

Integral counterpart of Gromov–Witten invariants

Shaoyun Bai
(Princeton University)

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(Based on joint work with Guangbo Xu)
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- ▶ $J : TX \rightarrow TX$ an ω -compatible almost complex structure:
 $J^2 = -Id$, $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$.

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- ▶ $J : TX \rightarrow TX$ an ω -compatible almost complex structure: $J^2 = -Id$, $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$.
- ▶ Given a (nodal) Riemann surface (Σ, j) , a map $u : \Sigma \rightarrow X$ is J -holomorphic if

$$du \circ j = J \circ du, \text{ equivalently } \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0,$$

e.g. algebraic curves in smooth projective varieties.

Gromov–Witten invariants in symplectic geometry

- ▶ Given $A \in H_2(X; \mathbb{Z})$, consider the moduli space

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- ▶ It is a singular space in general: non-triviality of automorphism group \Rightarrow “orbifold” singularity; failure of transversality of $\bar{\partial}_J \Rightarrow$ non-smoothness.

Gromov–Witten invariants in symplectic geometry

Theorem (Fukaya–Ono, Li–Tian, Ruan, Siebert, Pardon...)

The space $\overline{\mathcal{M}}_{g,k}(X, J, A)$ carries a \mathbb{Q} -valued virtual fundamental cycle $[\overline{\mathcal{M}}_{g,k}(X, J, A)]^{vir}$ of expected dimension.

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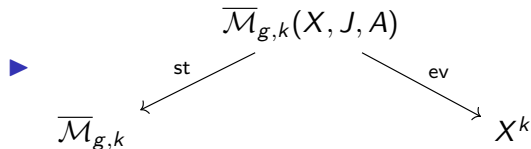
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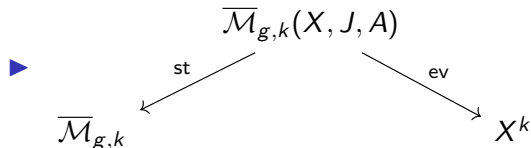


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- ▶ By pairing with classes in $H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$ and $H^*(X; \mathbb{Q})^{\otimes k}$ using st and ev , we obtain the so-called Gromov–Witten invariants.

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- ▶ Local model of $\overline{\mathcal{M}}_{g,k}(X, J, A) : s^{-1}(0)$ for $\pi \begin{pmatrix} E \\ \downarrow \\ D \end{pmatrix}^s$, where D

is an orbifold, E is an orbi-bundle, s is an “equivariant” section.

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- ▶ Fukaya–Ono, Li–Tian: use multi-valued perturbation of s to achieve transversality.
- ▶ Pardon: Poincaré duality for orbifolds holds only over \mathbb{Q} .

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Theorem (B–Xu, 2022)

Suppose (X, ω) is a closed symplectic manifold and $A \in H_2(X; \mathbb{Z})$. Fix a non-negative integer k . Then there is a well-defined integral homology class

$$[\overline{\mathcal{M}}_{0,k}(X, J, A)]_{free}^{vir} \in H_*(\overline{\mathcal{M}}_{0,k} \times X^k; \mathbb{Z})$$

defined by virtually “counting” J -holomorphic maps in $\overline{\mathcal{M}}_{0,k}(X, J, A)$ with trivial automorphism group.

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- ▶ Not true on the nose: we need to perturb the $\bar{\partial}_J$ -equation abstractly.
- ▶ This realizes a proposal of Fukaya–Ono back in the 1990s.
- ▶ Coincides with the ordinary fundamental class in the semi-positive case, which is known to be integral by Ruan–Tian.

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- ▶ In principle, such definitions would allow us to prove the Arnol'd conjecture over \mathbb{Z} , improving the best result so far by Abouzaid–Blumberg (over \mathbb{F}_p).

A general scheme of refining Gromov–Witten invariants

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- ▶ Step 1: Show that $\overline{\mathcal{M}}_{g,k}(X, J, A)$ defines an element in $\overline{\mathcal{M}}_{g,k} \times X^k$

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- ▶ Step 2: Construct natural transformations between generalized homology theories $\overline{\Omega}_*^{\mathbb{C}, \text{der}} \rightarrow MU_*, KU_*, \mathbb{Z}, \mathbb{Q} \dots$

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Definition

A *derived orbifold chart* is a triple (D, E, s) where D is a smooth orbifold, $E \rightarrow D$ is a smooth orbibundle and $s : D \rightarrow E$ is a smooth section. (D, E, s) is said to be compact if $s^{-1}(0)$ is compact.

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- ▶ The Kuranishi models of $\overline{\mathcal{M}}_{g,k}(X, J, A)$ present it locally as the zero locus of derived orbifold charts.
- ▶ It is possible to patch the local charts together to get a *global* derived orbifold chart for $\overline{\mathcal{M}}_{g,k}(X, J, A)$ using some recent results of Pardon, but there is a shortcut to take.

Moduli spaces as derived orbifolds

Proposition (Abouzaid–McLean–Smith, 2021)

After choosing certain auxiliary data, there exists a smooth derived orbifold chart (D, E, s) along with a map

$$\tilde{st} \times \tilde{ev} : D \rightarrow \overline{\mathcal{M}}_{0,k} \times X^k$$

such that the zero locus $s^{-1}(0)$ is isomorphic to $\overline{\mathcal{M}}_{0,k}(X, J, A)$ and the restriction of $\tilde{st} \times \tilde{ev}$ along $s^{-1}(0)$ coincides with the product of the stabilization map and the evaluation map.

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- ▶ Actually TD and E are complex vector bundles.
- ▶ The work in progress by Hirschi–Swaminathan generalizes this result to the higher genus moduli spaces.
- ▶ The quadruple $(D, E, s, \tilde{st} \times \tilde{ev})$ is independent of various choices as an element in $\overline{\Omega}_*^{\mathbb{C}, \text{der}}(\overline{\mathcal{M}}_{0,k} \times X^k)$.

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- ▶ Suppose M is a topological space. We consider quadruples (D, E, s, f) such that: (D, E, s) is a compact derived orbifold chart, $f : D \rightarrow M$ is a continuous map.
- ▶ Introduce the following relations:
 1. (*Restriction*) $(D, E, s, f) \sim (D', E', s', f')$ if $D' \subset D$ is an open subset with $s^{-1}(0) \subset D'$ and $E' = E|_{D'}$, $s' = s|_{D'}$, and $f' = f|_{D'}$.

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 3. (*Cobordism*) $(D, E, s, f) \sim (D', E', s', f')$ if there is a bordism between them.

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The *stable complex derived orbifold bordism* of M , denoted by $\overline{\Omega}_*^{\mathbb{C}, \text{der}}(Y)$, is defined to be the equivalence classes of (D, E, s, f) endowed with a stable complex structure modulo the relations introduced before.

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- ▶ This definition was first considered by Joyce and is developed further by Pardon.
- ▶ Abouzaid–McLean–Smith’s result actually shows that $\overline{\mathcal{M}}_{0,k}(X, J, A)$ uniquely defines an element in $\overline{\Omega}_*^{\mathbb{C}, \text{der}}(\overline{\mathcal{M}}_{0,k} \times X^k)$.

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- ▶ Work in progress: a decomposition of \mathbb{Q} -valued invariants into a weighted sum of integers.

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Theorem (B–Xu)

Denote by $X^{\text{free}} \subset X$ the suborbifold consisting of points with trivial isotropy group. Then there exist **normally complex** smooth sections $s : X \rightarrow \mathcal{E}$ such that $s^{-1}(0) \cap X^{\text{free}}$ defines a **pseudocycle**. Moreover, given a pair of such sections s_1 and s_2 , the pseudocycles $s_1^{-1}(0) \cap X^{\text{free}}$ and $s_2^{-1}(0) \cap X^{\text{free}}$ are cobordant.

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- ▶ We can drop the compactness of X by considering almost complex compact derived orbifold chart (D, E, s) . The section s is perturbed in a neighborhood of $s^{-1}(0)$.

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Lemma (Fukaya–Ono)

Given a finite group Γ , suppose V and W are finite dimensional complex Γ representations. If V is faithful, then there exists $d \gg 1$ such that for a generic $p \in \text{Poly}_d^\Gamma(V, W)$, the zero locus $p^{-1}(0)$ is a smooth algebraic variety over \mathbb{C} .

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- ▶ **Warning:** The genericity in Fukaya–Ono’s result depends on the degree d and the group Γ , so much of the hard work is to remove such dependence.
- ▶ To this end, we need to investigate Whitney stratifications on the universal zero locus

$$\mathcal{Z}_d^\Gamma(V, W) := \{(v, p) \in V \times \text{Poly}_d^\Gamma(V, W) \mid p(v) = 0\}.$$

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The rest is a bonus, which will be discussed only if time permits.

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- ▶ Assume $(V^{\Gamma'})^\perp$ is **not** $\{0\}$.
- ▶ Identifying $\text{Nbd}_0 T_x V$ with $\text{Nbd}_x V$, a smooth Γ -equivariant map $s : V \rightarrow W$ near x could be written as

$$s = s_{\text{inv}} \oplus s_\perp$$

under the decomposition $W = W^{\Gamma'} \oplus (W^{\Gamma'})^\perp$.

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- ▶ If we have a map $\mathfrak{s}_\perp : \text{Nbd}_{V^{\Gamma'}}(V) \rightarrow \text{Poly}_d^{\Gamma'}((V^{\Gamma'})^\perp, (W^{\Gamma'})^\perp)$, we can construct a section $s_\perp : \text{Nbd}_{V^{\Gamma'}}(V) \rightarrow (W^{\Gamma'})^\perp$ by composing \mathfrak{s}_\perp with the evaluation map. The map \mathfrak{s}_\perp is called a **complex polynomial lifting** of s_\perp .

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- ▶ A Γ -equivariant map $s : V \rightarrow W$ is called a Fukaya–Ono–Parker map near x if s_\perp from the decomposition $s = s_{\text{inv}} \oplus s_\perp$ has a complex polynomial lifting.

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If d is sufficiently large, any generic equivariant map s admitting a complex polynomial lifting satisfies:

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- ▶ This is the local version of our statement.
- ▶ There is a parametric version of the above lemma dealing with cobordism invariance.

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- ▶ Fukaya–Ono’s lemma is not sufficient to establish the full proof.
- ▶ The relevant definition of polynomial perturbation and transversality condition in their statement is not intrinsic enough: it depends on the cut-off degree d , the choice of tubular neighborhoods, and the choice of local uniformizer group of an orbifold chart.
- ▶ In other words, it was unclear about how to choose a complex polynomial section varying coherently along different strata indexed by isotropy groups, and it was unclear if the transversality is *open*.

Difficulties of local-to-global

- ▶ We overcome these problems by studying a “canonical” Whitney stratification on the universal zero locus

$$Z_d := \{(v, P) \in (V^{\Gamma'})^\perp \times \text{Poly}_d^{\Gamma'}((V^{\Gamma'})^\perp, (W^{\Gamma'})^\perp) \mid P(v) = 0\}$$

and study its behavior when varying d and Γ' .

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- ▶ Certain aspects of the proof are inspired by an unpublished work of Brett Parker.
- ▶ Once the openness of a suitable transversality condition is established, a good perturbation and the relevant parametric statement follow by the usual arguments in differential topology.