Integral counterpart of Gromov–Witten invariants

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Seminar in Real & Complex Geometry
April 28, 2022

(Based on joint work with Guangbo Xu)
Preprint available on arXiv:2201.02688

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- $J : TX \to TX$ an ω-compatible almost complex structure: $J^2 = -Id$, $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$.
- Given a (nodal) Riemann surface (Σ, j), a map $u : \Sigma \to X$ is J-holomorphic if

$$du \circ j = J \circ du, \text{ equivalently } \overline{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0,$$

e.g. algebraic curves in smooth projective varieties.

Gromov–Witten invariants in symplectic geometry

- Given $A \in H_2(X;\mathbb{Z})$, consider the moduli space

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Theorem (Fukaya–Ono, Li–Tian, Ruan, Siebert, Pardon...)

The space $\overline{\mathcal{M}}_{g,k}(X, J, A)$ carries a $\mathbb{Q}$-valued virtual fundamental cycle $[\overline{\mathcal{M}}_{g,k}(X, J, A)]^{vir}$ of expected dimension.

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- $$\begin{tikzcd} & \overline{\mathcal{M}}_{g,k}(X, J, A) \arrow[dl, "\mathrm{st}"'] \arrow[dr, "\mathrm{ev}"] & \\ \overline{\mathcal{M}}_{g,k} & & X^k \end{tikzcd}$$

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- By pairing with classes in $H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$ and $H^*(X; \mathbb{Q})^{\otimes k}$ using st and ev, we obtain the so-called Gromov–Witten invariants.

Gromov–Witten invariants in symplectic geometry

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- Local model of $\overline{\mathcal{M}}_{g,k}(X, J, A)$: $s^{-1}(0)$ for $\begin{tikzcd} E \arrow[d, bend right, "\pi"'] \\ D \arrow[u, bend right, "s"'] \end{tikzcd}$, where D is an orbifold, E is an orbi-bundle, s is an "equivariant" section.

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- Fukaya–Ono, Li–Tian: use multi-valued perturbation of s to achieve transverslity.
- Pardon: Poincaré duality for orbifolds holds only over $\mathbb{Q}$.

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$\mathbb{Z}$-valued Gromov–Witten invariants

Theorem (B–Xu, 2022)

Suppose (X, ω) is a closed symplectic manifold and $A \in H_2(X; \mathbb{Z})$. Fix a non-negative integer k. Then there is a well-defined integral homology class

$$[\overline{\mathcal{M}}_{0,k}(X, J, A)]^{vir}_{free} \in H_*(\overline{\mathcal{M}}_{0,k} \times X^k; \mathbb{Z})$$

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- Not true on the nose: we need to perturb the $\overline{\partial}_J$-equation abstractly.
- This realizes a proposal of Fukaya–Ono back in the 1990s.
- Coincides with the ordinary fundamental class in the semi-positive case, which is known to be integral by Ruan–Tian.

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- In principle, such definitions would allow us to prove the Arnol'd conjecture over $\mathbb{Z}$, improving the best result so far by Abouzaid–Blumberg (over $\mathbb{F}_p$).

A general scheme of refining Gromov–Witten invariants

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- Step 1: Show that

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- Step 2: Construct natural transformations between generalized homology theories $\overline{\Omega}_*^{\mathbb{C},\mathrm{der}} \to MU_*, KU_*, \mathbb{Z}, \mathbb{Q}...$

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Moduli spaces as derived orbifolds

Definition

A *derived orbifold chart* is a triple (D, E, s) where D is a smooth orbifold, $E \to D$ is a smooth orbibundle and $s : D \to E$ is a smooth section. (D, E, s) is said to be compact if $s^{-1}(0)$ is compact.

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- The Kuranishi models of $\overline{\mathcal{M}}_{g,k}(X, J, A)$ present it locally as the zero locus of derived orbifold charts.
- It is possible to patch the local charts together to get a *global* derived orbifold chart for $\overline{\mathcal{M}}_{g,k}(X, J, A)$ using some recent results of Pardon, but there is a shortcut to take.

Moduli spaces as derived orbifolds

Proposition (Abouzaid–McLean–Smith, 2021)

After choosing certain auxiliary data, there exists a smooth derived orbifold chart (D, E, s) *along with a map*

$$\widetilde{st} \times \widetilde{ev} : D \to \overline{\mathcal{M}}_{0,k} \times X^k$$

such that the zero locus $s^{-1}(0)$ *is isomorphic to* $\overline{\mathcal{M}}_{0,k}(X, J, A)$ *and the restriction of* $\widetilde{st} \times \widetilde{ev}$ *along* $s^{-1}(0)$ *coincides with the product of the stabilization map and the evaluation map.*

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- Actually TD and E are *complex* vector bundles.
- The work in progress by Hirschi–Swaminathan generalizes this result to the higher genus moduli spaces.
- The quadruple $(D, E, s, \widetilde{\mathrm{st}} \times \widetilde{\mathrm{ev}})$ is independent of various choices as an element in $\Omega_*^{\mathbb{C},\mathrm{der}}(\overline{\mathcal{M}}_{0,k} \times X^k)$.

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Stable complex derived orbifold bordism

- Suppose M is a topological space. We consider quadruples (D, E, s, f) such that: (D, E, s) is a compact derived orbifold chart, $f : D \to M$ is a continuous map.

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- Introduce the following relations:

1. (*Restriction*) $(D, E, s, f) \sim (D', E', s', f')$ if $D' \subset D$ is an open subset with $s^{-1}(0) \subset D'$ and $E' = E|_{D'}$, $s' = s|_{D'}$, and $f' = f|_{D'}$.

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3. (*Cobordism*) $(D, E, s, f) \sim (D', E', s', f')$ if there is a bordism between them.

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The *stable complex derived orbifold bordism* of M, denoted by $\overline{\Omega}_*^{\mathbb{C},\mathrm{der}}(Y)$, is defined to be the equivalence classes of (D, E, s, f) endowed with a stable complex structure modulo the relations introduced before.

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- This definition was first considered by Joyce and is developed further by Pardon.
- Abouzaid–McLean–Smith's result actually shows that $\overline{\mathcal{M}}_{0,k}(X, J, A)$ uniquely defines an element in $\overline{\Omega}_*^{\mathbb{C},\mathrm{der}}(\overline{\mathcal{M}}_{0,k} \times X^k)$.

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- Work in progress: a decomposition of $\mathbb{Q}$-valued invariants into a weighted sum of integers.

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Theorem (B–Xu)

Denote by $X^{\mathrm{free}} \subset X$ the suborbifold consisting of points with trivial isotropy group. Then there exist normally complex smooth sections $s : X \to \mathcal{E}$ such that $s^{-1}(0) \cap X^{\mathrm{free}}$ defines a pseudocycle. Moreover, given a pair of such sections s_1 and s_2, the pseudocycles $s_1^{-1}(0) \cap X^{\mathrm{free}}$ and $s_2^{-1}(0) \cap X^{\mathrm{free}}$ are cobordant.

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- $s^{-1}(0) \cap X^{\mathrm{free}}$ is a pseudocycle $\Leftrightarrow$ $s^{-1}(0) \cap X^{\mathrm{free}} \subset X^{\mathrm{free}}$ is a smooth submanifold and the boundary $\overline{s^{-1}(0) \cap X^{\mathrm{free}}} \setminus (s^{-1}(0) \cap X^{\mathrm{free}})$ could be covered by submanifolds with at least 2-dimensions lower.

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- We can drop the compactness of X by considering almost complex compact derived orbifold chart (D, E, s). The section s is perturbed in a neighborhood of $s^{-1}(0)$.

Overview of the proof

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Given a finite group Γ*, suppose* V *and* W *are finite dimensional complex* Γ *representations. If* V *is faithful, then there exists* $d \gg 1$ *such that for a generic* $p \in \mathrm{Poly}_d^{\Gamma}(V, W)$*, the zero locus* $p^{-1}(0)$ *is a smooth algebraic variety over* $\mathbb{C}$*.*

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- For p generic, the boundary of $p^{-1}(0)$ is of real codimension at least 2.

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- Warning: The genericity in Fukaya–Ono's result depends on the degree d and the group Γ, so much of the hard work is to remove such dependence.
- To this end, we need to investigate Whitney stratifications on the universal zero locus

$$\mathcal{Z}_d^\Gamma(V, W) := \{(v, p) \in V \times \mathrm{Poly}_d^\Gamma(V, W) | p(v) = 0\}.$$

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The rest is a bonus, which will be discussed only if time permits.

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- Assume $(V^{\Gamma'})^{\perp}$ is not $\{0\}$.
- Identifying $\mathrm{Nbd}_0 T_x V$ with $\mathrm{Nbd}_x V$, a smooth Γ-equivariant map $s : V \to W$ near x could be written as

$$s = s_{\mathrm{inv}} \oplus s_{\perp}$$

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- If we have a map $\mathfrak{s}_{\perp} : \mathrm{Nbd}_{V^{\Gamma'}}(V) \to \mathrm{Poly}_d^{\Gamma'}((V^{\Gamma'})^{\perp}, (W^{\Gamma'})^{\perp})$, we can construct a section $s_{\perp} : \mathrm{Nbd}_{V^{\Gamma'}}(V) \to (W^{\Gamma'})^{\perp}$ by composing $\mathfrak{s}_{\perp}$ with the evaluation map. The map $\mathfrak{s}_{\perp}$ is called a complex polynomial lifting of $s_{\perp}$.

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- A Γ-equivariant map $s : V \to W$ is called a Fukaya–Ono–Parker map near x if $s_{\perp}$ from the decomposition $s = s_{\mathrm{inv}} \oplus s_{\perp}$ has a complex polynomial lifting.

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- This is the local version of our statement.
- There is a parametric version of the above lemma dealing with cobordism invariance.

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- The relevant definition of polynomial perturbation and transversality condition in their statement is not intrinsic enough: it depends on the cut-off degree d, the choice of tubular neighborhoods, and the choice of local uniformizer group of an orbifold chart.
- In other words, it was unclear about how to choose a complex polynomial section varying coherently along different strata indexed by isotorpy groups, and it was unclear if the transversality is *open*.

Difficulties of local-to-global

- We overcome these problems by studying a "canonical" Whitney stratification on the universal zero locus

$$Z_d := \{(v, P) \in (V^{\Gamma'})^{\perp} \times \mathrm{Poly}_d^{\Gamma'}((V^{\Gamma'})^{\perp}, (W^{\Gamma'})^{\perp}) | P(v) = 0\}$$

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  and study its behavior when varying d and Γ'.
- Certain aspects of the proof are inspired by an unpublished work of Brett Parker.
- Once the openness of a suitable transversality condition is established, a good perturbation and the relevant parametric statement follow by the usual arguments in differential topology.