3 Homework

Excercise 3.1. Let $K \subset \mathbb{R}^n$ be a convex body.

1. Show that if $K = -K$ then (trivially) $\text{Vol}(K) \leq \left(\frac{\text{diam}(K)}{2}\right)^n \kappa_n$.

2. Show (using Steiner symmetrizations etc) that also without the symmetry assumption $\text{Vol}(K) \leq \left(\frac{\text{diam}(K)}{2}\right)^n \kappa_n$.

Excercise 3.2. 1. Show that one cannot expect a reverse inequality to that of the previous exercise, namely there does not exist a $C_n$ such that for all convex $K$ in $\mathbb{R}^n$ we have $(\text{diam}(K))^n \leq C_n \text{Vol}(K)$.

2. Assume a body $K$ has constant width $w$, namely $h_K(u) + h_K(-u) = w$ for all $u \in S^{n-1}$.
   (i) Show that a body $K$ has constant width if and only if $K - K$ is a euclidean ball
   (ii) Show that the diameter of a body of constant width $w$ is $w$.
   (iii) Show that there exists $C_n$ such that for all bodies of constant width one has $(\text{diam}(K))^n \leq C_n \text{Vol}(K)$.

Excercise 3.3. Recall the definition of an $1/\alpha$-concave function (assume in cases where it is needed that $f$ is upper-semi-continuous.)

\[ f \left( (1 - \lambda)x_0 + \lambda x_1 \right) \geq \left( (1 - \lambda)f^{1/\alpha}(x_0) + \lambda f^{1/\alpha}(x_1) \right)^\alpha. \]

1. Show that if $\alpha > \beta > 0$ then a function which is $1/\alpha$-concave is $1/\beta$-concave.

2. Show that using the above definition in general for non-zero $\alpha$ we have that if $\alpha > \beta$ then a function which is $\beta$-concave is $\alpha$-concave.

3. Another name for $(-1)$-concave functions?

4. Show that an indicator function of a convex set is $\alpha$-concave for all $\alpha \in \mathbb{R}$ (including $\alpha < 0$) and that conversly, if a function is $\alpha$-concave for all $\alpha$ then it is constant on its support (call this $\infty$-concave). To this end compute $\lim_{\alpha \to \infty} \left( (1 - \lambda)f^{1/\alpha}(x_0) + \lambda f^{1/\alpha}(x_1) \right)^\alpha$

5. Show that if a function $f$ is $\alpha$-concave for all $\alpha < \alpha_0$ then it is $\alpha_0$-concave.

6. Call a function $f : \mathbb{R}^n \to [0, \infty)$ log-concave if $f = \exp(-\varphi)$ for a convex $\varphi$. Show that log-concave should be called $0$-concave, in that $\lim_{\alpha \to 0^+} \left( (1 - \lambda)f^{1/\alpha}(x_0) + \lambda f^{1/\alpha}(x_1) \right)^\alpha = f(x)^{(1-\lambda)} g(y)^\lambda$. In particular, show that log-concave functions are $\alpha$-concave for $\alpha < 0$ and that if $\alpha > 0$ and $f$ is $\alpha$-concave it is log-concave.

7. Compute $\lim_{\alpha \to -\infty} \left( (1 - \lambda)f^{1/\alpha}(x_0) + \lambda f^{1/\alpha}(x_1) \right)^\alpha$. What is the reasonable definition for $-\infty$-concave functions? Give a geometric description for these functions.

Excercise 3.4. Using Prékopa Leindler, show that if a measure $\mu$ on $\mathbb{R}^n$ has a density $f$ which is continuous on its support then $f$ is a log-concave function if and only if $\mu$ is a log-concave measure, namely it satisfies

\[ \mu \left( (1 - \lambda)A + \lambda B \right) \geq (1 - \lambda)\mu(A) + \lambda \mu(B). \]