Higgs-Coulomb correspondence for abelian gauged linear sigma models

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1. Gauged linear sigma models (GLSMs)

The input data of a gauged linear sigma model (GLSM) is a 5-tuple $(V, G, \mathbb{C}_R^*, W, \omega)$

(1) (linear space) $V = \mathrm{Spec}\mathbb{C}[x_1, \ldots, x_m] \simeq \mathbb{C}^m$

(2) (gauge group) $G \subset GL(V) \simeq GL_m(\mathbb{C})$ linear reductive

(3) (R symmetries) $\mathbb{C}_R^* \subset GL(V)$, $\mathbb{C}_R^* \cong \mathbb{C}^*$.

$G, \mathbb{C}_R^*$ commute, $G \cap \mathbb{C}_R^* = \langle J \rangle = \mu_r$

$\mathbb{C}_R^*$ acts on V by weights $c_1, \ldots, c_m \in \mathbb{Z}$, R charges $q_j = \frac{2c_j}{r}$

(4) (superpotential) $W \in \mathbb{C}[x_1, \ldots, x_m]$

- G-invariant: $W(g \cdot x) = W(x) \, \forall g \in G \Leftrightarrow W \in \mathbb{C}[x_1, \ldots, x_m]^G$
- quasi-homogeneous: $W(t \cdot x) = t^r W(x) \, \forall t \in \mathbb{C}_R^*$

(5) (stability condition) $\omega \in \mathrm{Hom}(G, \mathbb{C}^*) \Leftrightarrow G$-linearization on V

assumption: $V_G^{ss}(\omega) = V_G^s(\omega)$

$$\mathcal{X}_\omega = [V_G^{ss}(\omega)/G] \quad \text{smooth DM stack}$$

$$\downarrow$$

$$\mathbb{C}_w^* \curvearrowright X_\omega = V_G^{ss}(\omega)/G = V/\!/_\omega G \quad \text{GIT quotient}$$

$$:= \mathbb{C}_R^*/\langle J \rangle \quad \downarrow \text{projective} \qquad w(t \cdot [x]) = tw([x]), t \in \mathbb{C}_w^*, [x] \in X_\omega$$

$$X_0 = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_m]^G) \xrightarrow{w} \mathbb{C}$$

A GLSM is abelian if the gauge group G is abelian
In most of this talk, $G = (\mathbb{C}^*)^\kappa$.
We have a short exact sequence of abelian groups (let $n = m - \kappa$)

$$1 \to G \xrightarrow{(D_1,\dots,D_{n+\kappa})} \widetilde{T} \simeq (\mathbb{C}^*)^{n+\kappa} \longrightarrow T \simeq (\mathbb{C}^*)^n \to 1$$

$$\cap \text{maximal torus}$$

$$GL_{n+\kappa}(\mathbb{C})$$

where $D_j \in \mathrm{Hom}(G, \mathbb{C}^*) = \mathbb{L}^\vee \simeq \mathbb{Z}^\kappa$. Then

- $\mathcal{X}_\omega$ is a smooth toric DM stack (Borisov-Chen-Smith)
- $X_\omega = V/\!/_\omega G$ is a semiprojective simplicial toric variety
- $\mathcal{X}_\omega = [\mu^{-1}(\omega)/G_\mathbb{R}]$ where $G_\mathbb{R} = U(1)^\kappa \subset G = (\mathbb{C}^*)^\kappa$, and $\mu : V = \mathbb{C}^{n+\kappa} \to \mathrm{Lie}(G_\mathbb{R}) \simeq \mathbb{L}^\vee_\mathbb{R} := \mathbb{L}^\vee \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^\kappa$ is the moment map of Hamiltonian $G_\mathbb{R}$-action on $\mathbb{C}^{n+\kappa}$.
- $\omega \in \mathbb{L}^\vee_\mathbb{R} \simeq \mathbb{R}^\kappa \supset$ secondary fan

Example: quintic

$$V = \mathbb{C}^6 = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_5,p], \quad G = \mathbb{C}^*, \quad \omega \in \mathbb{R} - \{0\}$$

gauge charges — G acts by weights $(1,\dots,1,-5)$
R charges — $\mathbb{C}^*_R$ acts by weights $(0,\dots,0,1)$
$\Big\}\ G \cap \mathbb{C}^*_R = \{1\}$

superpotential $\quad W = p(x_1^5 + \cdots + x_5^5) = pW_5(x)$

- $\omega > 0$: Calabi-Yau (CY)/geometric phase

$$\mathcal{X}_\omega = \big((\mathbb{C}^5 - \{0\})\times\mathbb{C}\big)/G = K_{\mathbb{P}^4}$$

$$\operatorname{Crit}(w) = \{W_5(x) = p = 0\} = X_5 \quad \text{Fermat quintic}$$

$$\subset \{p = 0\} = \mathbb{P}^4$$

GLSM invariants = Gromov-Witten (GW) invariants of X_5

- $\omega < 0$: Landau-Ginzburg (LG) phase

$$\mathcal{X}_\omega = \big[\big(\mathbb{C}^5\times(\mathbb{C} - \{0\})\big)/\mathbb{C}^*\big] = [\mathbb{C}^5/\mu_5]$$

$$\operatorname{Crit}(w)_{\mathrm{red}} = [0/\mu_5] \simeq B\mu_5$$

GLSM invariants = Fan-Jarvis-Ruan-Witten (FJRW) invariants of (W_5, μ_5)

Chiodo-Ruan (2008) LG/CY correspondence for quintic 3-folds:

GW invariants of $X_5 \longleftrightarrow$ FJRW invariants of (W_5, μ_5)

(1) (ϵ-wall-crossing) Givental style mirror theorems

- CY phase (Givental, Lian-Liu-Yau 1996-7):
  $J_+ = \frac{I_+}{I_+^0}$ under the mirror map
- LG phase (Chiodo-Ruan 2008): $J_- = \frac{I_-}{I_-^0}$ under the mirror map

$I_\pm$, $J_\pm$ are functions of 1 variable

take values in a 4-dimensional complex symplectic space

$$H(z)_\pm = zH_\pm^0 \oplus H_\pm^2 \oplus \frac{1}{z}H_\pm^4 \oplus \frac{1}{z^2}H_\pm^6$$

(2) (ω-wall-crossing) I_+ and I_- are related by analytic continuation and a $\mathbb{C}$-linear symplectic isomorphism $\phi : H(z)_+ \to H(z)_- \in Sp_4(\mathbb{C})$

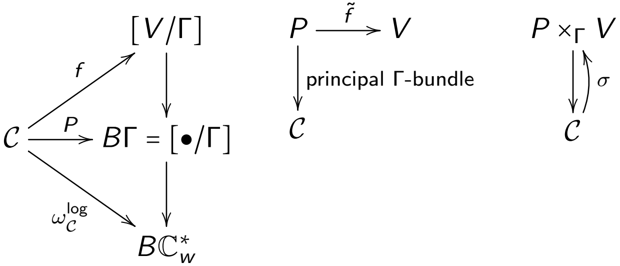
Question: ϵ-wall-crossing and ω-wall-crossing for general GLSM

2. Higgs branch

- Fan-Jarvis-Ruan, "A mathematical theory of the gauged linear sigma model" 2015, 2018, 2020
- Favero-Kim, "General GLSM invariants and their cohomological field theories," 2020
  - Polischuk-Vaintrob: affine LG models
  - Ciocan-Fontanine-Favero-Guéré-Kim-Shoemaker: convex hybrid models

Let $(V, G, \mathbb{C}^*_R, W, \omega)$ be the input data of a general GLSM, and let $\Gamma \subset GL(V)$ be the subgroup generated by G and $\mathbb{C}^*_R$ $\Rightarrow \Gamma/G = \mathbb{C}^*_R/\langle J \rangle = \mathbb{C}^*_w$.

GLSM invariants are virtual counts of LG quasimaps, which are birational maps from genus-g ℓ-pointed orbicurves $(\mathcal{C}, z_1, \ldots, z_\ell)$ to $[V^{ss}_G(\omega)/\Gamma]$ which extends to a morphism $f : \mathcal{C} \to [V/\Gamma]$ and satisfies certain stability conditions; enumerative geometry of $\mathrm{Crit}(w) \subset \mathcal{X}_\omega$



If $G = (\mathbb{C}^*)^\kappa$ then

$$\begin{array}{ccccccccc} & & & & & & \mathbb{C}^*_R & & \\ & & & & & \swarrow & \downarrow & & \\ 1 & \longrightarrow & G = (\mathbb{C}^*)^\kappa & \longrightarrow & \Gamma = (\mathbb{C}^*)^{\kappa+1} & \longrightarrow & \mathbb{C}^*_w & \longrightarrow & 1 \end{array}$$

$$H_2([V/\Gamma];\mathbb{Q}) = H_2(B\Gamma;\mathbb{Q}) = \mathbb{L}_{\mathbb{Q}} \oplus \mathbb{Q} \ni \deg f = (\beta, 2g-2+\ell)$$

$$P \times_\Gamma V = \bigoplus_{i=1}^{n+\kappa} \mathcal{L}_i, \quad \deg \mathcal{L}_i = \langle D_i, \beta \rangle + \frac{q_i}{2}(2g-2+\ell).$$

(Note that $\frac{q_i}{2} = \frac{c_i}{r}$ is the weight of the $\mathbb{C}^*_w$-action on x_i.)

$$\mathcal{X}_\omega = \bigcup_{I \in \mathcal{A}^{\min}_\omega} \mathcal{X}_I,$$ where $\mathcal{A}^{\min}_\omega$ is the set of minimal anticones,

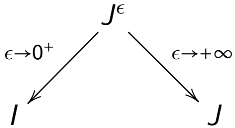
$I \subset \{1, \dots, n+\kappa\}$, $|I| = \kappa$, $\bar{I} := \{1, \dots, n+\kappa\} \setminus I$,

$$\mathcal{X}_I = \left[\left(\mathbb{C}^{\bar{I}} \times (\mathbb{C}^*)^I\right)/G\right] \simeq [\mathbb{C}^n/G_I] \supset p_I = [\{0\}/G_I] = BG_I$$

effective classes for $(g,\ell) = (0,1)$: $\mathbb{K}^\omega = \bigcup_{I \in \mathcal{A}^{\min}_\omega} \mathbb{K}^I$, where

$$\mathbb{K}^I = \{\beta \in \mathbb{L}_{\mathbb{Q}} : \deg \mathcal{L}_i = \langle D_i, \beta \rangle - q_i/2 \in \mathbb{Z}_{\geq 0} \ \forall i \in I\}.$$

D. Cheong, I. Ciocan-Fontanine, and B. Kim, "Orbifold Quasimap Theory": ϵ-stable quasimaps to $\mathcal{X}_\omega = [V_G^{ss}(\omega)/G]$, $\epsilon \in \mathbb{Q}_{>0}$.



quasimap wall-crossing (ϵ-wall-crossing)

$\Rightarrow$ Givental style mirror theorems

$\Rightarrow$ mirror theorem for smooth toric DM stacks (Coates-Corti-Iritani-Tseng)

Y. Zhou: quasimap wall-crossing in orbifold quasimap theory in all genera in full generality

It is expected that Y. Zhou's proof is generalizable to GLSM

$\Rightarrow$ Givental style mirror theorems for all GLSM in all phases

Clader-Janda-Ruan, "Higher-genus wall-crossing in the gauged linear sigma model", with an appendix by Y. Zhou: GLSM for complete intersections in weighted projective spaces

In orbifold quasimap theory, I-function is obtained by torus localization on stacky loop space (orbifold version of Givental's toric map spaces). We will consider the GLSM version.

The domain is $(\mathbb{P}[a,1], \infty = [1,0])$ where $a \in \mathbb{Z}_{>0}$, $(g,\ell) = (0,1)$.
M. Shoemaker "Towards a mirror theorem for GLSMs" $(g,\ell) = (0,2)$.
For $p = 0,1$, $a \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}$,

$$H^p\Big(\mathbb{P}^1, \mathcal{O}(m/a)\Big) := H^p\Big(\mathbb{P}[a,1], \mathcal{O}_{\mathbb{P}[a,1]}(m)\Big)$$

Given an effective class $\beta \in \mathbb{K}^\omega$,

$$V_\beta = \bigoplus_{i=1}^{n+\kappa} H^0\Big(\mathbb{P}^1, \mathcal{O}(\langle D_i, \beta\rangle - q_i/2)\Big), \quad W_\beta = \bigoplus_{i=1}^{n+\kappa} H^1\Big(\mathbb{P}^1, \mathcal{O}(\langle D_i, \beta\rangle - q_i/2)\Big)$$

degree β stacky loop space $\mathcal{X}_{\beta,\omega} = [V_\beta^{ss}(\omega)/G]$

degree β obstruction bundle $Ob_\beta = [\Big(V_\beta^{ss}(\omega) \times W_\beta\Big)/G]$

$\mathbb{C}_q^*$ rotates $\mathbb{P}^1$, $\widetilde{T}$ and $\mathbb{C}_q^*$ act linearly on V_β, W_β.
$\widetilde{T} \times \mathbb{C}_q^*$ acts on the smooth toric DM stack $\mathcal{X}_{\beta,\omega}$.
Ob_β is a $\widetilde{T} \times \mathbb{C}_q^*$-equivariant vector bundle over $\mathcal{X}_{\beta,\omega}$.
Caution: the superpotential W is not invariant under the $\widetilde{T}$-action

Following Okounkov, $\mathcal{X}^\circ_{\beta,\omega} = [V^\circ_\beta/G] \subset \mathcal{X}_{\beta,\omega} = [V^{ss}_\beta(\omega)/G]$ is the open substack such that the evaluation at ∞ is defined:

$$\mathrm{ev}_\infty : \mathcal{X}^\circ_{\beta,\omega} \longrightarrow \mathcal{X}_{\omega,v(\beta)}$$

where $\mathcal{X}_{\omega,v(\beta)}$ is a connected component of the inertia stack

$$I\mathcal{X}_\omega = \bigsqcup_{v\in\mathrm{Box}} \mathcal{X}_{\omega,v}, \quad \mathcal{X}_{\omega,v} = [V^{ss}_G(V)^{g(v)}/G].$$

$$\iota_{\beta\to v(\beta)} : \mathcal{F}_{\beta,\omega} := (\mathcal{X}^\circ_{\beta,\omega})^{\mathbb{C}^*_q} \hookrightarrow \mathcal{X}_{\omega,v(\beta)}.$$

Using the 4-tuple $(V, G, \mathbb{C}^*_R, \omega)$ and action of the diagonal torus $\widetilde{T} \subset GL(V)$, we define

$\widetilde{T}$-equivariant I-function $\quad I_{\widetilde{T}}(y,z) := \sum_{v\in\mathrm{Box}} I_{\widetilde{T},v}(y,z)\mathbf{1}_v$

where $I_{\widetilde{T},v}(y,z)$ takes values in $H^*_{\widetilde{T}}(\mathcal{X}_{\omega,v})$.

$$I_{\widetilde{T},v}(y,z) = e^{(\sum_{a=1}^{\kappa} \log y_a i_v^* p_a)/z} \sum_{\substack{\beta\in\mathbb{K}^\omega \\ v(\beta)=v}} y^\beta (\iota_{\beta\to v(\beta)})_* \left(\frac{1}{e_{\widetilde{T}\times\mathbb{C}^*_q}(N^{\mathrm{vir}}_\beta)} \right)$$

where $i_v : \mathcal{X}_{\omega,v} \hookrightarrow \mathcal{X}_\omega$, $p_a \in H^2_{\widetilde{T}}(\mathcal{X}_\omega)$, $N^{\mathrm{vir}}_\beta = N^{\mathrm{vir}}_{\mathcal{F}_\beta/\mathcal{X}^\circ_{\beta,\omega}}$.

Given $\mathcal{B} \in Coh_{\widetilde{T}}(\mathcal{X}_\omega)$, $[\mathcal{B}] \in K_{\widetilde{T}}(\mathcal{X}_\omega)$, define
$\widetilde{T}$-equivariant central charge

$$Z_{\widetilde{T}}([\mathcal{B}]) = \langle I_{\widetilde{T}}, \hat{\Gamma}_{\widetilde{T}} \mathrm{ch}_{\widetilde{T}}([\mathcal{B}]) \rangle = \sum_{I \in \mathcal{A}_\omega^{\min}} Z^I_{\widetilde{T}}([\mathcal{B}])$$

where $\hat{\Gamma}_{\widetilde{T}} \mathrm{ch}_{\widetilde{T}}([\mathcal{B}]) \in \bigoplus_{v \in \mathrm{Box}} H^*_{\widetilde{T}}(\mathcal{X}_{\omega,v}) \otimes_{R_{\widetilde{T}}} R_{\widetilde{T}}((z^{-1}))$,
$R_{\widetilde{T}} = H^*_{\widetilde{T}}(\bullet) = \mathbb{C}[\lambda_1, \dots, \lambda_{n+\kappa}]$.
Explicit formula for $Z^I_{\widetilde{T}}([\mathcal{B}])$ = contribution from $p_I = BG_I$.

Using the 5-tuple $(V, G, \mathbb{C}^*_R, W, \omega)$, define

$$\text{GLSM } I\text{-function} \quad I_w(y,z) = \sum_{v \in \mathrm{Box}} I_{w,v}(y,z) \mathbf{1}_v$$

where $I_{w,v}(y,z)$ takes values in $H_{w,v} = H^*(\mathcal{X}_{\omega,v}, \mathrm{Re}(i_v^* w_v) \gg 0)$.
Given $\mathcal{B} \in MF(\mathcal{X}_\omega, w)$, $[\mathcal{B}] \in K(MF(\mathcal{X}_\omega, w))$, define

$$\text{GLSM central charge} \quad Z_w([\mathcal{B}]) = \langle I_w, \hat{\Gamma}_w \mathrm{ch}_w([\mathcal{B}]) \rangle$$

where $\hat{\Gamma}_w \mathrm{ch}_w \in \bigoplus_{v \in \mathrm{Box}} H_{w,v} \otimes_{\mathbb{C}} \mathbb{C}((z^{-1}))$.

$K(MF(\mathcal{X}_\omega, w))$ is a module over the ring $K(\mathcal{X}_\omega)$, and there is a morphism of $K(\mathcal{X}_\omega)$-modules

$$\psi : K(MF(\mathcal{X}_\omega, w)) \to K(\mathcal{X}_\omega)$$

whose image is an ideal.
Any G character $t \in \mathbb{L}^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ defines a line bundle $\mathcal{L}_t$ on $[V/G]$. Let $\mathcal{L}_t^{\widetilde{T}}$ be the $\widetilde{T}$-equivariant line bundle over $[V/G]$ with the total space $[(V \times \mathbb{C})/G]$, where G acts on $\mathbb{C}$ by the character t and $\widetilde{T}$ acts trivially on $\mathbb{C}$.

$$\phi : K(\mathcal{X}_\omega) \to K_{\widetilde{T}}(\mathcal{X}_\omega), \quad \mathcal{L}_t \mapsto \mathcal{L}_t^{\widetilde{T}}.$$

If $G \subset SL(V)$ (Calabi-Yau) and there is a LG phase (e.g. Fermat Calabi-Yau hypersurfaces in finite quotients of weighted projective spaces) then

$$Z_w([\mathcal{B}]) = Z_{\widetilde{T}}(\phi \circ \psi([\mathcal{B}]))\Big|_{\lambda_j=0}.$$

3. Coulomb branch

(motivated by arXiv:1308.2438 by K. Hori and M. Romo)

Consider an abelian gauged linear sigma model $(V, G, \mathbb{C}^*_R, W, \omega)$
where $G \simeq (\mathbb{C}^*)^\kappa \subset SL(V)$ (Calabi-Yau)
$\theta = \omega + 2\pi\sqrt{-1}B \in \mathbb{L}^\vee_\mathbb{C}$ complexified/stringy Kähler class
ω = (extended) Kähler class, B = B-field
$\alpha = (\alpha_1, \ldots, \alpha_{n+\kappa}) \in \mathbb{R}^{n+\kappa}$, $\delta \in \mathbb{L}_\mathbb{R}$, $\langle D_i, \delta \rangle + \alpha_i > 0$ for $1 \le i \le n + \kappa$
Given $\mathcal{B} \in MF([V/G], w)$, define the
(α-perturbed) hemisphere/disk partition function

$$Z_{D^2}([\mathcal{B}]) = \frac{1}{(2\pi\sqrt{-1})^\kappa} \int_{\delta+\sqrt{-1}\mathbb{L}_\mathbb{R}} d\sigma \Gamma(\sigma) \mathrm{ch}[\mathcal{B}](\sigma) e^{\langle \theta, \sigma \rangle}$$

where $\Gamma(\sigma) = \prod_{i=1}^{n+\kappa} \Gamma(\langle D_i, \sigma \rangle + \alpha_i)$, and

$$\mathrm{ch}[\mathcal{B}](\sigma) = \sum_{t \in \mathbb{L}^\vee} c_t e^{2\pi\sqrt{-1}\langle t, \sigma \rangle} \quad \text{if } \psi([\mathcal{B}]) = \sum_{t \in \mathbb{L}^\vee} c_t \mathcal{L}_t \in K([V/G]).$$

- $Z_{D^2}([\mathcal{B}])$ is a multidimensional inverse Mellin transform of $\Gamma(\sigma)\mathrm{ch}[\mathcal{B}](\sigma)$.
- (R-wall-crossing) $\begin{cases} \alpha_i \to 0 : & \text{without superpotential} \\ \alpha_i \to q_i/2 : & \text{with superpotential} \end{cases}$

Proposition

There is an open subset $U \subset \mathbb{L}_{\mathbb{R}}^{\vee}$ such that

$$Z_{D^2}(\mathcal{L}_t) = \frac{1}{(2\pi\sqrt{-1})^{\kappa}} \int_{\delta+\sqrt{-1}\mathbb{L}_{\mathbb{R}}} d\sigma \Gamma(\sigma) e^{\langle \theta + 2\pi\sqrt{-1}t, \sigma \rangle}$$

is an analytic function in θ on

$$\{\theta = \omega + 2\pi\sqrt{-1}B \mid \omega \in \mathbb{L}_{\mathbb{R}}^{\vee}, B + t \in U\}.$$

Theorem 1 (Aleshkin-L)

Let C be a phase of the GLSM (i.e. C is the interior of a κ-dim'l cone in the secondary fan in $\mathbb{L}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}^{\kappa}$), and let $\omega_0 \in C$.

$\Rightarrow C = \bigcap_{I \in \mathcal{A}_{\omega_0}^{\min}} \angle_I \subset \mathbb{L}_{\mathbb{R}}^{\vee}$ where $\angle_I = \{\sum_{i \in I} a_i D_i \mid a_i \in (0, +\infty)\}$.

Then there is an open subset $U_C = \bigcap_{I \in \mathcal{A}_{\omega_0}^{\min}} U_I \subset \mathbb{L}_{\mathbb{R}}^{\vee}$ where

$$U_I = \{\sum_{i \in I} a_i D_i \mid a_i \in (N_i, +\infty)\} \ (N_i \gg 0) \quad = \text{shifted } \angle_I$$

such that if $\omega \in U_C$ then $Z_{D^2}(\mathcal{L}_t) = \sum_{I \in \mathcal{A}_{\omega_0}^{\min}} Z^I(\mathcal{L}_t)$, where

$$Z^I(\mathcal{L}_t) = \frac{1}{|G_I|} \sum_{m \in (\mathbb{Z}_{\geq 0})^I} \prod_{\bar{i} \in \bar{I}} \Gamma(\langle D_i, \sigma_m \rangle + \alpha_i) \prod_{i \in I} \frac{(-1)^{m_i}}{m_i!} e^{\langle \theta + 2\pi\sqrt{-1}t, \sigma_m \rangle}$$

$\sigma_m = -\sum_{i \in I} (m_i + \alpha_i) D_i^{*I}$ where $\{D_i^{*I} : i \in I\}$ is a basis of $\mathbb{L}_{\mathbb{Q}}$ dual to the basis $\{D_i : i \in I\}$ of $\mathbb{L}_{\mathbb{Q}}^{\vee}$.

The infinite series $Z^I(\mathcal{L}_t)$ converges absolutely and uniformally on $\{\theta = \omega + 2\pi\sqrt{-1}B : \omega \in U_I, B \in \mathbb{L}_{\mathbb{R}}^{\vee}\}$.

Moreover, we have the following Higgs-Coulomb correspondence

$$Z_{D^2}([\mathcal{B}])\Big|_{\theta=-\sum_{a=1}^{\kappa}(\log y_a)\xi_a,\ \alpha_i=\frac{\lambda_i}{z}+\frac{q_i}{2}} = Z_{\widetilde{\mathcal{T}}}([\mathcal{B}])$$

where $\{\xi_1, \ldots, \xi_\kappa\}$ is an integral basis of $\mathbb{L}^\vee$ and $1 \le i \le n+\kappa$.
Knapp-Romo-Scheidegger, "D-brane central charges and Landau-Ginzburg orbifolds," 2020.
Proof by careful manipulation of κ-dimensional cycles and convergence checks of integrals $\int$ and series $\sum$.

$$Z_{D^2}(\mathcal{L}_t) = \int_{\mathbb{R}^\kappa}(\cdots) = \sum_{\mathcal{A}_1}\sum_{m\in\mathbb{Z}_{\ge 0}}\int_{S^1\times\mathbb{R}^{\kappa-1}}(\cdots) = \cdots$$

$$= \sum_{\mathcal{A}_\ell}\sum_{m\in(\mathbb{Z}_{\ge 0})^\ell}\int_{(S^1)^\ell\times\mathbb{R}^{\kappa-\ell}}(\cdots) = \cdots = \sum_{\mathcal{A}_\kappa}\sum_{m\in(\mathbb{Z}_{\ge 0})^\kappa}\underbrace{\int_{(S^1)^\kappa}(\cdots)}_{\kappa\text{-dimensional residue}}$$

- $\mathcal{A}_1, \ldots, \mathcal{A}_\kappa = \mathcal{A}_{\omega_0}^{\min}$ are finite sets.
- Up to translation, $\mathbb{R}^{\kappa-\ell} \subset \sqrt{-1}\mathbb{L}_{\mathbb{R}}$.
- Use the Calabi-Yau condition.

4. Wall-Crossing

abelian GLSMs without superpotentials:

- Borisov-Horja "Mellin-Barnes integrals as Fourier-Mukai transforms"
- Coates-Iritani-Jiang "The Crepant Transformation Conjecture for Toric Complete Intersections."

Let C_+, C_- be two adjacent chambers in $\mathbb{L}^\vee_{\mathbb{R}}$ = space of stability conditions. Then $\bar{C}_\pm$ are κ-dimensional cones in the secondary fan, and the $(\kappa-1)$-dimensional cone $\bar{C}_+ \cap \bar{C}_-$ is contained in the hyperplane $(h^\perp)_{\mathbb{R}} := \{\omega \in \mathbb{L}^\vee_{\mathbb{R}} \mid \langle \omega, h\rangle = 0\}$ for some primitive $h \in \mathbb{L}$. Let $\omega_\pm \in C_\pm$, $\mathcal{X}_\pm := \mathcal{X}_{\omega_\pm}$. Then

$$C_\pm = \bigcap_{I \in \mathcal{A}^{\min}_{\omega_\pm}} \angle_I, \quad \mathcal{A}^{\min}_{\omega_\pm} = \mathcal{A}^{\mathrm{ess}}_\pm \cup \underbrace{\mathcal{A}^{\mathrm{noness}}}_{\mathcal{A}^{\min}_{\omega_+} \cap \mathcal{A}^{\min}_{\omega_-}}$$

$$\{1, \ldots, n+\kappa\} = I_+ \cup I_- \cup I_0, \text{ where } \begin{matrix} I_+ \\ I_- \\ I_0 \end{matrix} = \{i \mid \langle D_i, h\rangle \begin{matrix} > \\ < \\ = \end{matrix} 0\}$$

$$\mathcal{A}^{\mathrm{ess}}_\pm = \{\{i\} \cup J \mid i \in I_\pm, J \in \mathcal{A}_0\}, \quad J \in \mathcal{A}_0 \Rightarrow J \subset I_0, |J| = \kappa - 1.$$

Theorem 2 (Aleshkin-L)

In the setting above, if $t \in \mathbb{L}^\vee$ satisfies the Grade Restriction Rule

$$|\langle B + t, h\rangle| < \frac{1}{4}\sum_{i=1}^{n+\kappa} |\langle D_i, h\rangle| = \frac{1}{2}\eta$$

where $\eta = \sum_{i\in I_+}\langle D_i, h\rangle = \sum_{i\in I_-}\langle D_i, -h\rangle$. Then there exists an open subset $U \subset U_{C_\pm}$ such that for $\omega \in U$

$$Z_{D^2}(\mathcal{L}_t)_\pm = \sum_{J\in\mathcal{A}_0} Z_J^{\text{ess}}(\mathcal{L}_t) + \sum_{I\in\mathcal{A}^{\text{noness}}} Z_I(\mathcal{L}_t)$$

- $Z_J^{\text{ess}}(\mathcal{L}_t)$ is an explicit series of integrals over $(S^1)^{\kappa-1}\times\mathbb{R}$.
- $Z_I(\mathcal{L}_t)$ converges uniformly and absolutely on for $\omega \in U_I \supset U_{C_\pm}$.

The Grade Restriction Rule (GRR) $\langle B+t, h\rangle \in (-\frac{\eta}{2}, \frac{\eta}{2})$
defines equivalences

$$\mathrm{GR}: \begin{array}{ccc} D^b(\mathcal{X}_+) & \longrightarrow & D^b(\mathcal{X}_-) \\ D^b_T(\mathcal{X}_+) & \longrightarrow & D^b_T(\mathcal{X}_-) \\ D^b_{\widetilde{T}}(\mathcal{X}_+) & \longrightarrow & D^b_{\widetilde{T}}(\mathcal{X}_-) \\ D(MF(\mathcal{X}_+, w)) & \longrightarrow & D(MF(\mathcal{X}_-, w)) \end{array}$$

- Kawamata $\mathrm{FM}: D^b(\mathcal{X}_+) \xrightarrow{\simeq} D^b(\mathcal{X}_-)$ (Fourier-Mukai)
- Coates-Iritani-Jiang-Segal $\mathrm{GR} = \mathrm{FM}: D^b_T(\mathcal{X}_+) \xrightarrow{\simeq} D^b_T(\mathcal{X}_-)$
  (Grade Restriction Rule = Fourier-Mukai)
  Halpern-Leistner, Ballard-Favero-Katzarkov
- Baranovsky-Pecharich, ...

Theorem 2 $\Rightarrow Z_{D^2}([\mathcal{B}])_+$ and $Z_{D^2}(\mathrm{GR}[\mathcal{B}])_-$ are related by analytic continuation. $\mathrm{GR} \to$ symplectic transform