Birational minimal models of non-degenerate surfaces $S \subset (\mathbb{C}^*)^3$

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Plan

1. Motivation (case of curves)
2. Surfaces in $(\mathbb{C}^*)^3$. Examples
3. Fine interior. Applications.
4. Combinatorial formulas.

Danilov. V. I
Khovansky. A. G.

Toric varieties. $(\mathbb{C}^*)^d \hookrightarrow X$

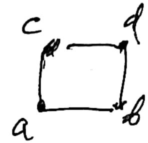
$(\mathbb{C}^*)^2 \supset C = \{f(x,y)=0\}$

$f(x,y) = \sum_{n,m} a_{n,m} x^n y^m$ $\quad$ $\mathrm{Conv}\{(n,m) \mid a_{n,m} \neq 0\}$ "$= P$

Newton. polytope of f.

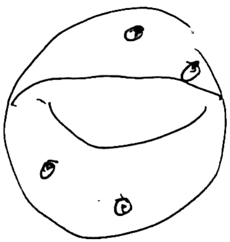
non-degenerate.

Zariski open.

Ex.: 

$a + bx + cy + dxy$

$ab\cdot cd\,(ad-bc) \neq 0$



$\mathbb{CP}^1 \setminus 4pt. = C \hookrightarrow \bar{C}$

Th. (Berker) 19th cent. $\quad g(C) = \#\{\mathrm{Int}(P) \cap \mathbb{Z}^2\}$

$e(\bar{C}) = 2-2g$.

$f(x,y,z) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$

$\dim \mathrm{Newt}\, f = 3$

birational invariant.

$S = \{f = 0\} \subset (\mathbb{C}^*)^3$

geom. genus.

$g(X)$
$\|$
$\dim H^0(\overline{X}, \Omega^d_{\overline{X}})$

$X \underset{bir.}{\sim} \overline{X}$ smooth proj.

$d = \dim X$.

$g(S) = \#(\mathrm{Int}\, P \cap \mathbb{Z}^3)$

$\overline{S} \qquad h^{2,0}(\overline{S})$

$P = \mathrm{Newt.}\, f$.

$\underline{g(K3) = 1}$

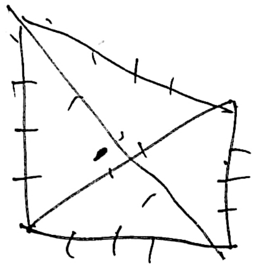
<u>Kodaira dimension.</u> $X \quad \dim H^0(X, (\Omega^d_X)^{\otimes m}) \sim m^{\kappa(X)}$

$\kappa = \underline{-\infty}, \underbrace{0, 1, 2}$ for surfaces

$\underline{\kappa \geq 0}$

$\exists!$ proj. minimal models. $S_{min} \overset{bir}{\sim} \overline{S}$

$e = c_2, \quad c_1^2 = K_X^2$



$S \hookrightarrow \mathbb{P}^3$

quartic surface. K3.

Def. Reflexive polytope. Lattice d-polytope. P $\mathbb{Z}^d = M$.

$0 \in \mathbb{Z}^d$ is unique int. point in P. $M_{\mathbb{R}}$.

$\langle , \rangle \quad M \times N \to \mathbb{Z}$

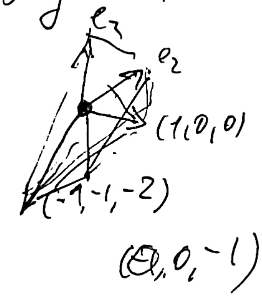
$\mathrm{Hom}(M, \mathbb{Z}) = N \subset N_{\mathbb{R}}$.

$P^* := \{ y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \ \forall x \in P \}$.

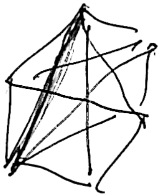
dual polytope.

We say P is reflexive if P^* lattice polytope.

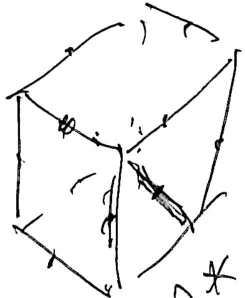
P



P^*



$P \supset \Theta$



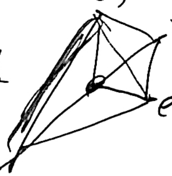
$P^* \supset \Theta^*$

$$\sum_{\Theta \subset P} \ell(\Theta) \cdot \ell(\Theta^*) = 24$$

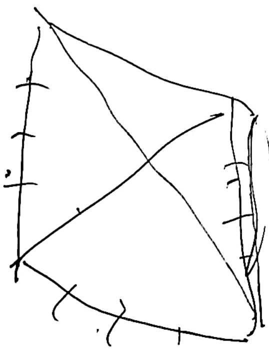
$12 \quad 1 \cdot 2$

$\dim \Theta = 1$

1



P



4

P^*

$$6 \cdot 1 \cdot 4 = 24 = e(K3)$$

Fine interior of P. J. Fine M. Reid 80-ies

$M, \; N = M^*$.

Def.

$$F(P) = \{ x \in P \mid \langle x, l \rangle \geq \mathrm{Min}_P(l) + 1 \;\; \forall l \in N \setminus \{0\} \}$$

dim P = max.

Fine interior of P

$F(P) \subset P$ rational subpolytope

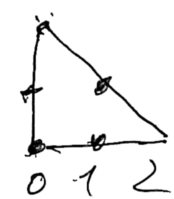
or $\emptyset$

Ex. $d = 2$. P lattice polygone $\Rightarrow$ $F(P) = \mathrm{Conv}(\mathrm{Int} P \cap M)$

$x \in \mathrm{Int}(P) \cap M \Rightarrow x \in F(P)$

$F(P) \supseteq \mathrm{Conv}(\mathrm{Int} P \cap M)$

$= \dim 1, 2$



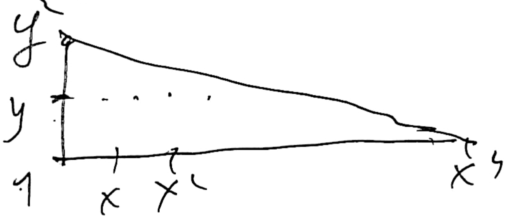
Th. $\kappa(Z) = \min\{\dim F(P), d-1\}$. $\quad P = \mathrm{Newt}(f)$

$Z = \{f = 0\}$

$d = \dim M_{\mathbb{R}}$.

$d=2 \quad \dim F(P) = 1 \qquad Z \subset (\mathbb{C}^*)^2$

↑ hyperelliptic



$\dim F(P) = 0$

$d=3 \qquad \dim F(P) = 0$. K3, Enriques.

1 - elliptic surfaces $\kappa(S) = 1$

$\left.\begin{matrix} 2 \\ 3 \end{matrix}\right\}$ surfaces of general type.

Cor. $S \subset (\mathbb{C}^*)^3 \qquad \kappa(S) \geq 0 \iff F(P) \neq \emptyset$

Constructing minimal model. ($F(P) \neq \emptyset$)

Def. $S_F(P) := \{ \ell \in N \mid -\mathrm{Min}_P(\ell) + \mathrm{Min}_{F(P)}(\ell) = 1 \}$

$F(P) \subset P$

$\mathrm{Min}_{F(P)}(\ell) = \mathrm{Min}_P(\ell) + 1$

$|S_F(P)| < \infty$.

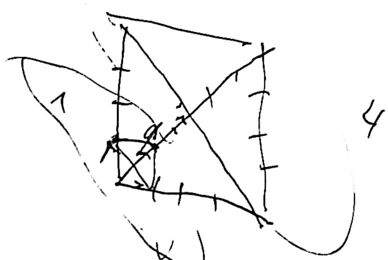
Def. $C(P) = \{ x \in M_{\mathbb{R}} \mid \langle x, \ell \rangle \geq \mathrm{Min}_P(\ell) \ \forall \ell \in S_F(P) \}$

$P \subseteq C(P)$

$d=2$. $\dim P = 2 \Rightarrow C(P) = P$

$F(P) = \{(1,1,1)\}$

$d=3$



$x_i \geq 0$ $\sum x_i \leq 4$

$\sum x_i \geq 1$

$(1,1,1) \notin S(P)$

$\{ e_1, e_2, e_3, -e_1 - e_2 - e_3 \} = S_F(P)$

Algorithm

$$C(P) + F(P) = \tilde{P} \qquad \underline{\text{3-dim.}}$$

$$\begin{array}{c} C(P) \\ \cup \\ P \end{array}$$

Σ normal fan of $\tilde{P}$ $\qquad$ $\tilde{V}$ 3-dim toric variety.

$S \subset (\mathbb{C}^*)^3$ $\qquad$ $\tilde{S}$ closure in $\tilde{V}$

$$\begin{array}{c} (\mathbb{C}^*)^3 \\ \cap \\ \tilde{V} \end{array}$$

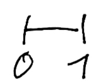
Th: sings. $\tilde{S}$ only A_n-sings.

Du Val.

$\bar{S}$ min. resolut. of $\tilde{S}$

Rem. if P is reflexive

$\underline{d=4} = \dim P =$ $\qquad$ $(P^*)^* = P$



$C(P) = P \quad F(P) = \{0\}$

$\mathcal{S}_F(P) = N \cap \partial P^*$

$\pi_1(\hat{S}) \simeq \mathbb{Z}/9\mathbb{Z}.$

$\dim F(P) = 3$

$\dim F(P) = 2$

c_1^2, c_2

$$c_1^2 = \begin{cases} 2\,\mathrm{vol}_2(F(P)) \\ 0 & 0 \le \dim F \le 1 \end{cases}$$

$$\mathrm{Vol}_3(F(P)) + \sum_{\Theta} \mathrm{vol}_2(\Theta) = c_1^2.$$

$$\frac{c_1^2 + c_2}{12} = \chi(\mathcal{O}_S) = 1 + |F(P) \cap M|$$

Noether's formula

c_2

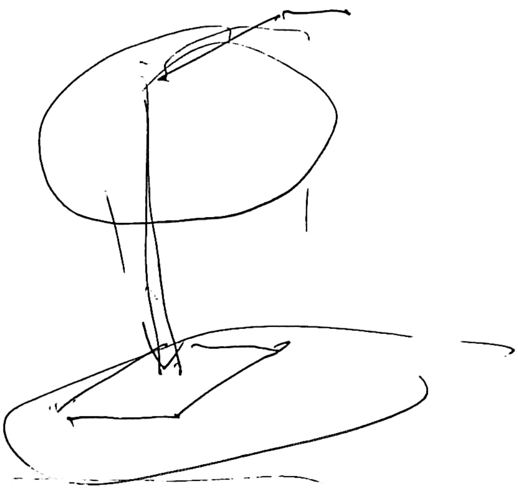
$F(P) \neq \emptyset$

K3 $\quad 24 = \sum \ldots$

$\boxed{\text{elliptic surfaces}}$

$c_1^2 = 0$

$e(S) = 12m.$

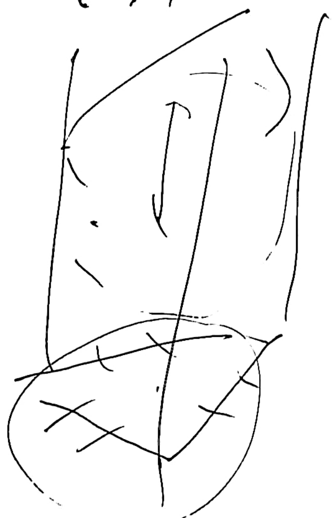


$\mathbb{P}^1 \times \mathbb{P}^2$

$(m, 3)$

n

16



$\dim F(P) = 2$

Horikawa surfaces.

$S \xrightarrow{2:1} \text{rat.}$