CONVEX BODIES IN HIGH DIMENSIONS HWK 2

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Exercise 2.1

1. Let \( u \in \mathbb{R}^n \), and look at \( h_K(u) \). By Caratheodory theorem, every vector in \( K \) is a convex combination of \( n+1 \) elements of \( A \). Writing this down we have:

\[
h_K(u) = \sup_{k \in K} \langle k, u \rangle = \sup_{a_i \in A} \left\{ \sum_{i=1}^{n+1} \lambda_i a_i, u \right\} = \sup_{a_i \in A} \left\{ \sum_{i=1}^{n+1} \lambda_i \langle a_i, u \rangle \right\}
\]

\[
\leq \sum_{i=1}^{n+1} \lambda_i \sup_{a_i \in A} \langle a_i, u \rangle = \sum_{i=1}^{n+1} \lambda_i h_A(u) = h_A(u)
\]

where the inequality follows from "supremum of a sum is not greater than sum of suprema". The other side of the inequality follows since \( A \subset K \) and the supremum over a larger set can not be smaller.

2. We wish to show that \( K = \{ x : \langle x, u \rangle \leq h_K(u) \forall u \} \subset A \). It is clear that \( K \subset A \). In addition, if \( x \in \partial K \), there is a sequence \( x_n \rightarrow x \) where \( x_n \in K \) and from continuity of inner product, \( \langle x, u \rangle \) is not greater than the supremum over \( y \in K \), and so \( x \in A \). For the other inclusion, we can show \( \overline{K} \subset \partial A \). Let \( p \in \overline{K} \). By the separation lemma, there exists \( \alpha \in \mathbb{R} \), \( w \in \mathbb{R}^n \) such that \( \langle p, w \rangle < \alpha < \langle x, w \rangle \) for all \( x \in \overline{K} \). Alternatively, \( \langle p, -w \rangle > -\alpha > \langle x, -w \rangle \) , and so \( p \in \partial A \) and the result follows.

3. \( K \subset T \) means \( \sup_{x \in K} \langle x, u \rangle \leq \sup_{y \in T} \langle y, u \rangle \) for all \( u \in \mathbb{R}^n \) since the supremum is taken over a larger set, which implies \( h_K(u) \leq h_T(u) \).

Now, assume \( h_K(u) \leq h_T(u) \). Using questions (1),(2) repeatedly we can write:

\[
\text{conv}(K) = \{ x \langle x, u \rangle \leq \sup_{x \in K} \langle x, u \rangle \forall u \} = \{ x \langle x, u \rangle \leq h_K(u) \forall u \}
\]

\[
\subset \{ x \langle x, u \rangle \leq h_T(u) \forall u \} = \{ x \langle x, u \rangle \leq \sup_{x \in T} \langle x, u \rangle \forall u \} = \text{conv}(T)
\]

and the result is proven.

4. Since \( 0 \in \text{int}(K) \) we have a ball \( B_{2\varepsilon} \) around 0 in \( K \), and so for every \( u \in \mathbb{R}^n \), we can take a vector \( x \in B_{2\varepsilon} \) of length \( \varepsilon \) in the direction of \( u \). \( \langle x, u \rangle = |x||u| = \varepsilon|u| > 0 \) hence \( h_K(u) > 0 \) for all \( u \in \mathbb{R}^n \).

5. Let \( K \) be bounded. We have

\[
|h_k(u) - h_k(v)| = |\sup_{x \in K} \langle x, u \rangle - \sup_{x \in K} \langle x, v \rangle| = |\sup_{x \in K} \langle x, u - v + v \rangle - \sup_{x \in K} \langle x, v \rangle|
\]

\[
= |\sup_{x \in K} \{ \langle x, u - v \rangle + \langle x, v \rangle \} - \sup_{x \in K} \langle x, v \rangle|
\]

\[
\leq |\sup_{x \in K} \langle x, u - v \rangle + \sup_{x \in K} \langle x, v \rangle - \sup_{x \in K} \langle x, v \rangle|
\]

\[
= |\sup_{x \in K} \langle x, u - v \rangle| \leq |\sup_{x \in K} |x||u - v|| = |u - v| \sup_{x \in K} |x|
\]
Since $K$ is bounded, $\sup_{x \in K} |x|$ is some positive constant, and in fact $h_K$ is lipschitz with that constant.

6. First, we have $K_i \subset \bigcup K_i \subset \text{conv}(\bigcup K_i)$ and so $h_{K_i} \leq h_{\text{conv}(\bigcup K_i)}$ for all $i$, and so the inequality holds for the supremum.

For the other side we have, similarly to question (1):

$$h_{\text{conv}(\bigcup K_i)}(u) = \sup_{k \in \bigcup K_i} \sum_{j=1}^{m} \lambda_j \langle x_j, u \rangle \leq \sup_k \sup_{i} \langle x_i, u \rangle = \sup_i h_{K_i}$$

7. It is enough to describe $h_K$ on the unite sphere since it is positively homogeneous. Let $u \in S^{n-1}$. For $K = [-1,1]^n$, the maximum will be achieved by the sum $|u_1| + |u_2| + ... + |u_n|$ since for all other vectors in $K$, one coordinate is smaller then 1, and thus will decrease this sum. This sum can be achieved by the vector $x = (\text{sgn}(u_1), ..., \text{sgn}(u_n)) \in K$.

For $K = B^n_1$, using the Hölder inequality, we have $\sum_{j=1}^{m} |x_i u_i| \leq \|x\|_p \|u\|_q$ and equality holds if $x, u$ are linearly dependent and so $x$ can be chosen such that $h_K(u) = \|u\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Exercise 2.2

1. Assume $K \subset T$, then for all $x \in u^\perp$ we have $(x + Ru \cap K) \subset (x + Ru \cap T)$ and so the respective intervals centred around $x$ carry the same containment relations, meaning $S_u(K) \subset S_u(T)$.

2. Both radius arguments follow by observing that the symmetrization of a ball is a ball of the same radius, combined with (1.), hence a circumscribed ball still contains $K$ after symmetrization, and a ball contained in $K$ is also contained in $S_u(K)$ which shows both inequalities.

As for the diameter, we can reduce to the two dimensional case, and look at $k_1, k_2$ attaining the diameter of $K$. The Trapezoid created by taking the convex hull of $K \cap k_1 + Ru$ and $K \cap k_2 + Ru$ has a large diagonal which is the line between $k_1$ and $k_2$.

After symmetrization, the bases will be now centralized around a common vector, hence the large diagonal can only decrease in length, and so does the diameter of $S_u(K)$.

3. As shown in class, if $z = x + y$, where we can write $x = x_1 + \lambda u$ , $y = y_1 + \gamma u$ , such that $x_1, y_1 \in u^\perp$ and $\lambda \leq \frac{1}{2} |K \cap (x + Ru)|$ and similarly for $y$. To see the inclusion, we need to show that the length of the sums of these intersections is not greater than $(|K + T| \cap (x + y + Ru))$ but this is clearly the case since we have

$$(K + T) \cap (x + y + Ru) \subset (K \cap (x + Ru)) + (T \cap (y + Ru))$$

4. Assume $K_m \rightarrow K$ with non empty interior. Convergence in Hausdorff distance is equivalent to the fact that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > N$ it holds that $(1 - \varepsilon)K \subset K_m \subset (1 + \varepsilon)K$. Now, since $S_u(\lambda K) = \lambda S_u(K)$ (trivial by definition) , we get $(1 - \varepsilon)S_u(K) \subset S_u(K_m) \subset (1 + \varepsilon)S_u(K)$ and so $S_u$ is continuous with respect to the Hausdorff distance.

5. Take a series of vectors $v_n \rightarrow v$ that approach a vector perpendicular to $u$. In each step, $S_u(v_n) = u$ since the intersection has zero length. However $S_u(v) = u \cup v$ since now, the intersection is all of $v$. 
6. The previous example shows that the statement is not true, as the image of some \( v_n \) is \( u \cup v \) which is very far from the distance between \( v_n \) and \( v \) which can be as small as we wish.

If, however, \( T = B^n_2 \) the result is true. The distance to the unit ball is exactly \( \max\{R - 1, 1 - r\} \) where \( r, R \) are the inner and outer radii of \( K \). Since we proved that the symmetrization only decreases \( R \) and only increases \( r \), we have

\[
\delta^H(S_u(K), S_u(B^n_2)) \leq \delta^H(K, B^n_2).
\]

7. Let \( E \) be an ellipsoid. We claim that the middle points of the lines of the form \( l = E \cap (x + Ru) \) define a hyperplane. To prove this, we note that \( E = AB \), where \( A \) is a positive definite matrix, and \( B \) is the unit ball. Since linear transformations map the middle point of a segment to the middle point of the resulting image segment, we get that the respective middle points in the unit ball, which clearly form a hyperplane that goes through the centre of the ball, are mapped to the desired set of middle points in the ellipsoid, which in turn, is another hyperplane.

After establishing this fact, we now note that \( S_u(E) \) is, by definition, is what we obtain after aligning the middle points with \( u^\perp \), and so is some invertible, linear transformation. Since the image of an ellipsoid under a linear transformation is another ellipsoid, we get that \( S_u(E) \) is an ellipsoid.

**Exercise 2.3**

1. In the previous homework we showed that the Minkowski sum of two convex sets is again convex. Reflection of a convex set is again convex, and thus \( M_u(K) \) is a sum of two convex sets. To see that \( S_u M_u(K) = M_u(K) \) it is enough to show that \( M_u(K) \) is symmetric with respect to \( u \). This is evident by noting that \( R_u \in O(n) \) and so

\[
R_u(A + B) = R_u(A) + R_u(B)
\]

and then

\[
R_u(M_u(K)) = R_u \left( \frac{K + R_u(K)}{2} \right) = \frac{R_u(K) + K}{2} = M_u(K)
\]

2. Assume \( K \subset T \). It is of course true that \( R_u(K) \subset R_u(T) \) and since Minkowski sum also preserve the subset relation, we have \( K + R_u(K) \subset T + R_u(T) \), hence the desired inclusion.

3. We note the following fact:

let \( l \) be the line defined by \( K \cap (x + Ru) \), where \( x \in u^\perp \), and let \( R_u(l) \) be it’s reflection through \( u \). Then \( l + R_u(l) \) is exactly the line segment with centre on \( u \) and with length \( ||l|| \), and so, restricting the addition to be made only on these cuts, we can create \( S_u(K) \), which means \( S_u(K) \subset M_u(K) \).

4. Due to linearity of \( R_u \), and commutativity of the Minkowski sum we can simply write

\[
M_u(K) + M_u(T) = \frac{K + R_u(K)}{2} + \frac{T + R_u(T)}{2} = \frac{(K+T) + R_u(K+T)}{2} = M_u(K + T)
\]

5. We’ll use the following version of the Brunn–Minkowski Inequality:

\[
Vol(\lambda A + (1 - \lambda)B)^\frac{1}{2} \geq \lambda Vol(A)^\frac{1}{2} + (1 - \lambda)Vol(B)^\frac{1}{2}
\]

Plugging in \( \lambda = \frac{1}{2} \), \( A = K \), \( B = R_u(K) \) and using the fact that \( Vol(K) = Vol(R_u(K)) \) we get \( Vol(M_u(K)) \geq Vol(K) \), as desired.
6. Reflection is just a linear transformation, hence continuous. Minkowski sum is continuous as if \((1-\varepsilon)X \subset Y \subset (1+\varepsilon)X\) we have, using properties of Minkowski sums of convex sets:

\[
X + Y \subset (1+\varepsilon)X \subset (1+\varepsilon)X + (Y + \varepsilon X) = (X + Y) + 2\varepsilon X
\]

and the other inclusion follows similarly. Choosing appropriate \(\varepsilon\) shows Minkowski Sums of convex sets is continuous, and so composition of reflection and sum is also continuous, which is exactly \(M_u(K)\).

7. Similar to what we did in class, we consider the class \(\mathcal{S}\) of all bodies obtained by successive symmetrizations of the original body \(K\). Denote \(R_0\) to be the infimum of the circumradii of bodies in the family \(\mathcal{S}\). Let \(K_n\) be a sequence whose circumradii converge to \(R_0\). There exists a convergent subsequence, by Blaschke’s Selection Theorem, denoted \(K_j \to L\). \(R(L) = R_0\). Assume that \(L\) is not a ball, then it is included in \(R_0B_2^3\), but misses some cap of radius \(\delta > 0\). By compactness, we can cover the boundary of \(R_0B_2^3\) with finitely many caps of the same radius by reflecting it through some hyperplanes. Now, if we use the Minkowski symmetrization through these planes, we will get a new body in \(\mathcal{S}\) which has a smaller circumradius. This is evident by noting that the Minkowski sum of a ball without two caps is a ball of double the radius which is missing the respective caps. After dividing by two, we get the original ball with missing cap of radius \(\frac{\delta}{2}\). This contradicts the definition of \(R_0\) since we can take \(K_j\) for large enough \(j\) such that after these symmetrizations we will get a body \(K'_j \in \mathcal{S}\) with a smaller circumradius. This shows that we can take a finite sequence of symmetrizations that brings us \(\varepsilon\) -close to a ball. Defining the sequence inductively with \(\varepsilon \to 0\) we can construct the desired sequence of symmetrizations.

8. False. Take a series of ellipsoids in \(\mathbb{R}^2\), \(E_n\) that tends to a line defined by \([(-1,-1),(1,1)]\). From continuity, if we reflect by the perpendicular line defined by \([(-1,1),(1,-1)]\), we get the unit square, however if the symmetrization in every step were also an ellipsoid we would get a new series of ellipsoids \(E'_n\) that tends to a unit square, which is impossible.