1 Homework

1.1 Exercise

1. \(\Rightarrow\) \(\alpha K + \beta K = \{\alpha x + \beta y | x, y \in K\}. \) If \(\alpha = 0\) or \(\beta = 0\) the statement is trivially true. Otherwise, let \(\lambda = \frac{\alpha}{\alpha + \beta}\) and then:
\[
\alpha K + \beta K = \{(\alpha + \beta)(\lambda x + (1 - \lambda)y) | x, y \in K\} = (\alpha + \beta)(\lambda x + (1 - \lambda)y) \Rightarrow (\alpha + \beta)K
\]
\(\Leftarrow\) Choose \(\lambda \in [0,1]\) and then set \(\alpha = \lambda \beta = (1 - \lambda),\) and then get that for all \(x, y \in K,\)
\[
\lambda x + (1 - \lambda)y \in \lambda K + (1 - \lambda)K = K.
\]

2. \(\Rightarrow\) Is given by choosing \(\alpha = \beta = 1\) in (1.).
\(\Leftarrow\) Assume the contrary: \(\frac{1}{2}A + \frac{1}{2}A = A,\) but \(A\) is not convex. Then for some \(x_0, y_1 \in A,\)
the line \(l = l_{0,1} = \lambda x_0 + (1 - \lambda)y_1\) is not in \(A.\) By the assumption, \(z_\frac{1}{2} = \frac{1}{2}x_0 + \frac{1}{2}y_0 \in A,\)
and we now proceed by induction: Mark \(x_\frac{1}{2} = y_\frac{1}{2} = z_\frac{1}{2}\), and look at the lines
\(l_{0,\frac{1}{2}} = \lambda x_0 + (1 - \lambda)y_\frac{1}{2}, l_{\frac{1}{2},1} = \lambda x_\frac{1}{2} + (1 - \lambda)y_1,\) by the assumption, \(z_\frac{1}{2} = \frac{1}{2}x_0 + \frac{1}{2}y_\frac{1}{2} \in A,\)
\(z_\frac{1}{2} = \frac{1}{2}x_\frac{1}{2} + \frac{1}{2}y_0 \in A.\) We will proceed by induction and find that the infinite set of elements \(C =\{z_n | n \leq 2^n\} \subset A\) is dense in both \(A\) and \(l,\) and since \(A\) is a closed set, we can deduce that \(l = \overline{C} \subset A,\) a contradiction!

3. If \(A\) and \(B\) are convex, then for all \(\lambda \in [0,1]\) and for all \((a_1 + b_1), (a_2 + b_2) \in A + B\) we have \(\lambda(a_1 + b_1) + (1 - \lambda)(a_2 + b_2) = (\lambda a_1 + (1 - \lambda)a_2) + (\lambda b_1 + (1 - \lambda)b_2) \in A + B\) hence \(A + B\) is convex.

4. Example: \(A = \{(x, y) | x > 0, y \geq \frac{1}{2}\} \) and \(B = \{(x, y) | x = 0\} \) both are closed yet \(A + B = \{(x, y) | x > 0, \forall xy \geq \frac{1}{2}\} \neq \{(x, y) | x = 0\}\) is open.
An example does not exist for 2 compact sets. Let \(A, B\) as before, w.l.o.g \(A\) is compact.
Let \(\{x_n\}_{n=1}^\infty \subset A + B\) s.t. \(x_n \rightarrow x = a + b.\) For each \(x_n, x_n = a_n + b_n\) and for \(\{a_n\}_{n=1}^\infty\)
there is a subset \(\{a_n\}_{k=1}^\infty\) that converges to \(a \in A,\) by the compactness of \(A.\) Now
\(b_n = x_n - a_n \rightarrow x - a = b \in B.\) So, \(x = a + b \in A + B,\) hence \(A + B\) is closed.

1.2 Exercise

1. Remember that the orthogonal projection is a linear transformation, let \(x, y \in P_EK\) s.t. \(x = P_E(\tilde{x}) y = P_E(\tilde{y})\) for some \(\tilde{x}, \tilde{y} \in K\) and \(\lambda \in [0,1].\) Then \(\lambda x + (1 - \lambda)y = \lambda P_E(\tilde{x}) + (1 - \lambda)P_E(\tilde{y}) = P_E(\lambda x + (1 - \lambda)y) \in P_EK.
\]

2. False! Let \(K = S^{n-1},\) for all subspace \(E\) of \(K\) of dimensions \(m < n,\) \(P_EK\) is the convex ball \(B_0^n .\)

3. Any subspace is convex by its linearity, so we must show that the intersection between two convex bodies is convex: Let \(x, y \in K_1 \cap K_2,\) then \(\lambda x + (1 - \lambda)y \in K_1, K_2\) and so \(\lambda x + (1 - \lambda)y \in K_1 \cap K_2.
\]

4. True! Let \(x, y \in K,\) \(K\) is convex \(\iff\) \(\{\lambda x + (1 - \lambda)y | \lambda \in [0,1]\} \subset K.\) If we are given some \(m < n\) in the statement of the exercise, we can intersect it with any affine subspace of dimension \(m = 1,\) and get a convex intersection of dimension \(m = 1.\) So, it will suffice to show that the statement is true for \(m = 1.\)
Let \(K\) be a convex body, assume that for all one-dimensional affine subspaces of \(\mathbb{R},\)
\(K \cap (E + a)\) is convex. Let \(x, y \in K,\) and let \((E + a) = \{tx + (1 - t)y | t \in \mathbb{R}\}\) be
an affine subspace. $K \cap (E + a)$ is convex by the assumption, hence \( \{ \lambda x + (1 - \lambda)y | \lambda \in [0, 1] \} \subset K \cap (E + a) \subset K \).

1.3 Exercise

1. Positive: 
   \( p(x) = 0 \iff \inf \{ r | \frac{x}{r} \in K \} = 0 \iff x = 0 \)
   
   Scalar: 
   \( p(\lambda x) = \inf \{ r | \frac{\lambda x}{r} \in K \} = \inf \{ |\lambda| r | \frac{x}{r} \in K \} = |\lambda| p(x) \)
   
   \( \triangle \) inequality: Let \( x, y \in K \) and mark \( r_x = \inf \{ r | \frac{x}{r} \in K \} \) \( r_y = \inf \{ r | \frac{y}{r} \in K \} \) \( r = \inf \{ r | \frac{x+y}{r} \in K \} \). Now, by the symmetry and convexity of \( K \), we know that if for all \( \varepsilon > 0, r + \varepsilon > 0 \frac{x}{r} \in K \) then for all \( m \geq r + \varepsilon \frac{x}{m} \in K \) (since \( \frac{x}{m} \) is on the line \( [r, \frac{-x}{r}] \)), and so we get \( \frac{x+y}{r_1 \varepsilon + r_2 \varepsilon} \in K \) and \( \frac{r_1 \varepsilon + r_2 \varepsilon}{r_1 \varepsilon + r_2 \varepsilon} \in K \). By the property mentioned above we get the inequality: \( r + \varepsilon \leq r_1 + r_2 + \varepsilon \), so \( r \leq r_1 + r_2. \)

2. \( B \) is centrally symmetric: \( x \in B \iff \|x\| \leq 1 \iff | -1 ||x|| \leq 1 \iff \| -x \| \leq 1. \)

\( B \) is convex: Let \( x, y \in B \) so \( \|x\|, \|y\| \leq 1. \) \( \lambda x + (1 - \lambda)y \in B \iff \|\lambda x + (1 - \lambda)y\| \leq 1 \) but \( \|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = \lambda \|x\| + (1 - \lambda)\|y\| \leq \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1. \)

1.4 Exercise
1.5 Exercise

1. Let \( x, y \in \text{conv}(A), B = \{ B | A \subset B, B \text{ is convex} \} \) then \( \forall B \in B, x, y \in B \) and \( \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in B \) by convexity of \( B \), so \( \lambda x + (1 - \lambda)y \in \bigcap B = \text{conv}(A) \).

2. Mark \( C = \{ \sum_{i=1}^{m} \lambda_i x_i | 1 \leq m \in \mathbb{N}, \sum_{i=1}^{m} \lambda_i = 1, x_i \in A \text{ for } 1 \leq i \leq m \} \).

   \( \text{conv}(A) \subset C \): We will simply show it is convex. Let \( x = \sum_{i=1}^{m} \lambda_i x_i, y = \sum_{j=1}^{k} \lambda'_j y_j \), and choose \( \delta \in [0, 1] \), then \( \delta x + (1 - \delta)y = \sum_{i=1}^{m+k} \gamma_i z_i \) where \( z_i = x_i \gamma_i = \delta \lambda_i \) for \( 1 \leq i \leq m \) and \( z_i = y_i - m \gamma_i = \delta \lambda'_j - \gamma_i \) for \( m + 1 \leq i \leq k \). In this construction, \( \sum_{i=1}^{m+k} \gamma_i = 1 \), so \( C \) is convex and contains \( A \), therefore \( \text{conv}(A) \subset C \).

   \( C \subset \text{conv}(A) \): Mark \( C_m = \{ \sum_{i=1}^{m} \lambda_i x_i | \sum_{i=1}^{m} \lambda_i = 1, x_i \in A \text{ for } 1 \leq i \leq m \} \), and \( C = \bigcup_{m \in \mathbb{N}} C_m \), so it is enough to show \( C_m \subset \text{conv}(A) \), and we can do so by induction. Firstly \( m = 1 \), \( A \subset \text{conv}(A) \). For arbitrary \( m \) and a combination \( x = \sum_{i=1}^{m} \lambda_i x_i \) take \( \delta = \lambda_1 \). By the induction assumption, \( y = \sum_{i=2}^{m} \frac{\lambda_i}{\delta} x_i \in \text{conv}(A) \), so \( x = \delta x_1 + (1 - \delta)y \in \text{conv}(A) \).

3. \( \text{conv}(A + B) \subset \text{conv}(A) + \text{conv}(B) \): \( \text{conv}(A), \text{conv}(B) \) are both convex, and contain \( A, B \) respectively. Therefore, their Minkowski sum is convex and contains \( A + B \), therefore it must contain \( \text{conv}(A + B) \).

   \( \text{conv}(A) + \text{conv}(B) \subset \text{conv}(A + B) \): For \( x = \sum_{i=1}^{m} \lambda_i x_i, y = \sum_{j=1}^{k} \lambda'_j y_j \) notice \( x + y = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \lambda'_j (x_i + y_j) \in \text{conv}(A + B) \).

1.6 Exercise*

1. Radon’s theorem: Since \( \{ x_i | i = 1 \} \) are dependent, there exist nonzero \( \{ b_i | i = 1 \} \) such that \( \sum_{i=1}^{m} b_i x_i = 0 \), and also \( \sum_{i=1}^{m} b_i = 0 \). Partition \( m \) into two sets \( J = \{ i | b_i > 0 \} \) and \( I = \{ i | b_i \leq 0 \} \). Mark \( A = \sum_{j \in J} b_j = -\sum_{i \in I} b_i \), then the point \( x = \sum_{j \in J} b_j x_j = -\sum_{i \in I} b_i x_i \) is in both \( \text{conv}(\{ x_i | i \in I \}) \) and \( \text{conv}(\{ x_j | j \in J \}) \).

2. Carathéodory’s theorem: Let \( x \in \text{conv}(A) \), so \( x = \sum_{i=1}^{m} \lambda_i x_i \) and \( \sum_{i=1}^{m} \lambda_i = 1 \) for some group of integers \( \{ \lambda_i | i \in I \} \). If \( |I| = n + 1 \) then there is nothing to prove. Otherwise, look at the sets \( \{ A_\alpha | \alpha \in J \} : A_\alpha = \{ x_i | i \in I_\alpha \} \) that satisfy \( \sum_{i=1}^{m} \lambda_i x_i = x \) for some \( \sum_{i=1}^{m} \lambda_i = 1 \). By Axiom of choice we can choose one \( A_\alpha \) such that \( |I_\alpha| = \min B \). Assume that \( |\Lambda| > n + 1 \), then its elements are affinely dependent, hence there is a set \( \{ \gamma_i | i \in I_\alpha \} \) so that \( \sum_{i \in I_\alpha} \gamma_i x_i = 0 \) and \( \sum_{i \in I_\alpha} \gamma_i = 0 \). Take the minimal \( \frac{\lambda_j}{\gamma_j} \) and we have \( x = \sum_{i \in I_\alpha} \lambda_i x_i = \sum_{i \in I_\alpha} \lambda_i x_i - \frac{\lambda_j}{\gamma_j} \sum_{i \in I_\alpha} \gamma_i x_i = \sum_{i \in I_\alpha} (\lambda_i - \frac{\lambda_j}{\gamma_j} \gamma_i) x_i \). Now the \( j \)th coefficient is 0, and we find that \( \Lambda \setminus \{ x_j \} \) contains enough to span \( x \), which is a contradiction to the assumption that \( \Lambda \) has minimal size \( I_\alpha \).