otimes u \rangle \]
for any \( u \) which is a contact point. This means exactly
\[ \frac{\text{tr} T}{n} > c > \langle Tu, u \rangle \]
which contradicts Lemma 6.19.

7 Reverse isoperimetric inequality

Modulo affine transformations, among all convex bodies of a given volume in \( \mathbb{R}^n \), the \( n \)-dimensional simplex has “largest” surface area, while among centrally symmetric convex bodies, the cube is the extremal body.

**Theorem 7.1** (Ball). Let \( K \) be a convex body in \( \mathbb{R}^n \) and \( T \) a regular \( n \)-dimensional solid simplex. Then there is an affine image \( \tilde{K} \) of \( K \) satisfying
\[ \text{Vol}(\tilde{K}) = \text{Vol}(T) \quad \text{and} \quad \text{Vol}_{n-1}(\partial(\tilde{K})) \leq \text{Vol}_{n-1}(\partial(T)). \]
If \( K \) is a centrally symmetric convex body in \( \mathbb{R}^n \) and \( Q \) an \( n \)-dimensional cube then there is an affine image \( \tilde{K} \) of \( K \) satisfying
\[ \text{Vol}(\tilde{K}) = \text{Vol}(Q) \quad \text{and} \quad \text{Vol}_{n-1}(\partial(\tilde{K})) \leq \text{Vol}_{n-1}(\partial(Q)). \]

A main tool in the proof of the above theorem is the Brascamp-Lieb inequality. In Section ?? we state and prove its “normalized form” put forward by K. Ball together with its reverse form, which is due to Barthe.

**Theorem 7.2** (Ball). Let \( u_1, \ldots, u_m \in S^{n-1} \) and \( c_1, \ldots, c_m > 0 \) satisfy
\[ \text{Id} = \sum_{j=1}^m c_j u_j \otimes u_j. \]
If \( f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}^+ \) are measurable functions, then
\[ \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(\langle u_j, x \rangle) dx \leq \prod_{j=1}^m \left( \int_{\mathbb{R}} f_j(t) dt \right)^{c_j}. \]
### 7.1 Maximal volume ratio

**Definition 7.3.** The *volume ratio* of a convex body $K$ is defined to be

$$ vr(K) = \inf_{E \subseteq K} \left( \frac{\Vol_n(K)}{\Vol_n(E)} \right)^{1/n} $$

where the infimum is taken over all ellipsoids $E$ inside $K$.

**Theorem 7.4** (Ball). (i) Among centrally symmetric convex bodies in $\mathbb{R}^n$, the cube has the largest volume ratio.

(ii) Among convex bodies in $\mathbb{R}^n$, the simplex has the largest volume ratio.

**Proof.** (i) One has to show that if a centrally symmetric convex body $K$ is in John position then $\Vol_n(K) \leq 2^n$. But when a body is in John position, by Theorem 6.7 there exist contact points $u_j$ of it and $B_n^2$ and $c_j > 0$ such that

$$ \text{Id} = \sum_{j=1}^m c_j u_j \otimes u_j. $$

Since $K$ is symmetric, for every contact point $u_j$ of $K$ and $B_n^2$ we have that $-u_j$ is also a contact point. Thus the body is contained in the intersection of the strips $C := \{x : |\langle x, u_j \rangle| \leq 1\}$ (because for all contact points $u_j$ we have that $h_K(u_j) = 1$), and its volume is at most

$$ \Vol_n(C) = \int_{\mathbb{R}^n} \prod_{j=1}^m 1_{[-1,1]}(\langle x, u_j \rangle)^{c_j} dx \leq \prod_{j=1}^m \left( \int_{\mathbb{R}} 1_{[-1,1]} \right)^{c_j} = 2^n $$

by the Brascamp-Lieb inequality and the fact that $\sum_{j=1}^m c_j = n$.

(ii) Moving to the non-symmetric case, we note that the $n$-dimensional simplex that circumscribes $B_n^2$ has volume

$$ \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} $$

(it is easier to check this working in $\mathbb{R}^{n+1}$ on the hyperplane $\sum x_i = 1$, say). We thus need to establish this bound for the volume of a body in John position. Use John’s theorem in the non-symmetric case to find contact points $u_j$ and $c_j > 0$ such that $\sum_j c_j u_j = 0$ and $\text{Id} = \sum_j c_j u_j \otimes u_j$. As above, the body $K$ lies inside the possibly larger body

$$ L := \{x : \langle x, u_j \rangle \leq 1, i = j, \ldots, m\} $$

(this is a bounded set because $\sum_j c_j u_j = 0$), so it would suffice to bound this body’s volume by the desired bound.

We construct a new sequence of vectors $(v_i)_{i=1}^m$ in $\mathbb{R}^{n+1}$ which would be orthogonal in the extreme case where $K$ is a simplex. The estimate follows from an application of the Brascamp-Lieb inequality to a family of functions whose product is supported on a cone in $\mathbb{R}^{n+1}$ that has cross-sections similar to $K$: regard $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$ and for each $j$, let

$$ v_j = \sqrt{\frac{n}{n+1}} \left( -u_j, \frac{1}{\sqrt{n}} \right) \in \mathbb{R}^{n+1} \quad \text{and} \quad d_j = \frac{n+1}{n} c_j \in \mathbb{R}^+. $$

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Direct computation shows that
\[ \text{Id}_{n+1} = \sum_{j=1}^{m} d_j v_j \otimes v_j. \]

Take \( f_j = e^{-t}, t \geq 0 \). Apply the Brascamp-Lieb inequality to get
\[ \int_{\mathbb{R}^{n+1}} \prod_{j} f_j^{d_j} (\langle x, v_j \rangle) dx \leq \prod_{j=1}^{m} \left( \int_{\mathbb{R}} f_j \right)^{d_j} = 1, \]

because \( \int f_j = 1 \). Next, we compute the same integral over each hyperplane \( x_{n+1} = r \). One can check that the function \( \prod f_j^{d_j} (\langle x, v_j \rangle) \) is non-zero precisely when \( r \geq 0 \) and the point \( x \) is in \( \frac{r}{\sqrt{n}} L \times \{ r \} \) (given that each \( f_j \) is defined to be non-zero precisely at the non-negative \( t \in \mathbb{R} \)), and in that case it equals exactly \( e^{-r\sqrt{n+1}} \) (because we know that \( \sum_j c_j u_j = 0 \)). In other words,
\[ \int_{\{x_{n+1} = r\}} \prod_{j} f_j^{d_j} (\langle x, v_j \rangle) dx = e^{-r\sqrt{n+1}} \text{Vol}_n \left( \frac{r}{\sqrt{n}} L \right) = e^{-r\sqrt{n+1}} \left( \frac{r}{\sqrt{n}} \right)^n \text{Vol}_n(L). \]

Then, (19) implies
\[ 1 \geq \text{Vol}_n(L) \int_0^{\infty} e^{-r\sqrt{n+1}} \left( \frac{r}{\sqrt{n}} \right)^n dr \geq \frac{\text{Vol}_n(L) \cdot n!}{n^{n/2} (n+1)(n+1)/2}. \]

We conclude the proof noticing that \( \text{Vol}_n(K) \leq \text{Vol}_n(L) \) and that the upper bound for \( \text{Vol}_n(K) \) is no other than the number we had equality for in the case of the simplex. \( \square \)

### 7.2 Proof of the reverse isoperimetric inequality

It is perhaps surprising that in proving a statement about the maximal “minimal surface invariant”, we shall not use the minimal surface area position but rather the John position. Of course, in the case of the extremal bodies for the required inequality, these two positions coincide. We shall prove only the second part of Theorem 7.1, as the proof of the first part is identical.

**Proof of Theorem 7.1.** We have to find an affine image of a given body \( K \) such that
\[ \text{Vol}_{n-1}(\partial(\tilde{K})) \leq c_n \text{Vol}(\tilde{K})^{\frac{n-1}{n}} \]
where \( c_n \) is determined so that for the (say, side-length-1) cube there is equality, that is, \( c_n = 2n \). This position will be the John position of \( K \): indeed, for this position we have
\[ \text{Vol}_{n-1}(\partial(K)) = \lim_{t \to 0} \frac{\text{Vol}_n(K + tB_2^n) - \text{Vol}_n(K)}{t} \leq \lim_{t \to 0} \frac{\text{Vol}_n((1+t)K) - \text{Vol}_n(K)}{t} = n \text{Vol}_n(K) \leq 2n \text{Vol}(K)^{\frac{n-1}{n}}, \]
where we make use of Theorem 7.4 for the last inequality.

For the simplex, we again take the body $K$ in John-position so that now

$$\text{Vol}_{n-1}(\partial(K)) = \lim_{t \to 0} \frac{\text{Vol}_n(K + tB^n_2) - \text{Vol}_n(K)}{t} \leq \lim_{t \to 0} \frac{\text{Vol}_n((1 + t)K) - \text{Vol}_n(K)}{t} = n \text{Vol}_n(K) = n \text{Vol}(K)^{\frac{n-1}{n}} \text{Vol}_n(K)^{\frac{1}{n}} \leq \frac{n^{3/2}(n + 1)^{(n+1)/2n}}{(n!)^{1/n}} \text{Vol}(K)^{\frac{n-1}{n}} ,$$

where we make use of Theorem 7.4 for the last inequality. We should check that the simplex gives equality in this inequality, to this end we need to find the surface area of a simplex in John position, which is $(n+1)$ times the area of a simplex with side length which we computed in class; check.

7.3 Proof for Ball’s normalized form for the Brascamp-Lieb inequality and its inverse

We are now going to prove the Brascamp-Lieb theorem we used for the proof above, namely

**Theorem [Ball]** Let $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ satisfy

$$\text{Id} = \sum_{j=1}^m c_j u_j \otimes u_j.$$ 

If $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, then

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(\langle u_j, x \rangle) dx \leq \prod_{j=1}^m \left( \int_{\mathbb{R}} f_j(t) dt \right)^{c_j} .$$

As stated in class, the inequality is actually more general, and states that when you have some vectors $u_j$ and constants $p_j$ such that $\sum \frac{1}{p_j} = n$ then

$$\sup \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\langle u_j, x \rangle) dx}{\prod_{j=1}^m \left( \int_{\mathbb{R}} f_j^{p_j}(t) dt \right)^{1/p_j}}$$

is maximized for Gaussian functions $f_j$. The advantage of the normalization condition $\text{Id} = \sum_{j=1}^m c_j u_j \otimes u_j$ (where $c_j = 1/p_j$) is that the constant in the inequality, which is (always) attained by Gaussian functions, is 1.

Our method of proof will actually prove the theorem along with a reverse form, which we state in Section 7.3.3, and then prove both of them simultaneously. But before that, let examine some nice case and connect it to other things in life.

7.3.1 Young inequality

Recall the usual convolution of two bounded integrable functions $f, g : \mathbb{R} \to \mathbb{R}$ is

$$f * g(x) = \int f(y)g(x - y)dy.$$
Young’s inequality states that if \( f \in L^p \) and \( g \in L^q \) and \( p^{-1} + q^{-1} = 1 + s^{-1} \) then

\[
\| f * g \|_s \leq \| f \|_p \| g \|_q.
\]

It actually holds on all locally compact groups (so \( \mathbb{R} \) is just one case), and the proof is just a bunch of cleverly applied Hölder inequalities. When the group is actually compact, like the circle, constant functions give an example of equality; But in \( \mathbb{R} \), there is no equality, and a sharper constant may be inserted. It was Beckner, and also Brascamp and Lieb, who showed that in \( \mathbb{R} \) the best constant is attained by Gaussians. [The actual percise constant is \( A_p A_q A_r \) where \( A_p = p^{1/p} / (p')^{1/p'} \).]

How is this connected to our previous discussions? Easy. to find \( \| f * g \|_s \) we need to take

\[
\sup \{ \int (f * g) h : h \in L^r, \| h \|_r = 1 \}
\]

where \( r^{-1} + s^{-1} = 1 \) (so that \( L_r \) is the dual space to \( L_s \)). Restating Young inequality is simply saying that if \( p^{-1} + q^{-1} + r^{-1} = 2 \) then

\[
\int (f * g) h = \int \int f(y)g(x-y)h(x)dxdy \leq \| f \|_p \| g \|_q \| h \|_r.
\]

Define \( u_1 = e_2, u_2 = e_1 - e_2, u_3 = e_1 \) and this is very much what we wanted to talk about before. Of course, this is not a representation of identity exactly, but it is still in the form of the more general theorem stated above, and indeed the sharp constant in Young’s inequality is due to them.

We only use the normalized form in this text, so we stick to it from here onward.

### 7.3.2 Gaussian functions

We start with the technical fact, which is linear algebra. We quote here the Cauchy-Binet formula from linear algebra:

Let \( A \in M_{n \times m} \) and \( B \in M_{m \times n} \). For \( 1 \leq j_1, j_2, \ldots, j_n \leq m \) let \( A^{j_1 j_2 \ldots j_n} \) denote the \( n \times n \) matrix consisting of columns \( j_1, j_2, \ldots, j_n \) of \( A \) and \( B_{j_1 j_2 \ldots j_n} \) denote the \( n \times n \) matrix consisting of rows \( j_1, j_2, \ldots, j_n \) of \( B \). Then

\[
\det (AB) = \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq m} \det (A^{j_1 j_2 \ldots j_n}) \det (B_{j_1 j_2 \ldots j_n}).
\]

We shall use the formula as follows: Let \( \lambda_j > 0, j = 1, \ldots, m \). For every \( I \subseteq \{1, \ldots, m\} \) with cardinality \( |I| = n \) we define

\[
\lambda_I = \prod_{i \in I} \lambda_i \quad \text{and} \quad U_I = \left( \det \left( \sum_{j \in I} c_j u_j \otimes u_j \right) \right)^2.
\]

Then

**Lemma 7.5.** Under the above assumptions and notations, we have

\[
\det \left( \sum_{j=1}^m c_j \lambda_j u_j \otimes u_j \right) = \sum_{|I|=n} \lambda_I U_I.
\]
Proof. Let \( A \in M_{n \times m} \) have columns \( \lambda_jc_ju_j \) and let \( B \) have rows \( u_j \). Then

\[
ABx = \sum_{j=1}^{m} \lambda_jc_ju_j \langle u_j, x \rangle.
\]

In other words, \( AB = \left( \sum_{j=1}^{m} \lambda_j(\sqrt{c_j}u_j) \otimes (\sqrt{c_j}u_j) \right) \). By the Cauchy-Binet formula we have

\[
det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) = \sum_{|I|=n} \det \left( \sum_{j \in I} c_j \lambda_j u_j \otimes u_j \right).
\]

However, we now simply have a diagonal matrix multiplying a square one, so that

\[
det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) = \sum_{|I|=n} \lambda_I det U_I.
\]

Next we give show that when restricted to one dimensional Gaussian functions, the Brascamp-Lieb inequality holds with constant 1, and it is sharp (in other words, if we supermize the ratio in the Brascamp-Lieb theorem, over Gaussians, we get 1.

**Proposition 7.6.** Let \( u_1, \ldots, u_m \in S^{n-1} \) and \( c_1, \ldots, c_m > 0 \) satisfy \( Id = \sum_{j=1}^{m} c_j u_j \otimes u_j \). Then,

\[
\sup \left\{ \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j^{c_j}(\langle x, u_j \rangle) dx}{\prod_{j=1}^{m} (\int_{\mathbb{R}^n} g_j(t) dt)^{c_j}} : g_j(t) = e^{-\lambda_j t^2}, \lambda_j > 0 \right\}
= \inf \left\{ \frac{\det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)}{\prod_{j=1}^{m} \lambda_j^{c_j}} : \lambda_j > 0 \right\} = 1.
\]

**Proof.** Let \( g_j(t) = \exp(-\lambda_j t^2), j = 1, \ldots, m \), where \( \lambda_j \) are positive reals. Then,

\[
\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j^{c_j}(\langle x, u_j \rangle) dx = \int_{\mathbb{R}^n} \exp \left( -\sum_{j=1}^{m} c_j \lambda_j \langle x, u_j \rangle^2 \right) dx
= \int_{\mathbb{R}^n} \exp \left( -\left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)(x), x \right) \right) dx
= \pi^{n/2} \sqrt{\det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)}.
\]
On the other hand,
\[ \prod_{j=1}^{m} \left( \int_{\mathbb{R}} g_j \right)^{c_j} = \prod_{j=1}^{m} \left( \int_{\mathbb{R}} \exp(-\lambda_j t^2) dt \right)^{c_j} = \prod_{j=1}^{m} \left( \frac{\sqrt{\pi}}{\sqrt{\lambda_j}} \right)^{c_j} \]
\[ = \frac{\pi^{n/2}}{\sqrt{\prod_{j=1}^{m} \lambda_j^{c_j}}} \]
since \( c_1 + \cdots + c_m = n \).

It follows that
\[ \inf \left\{ \left( \frac{\prod_{j=1}^{m} \left( \int_{\mathbb{R}} g_j \right)^{c_j}}{\prod_{j=1}^{m} g_j^{c_j} (\langle x, u_j \rangle)} \right)^2 : g_j(t) = e^{-\lambda_j t^2}, \lambda_j > 0 \right\} = \inf \left\{ \det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) / \prod_{j=1}^{m} \lambda_j^{c_j} : \lambda_j > 0 \right\}. \]

We turn to showing that this constant is 1.

By Lemma 7.5 we know that letting
\[ \lambda_I = \prod_{i \in I} \lambda_j \text{ and } U_I = \left( \det \left( \sum_{j \in I} c_j u_j \otimes u_j \right) \right)^2, \]
one has
\[ \det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) = \sum_{|I|=n} \lambda_I U_I. \]

Setting \( \lambda_j = 1 \) in (20) we see that in particular
\[ \sum_{|I|=n} U_I = 1. \]

By the arithmetic-geometric means inequality,
\[ \sum_{|I|=n} \lambda_I U_I \geq \prod_{|I|=n} \lambda_I U_I = \prod_{j=1}^{m} \lambda_j^{\sum_{\{I:j \in I\}} U_I}. \]

Applying the Cauchy-Binet formula again (this time for all \( \lambda = 1 \) except the \( j^{th} \) which is 0, say), we have
\[ \sum_{\{I:j \in I\}} U_I = \sum_{|I|=n} U_I - \sum_{\{I:j \notin I\}} U_I = 1 - \det \left( \mathbb{I} - (\sqrt{c_j} u_j) \otimes (\sqrt{c_j} u_j) \right) \]
\[ = 1 - (1 - c_j |u_j|^2) = c_j. \]
Going back to (20) and (22) we see that

\[(23) \quad \det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) \geq \prod_{j=1}^{m} \lambda_j^{c_j} \]

and thus

\[\inf \left\{ \frac{\det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)}{\prod_{j=1}^{m} \lambda_j^{c_j}} : \lambda_j > 0 \right\} \geq 1.\]

The choice \( \lambda_j = 1 \) gives equality in (23), which completes the proof. \( \square \)

### 7.3.3 The reverse inequality, and proof of both

We set

\[I(f_1, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^c(\langle x, u_j \rangle) \, dx.\]

By considering the Gaussians, we have shown that,

\[(24) \quad \sup \left\{ I(f_1, \ldots, f_m) : \int_{\mathbb{R}} f_j = 1, \ j = 1, \ldots, m \right\} \geq 1.\]

The following is a reverse form of Theorem 7.2.

**Theorem 7.7** (Barthe). Let \( u_1, \ldots, u_m \in S^{n-1} \) and \( c_1, \ldots, c_m > 0 \) satisfy \( \text{Id} = \sum_{j=1}^{m} c_j u_j \otimes u_j \).

If \( h_1, \ldots, h_m : \mathbb{R} \rightarrow \mathbb{R}^+ \) are measurable functions, we set

\[K(h_1, \ldots, h_m) = \int_{\mathbb{R}^n} \sup \left\{ \prod_{j=1}^{m} h_j^c(\theta_j) : \theta_j \in \mathbb{R}, \ x = \sum_{j=1}^{m} \theta_j c_j u_j \right\} \, dx.\]

Then,

\[\inf \left\{ K(h_1, \ldots, h_m) : \int_{\mathbb{R}} h_j = 1, \ j = 1, \ldots, m \right\} = 1.\]

Before we continue, let us note that in dimension 1 this is Prékopa-Leindler. Indeed, we have a simple representation of identity by \( u_1 = e_1 \) and \( u_2 = -e_2 \) and \( c_i = 1/2 \). Then the function integrated is \( \sup \{ f(y)^{1/2} g(z)^{1/2} : x = (y - z)/2 \} \). It is compared with the (square root of - if there was normalization of integrals equal to 1) product of the integrals.

Again as a first step we check what happens when we test the inequality on centered Gaussian functions. This will give us the easy part of this reverse Brascamp-Lieb inequality.

**Proposition 7.8.** With the notation of Theorem 7.7 we have

\[\inf \left\{ K(h_1, \ldots, h_m) : \int_{\mathbb{R}} h_j = 1, \ j = 1, \ldots, m \right\} \leq 1.\]

**Proof.** Let \( \lambda_j > 0, \ j = 1, \ldots, m \) and consider the functions \( h_j(t) = \exp(-t^2/\lambda_j) \). Then, the function

\[m(x) := \sup \left\{ \prod_{j=1}^{m} h_j^c(\theta_j) : x = \sum_{j=1}^{m} \theta_j c_j u_j \right\}\]

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is given by

\[ m(x) = \exp \left( -\inf \left\{ \sum_{j=1}^{m} \frac{c_j \theta_j^2}{\lambda_j} : x = \sum_{j=1}^{m} \theta_j c_j u_j \right\} \right). \]

Note we claim that \( m \) is a quadratic form in \( x \) and in fact that it is given by \( \langle Bx, x \rangle \) where \( B = A^{-1} \) and \( A := \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \) (both are positive definite). To show this define

\[ \|x\|^2 = \sum_{j=1}^{m} c_j \lambda_j \langle x, u_j \rangle^2 = \langle Ax, x \rangle \]

and check that the dual norm is exactly

\[ \|y\|_*^2 = \inf \left\{ \sum_{j=1}^{m} \frac{c_j \theta_j^2}{\lambda_j} : y = \sum_{j=1}^{m} \theta_j c_j u_j \right\}. \]

You shall do this in the exercise sheet. Therefore,

\[ \|y\|_*^2 = \langle By, y \rangle, \]

where \( B = A^{-1} \). It follows that

\[ \int_{\mathbb{R}^n} m(x) dx = \frac{\pi^{n/2}}{\sqrt{\det B}} = \frac{\pi^{n/2}}{\sqrt{\det A}}. \]

On the other hand,

\[ \prod_{j=1}^{m} \left( \int_{\mathbb{R}} \exp(-t^2/\lambda_j) dt \right)^{c_j} = \pi^{n/2} \prod_{j=1}^{m} \lambda_j^{c_j/2}. \]

This shows that

\[ \inf \left\{ K^2(h_1, \ldots, h_m) : \int_{\mathbb{R}} h_j = 1 \right\} \leq \inf \left\{ \det \left( \sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right) : \lambda_j > 0 \right\} = 1 \]

and the proof is complete. \( \square \)

The main step in Barthe’s argument is the following proposition.

**Proposition 7.9.** Let \( f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}^+ \) and \( h_1, \ldots, h_m : \mathbb{R} \to \mathbb{R}^+ \) be integrable functions with

\[ \int_{\mathbb{R}} f_j(t) dt = \int_{\mathbb{R}} h_j(t) dt = 1, \quad j = 1, \ldots, m. \]

Then,

\[ I(f_1, \ldots, f_m) \leq K(h_1, \ldots, h_m). \]

**Proof.** We may assume that \( f_j, h_j \) are continuous and strictly positive. We use the transportation of measure idea that was used for the proof of the Prékopa-Leindler inequality: For every
\[ j = 1, \ldots, m \] we define \( T_j : \mathbb{R} \to \mathbb{R} \) by the equation
\[
\int_{-\infty}^{T_j(t)} h_j(s) \, ds = \int_{-\infty}^{t} f_j(s) \, ds.
\]
Then, each \( T_j \) is strictly increasing, 1-1 and onto, and
\[ T_j'(t)h_j(T_j(t)) = f_j(t), \quad t \in \mathbb{R}. \tag{25} \]
We now define \( W : \mathbb{R}^n \to \mathbb{R}^n \) by
\[ W(y) = \sum_{j=1}^{m} c_j T_j(\langle y, u_j \rangle)u_j. \tag{26} \]
To compute the differential of this map we must differentiate \( D(F(\langle x, u \rangle)u) \), and standard Calculus course tells us that this is \( F'((x, u))u \otimes u \). Thus
\[
D(W)(y) = \sum_{j=1}^{m} c_j T_j'(\langle y, u_j \rangle)u_j \otimes u_j.
\]
This implies
\[ \langle [D(W)(y)](v), v \rangle > 0 \quad \text{if} \quad v \neq 0 \]
and hence \( W \) is in particular injective. Consider the function
\[ m(x) = \sup \left\{ \prod_{j=1}^{m} h_j^j(\theta_j) : x = \sum_{j=1}^{m} \theta_j c_j u_j \right\}. \]
Then, (26) shows that
\[ m(W(y)) \geq \prod_{j=1}^{m} h_j^j(T_j(\langle y, u_j \rangle)) \]
for every \( y \in \mathbb{R}^n \). It follows that
\[
\int_{\mathbb{R}^n} m(x) \, dx \geq \int_{W(\mathbb{R}^n)} m(x) \, dx = \int_{\mathbb{R}^n} m(W(y)) \cdot |\det D(W)(y)| \, dy
\geq \int_{\mathbb{R}^n} \prod_{j=1}^{m} h_j^j(T_j(\langle y, u_j \rangle)) \det \left( \sum_{j=1}^{m} c_j T_j'(\langle y, u_j \rangle)u_j \otimes u_j \right) \, dy
\geq \int_{\mathbb{R}^n} \prod_{j=1}^{m} h_j^j(T_j(\langle y, u_j \rangle)) \prod_{j=1}^{m} (T_j'(\langle y, u_j \rangle))^c_j \, dy,
\]
where in the last inequality we have used Proposition 7.6. Therefore, taking (25) into account we have
\[
\int_{\mathbb{R}^n} m(x) \, dx \geq \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^c(\langle y, u_j \rangle) \, dy = I(f_1, \ldots, f_m).
\]
In other words, \( I(f_1, \ldots, f_m) \leq K(h_1, \ldots, h_m). \)

**Proof of Theorem 7.7 and Theorem 7.2.** Let \( f_1, \ldots, f_m \) and \( h_1, \ldots, h_m : \mathbb{R} \to \mathbb{R}^+ \) be integrable functions with

\[
\int_{\mathbb{R}} f_j(t)dt = \int_{\mathbb{R}} h_j(t)dt = 1, \quad j = 1, \ldots, m.
\]

Then,

\[ I(f_1, \ldots, f_m) \leq K(h_1, \ldots, h_m). \]

Taking the supremum over all such functions \( f_i \) and the infimum over all such functions \( h_i \) we get that

\[ 1 \leq \sup \{ I(f_1, \ldots, f_m) \} \leq \inf \{ K(h_1, \ldots, h_m) \} \leq 1, \]

so that there must be equality throughout. \( \square \)

### 8 Some remarks on Measure Concentration

Concentration of measure is a phenomenon of high dimensions which is responsible to many of the counter-intuitive results. Let me approach it from a very non-standard angle:

**Lemma 8.1 (Cauchy formula).** Let \( K \in \mathcal{K}^n \). Then

\[
\text{Vol}_{n-1}(\partial K) = n \frac{\kappa_n}{\kappa_{n-1}} \int_{S^{n-1}} \text{Vol}_{n-1}(P_{u \perp} K)d\sigma(u).
\]

Sometimes one uses integration on the sphere with respect to usual Lebesgue measure, the relation being \( d\sigma(u) = du/\text{Vol}_{n-1}(S^{n-1}) = du/(n\kappa_n) \) so we arrive here at the formula

\[
\text{Vol}_{n-1}(\partial K) = \frac{1}{\kappa_{n-1}} \int_{S^{n-1}} \text{Vol}_{n-1}(P_{u \perp} K)du
\]

which may be more familiar to you as “Cauchy formula”.

**Proof.** We work with a polytope \( P \) first. For a generic \( u \in S^{n-1} \), each facet \( F_i \) of \( P \) has some angle \( \theta_i \) between its normal and \( u \) which is not \( \pi/2 \). When such a facet is projected onto \( u^\perp \), its area when projected is \( |\cos(\theta)| \) times its original area. Clearly the projection \( P_{u^\perp}(P) \) is covered twice by the projections of the facets of \( P \). We get that

\[
\text{Vol}_{n-1}(P_{u^\perp}(P)) = \frac{1}{2} \sum_{i=1}^m |\cos(\theta_i)| \text{Vol}_{n-1}(P_i).
\]

When integrated over the sphere, we get

\[
\int_{S^{n-1}} \text{Vol}_{n-1}(P_{u^\perp}(P))d\sigma(u) = \frac{1}{2} \sum_{i=1}^m \text{Vol}_{n-1}(P_i) \int_{S^{n-1}} |\cos(\theta(n_i, u))|d\sigma(u).
\]

By rotation invariance, the latter integral does not depend on \( n_i \) and in simply a constant depending on the dimension (for example, let \( n = (1, 0, \ldots, 0) \), so that \( \cos(\theta) = u_1 \)). We have thus shown that there exists some constant \( c_n \) such that for any polytope \( P \) we have

\[
\int_{S^{n-1}} \text{Vol}_{n-1}(P_{u^\perp}(P))d\sigma(u) = c_n \text{Vol}_{n-1}(\partial P).
\]