Caporaso - Harris

for Cusps.

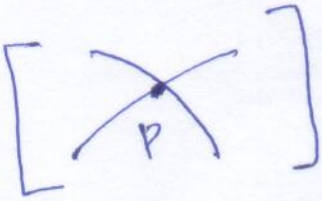
Setup

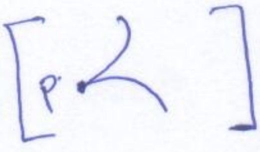
We look at degree d- curves in $\mathbb{P}^2$ $\approx$ Homogeneous deg d poly, non zero, upto scaling

$$= \mathcal{D}_d = \mathbb{P}^{\delta_d}$$

where $\delta_d := \dfrac{d(d+3)}{2}$

Define

$A_1 := \{ (\mathcal{C}_d, p) \in \mathcal{D}_d \times \mathbb{P}^2 \mid$ p is a nodal point of the curve $\mathcal{C}_d \}$. $=$ 

$A_2 := \{ (\mathcal{C}_d, p) \in \mathcal{D}_d \times \mathbb{P}^2 \mid$ p is a cuspidal point of the curve $\mathcal{C}_d \}$. $=$ 

$$\left.\begin{aligned} y_d &\in H^*(\mathcal{D}_d), \\ b &\in H^*(\mathbb{P}^2) \end{aligned}\right\} \text{ Respective Hyperplane Classes.}$$

Goal :- Compute all intersection numbers involving the class $[A_1]$ & $[A_2]$.

Eg: $[A_1] \cdot y_d^{\delta_d - 1}$

= # of degree d curves in $\mathbb{P}^2$ passing thru $\delta_d - 1$ generic points & having a node.

///ly $[A_2] \cdot y_d^{\delta_d - 2}$

= ______ having a cusp.

Let us focus on $d = 3$ for simplicity.

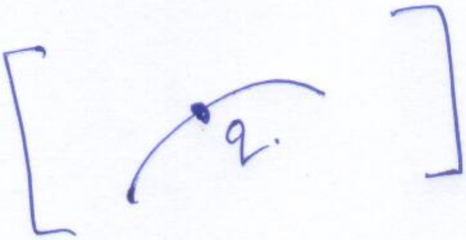
Q. What is

$$\left.\begin{aligned} [A_1] \cdot y_3^8 &= ? \\ [A_2] \cdot y_3^7 &= ? \end{aligned}\right\}.$$

We will start with the computation] $[A_1] \cdot y_3^8$

Recap: Incidence Variety

$$I := \{ (\mathcal{C}_d, q) \in \mathcal{D}_d \times \mathbb{P}^2 \mid q \in \mathcal{C}_d \}.$$

$=$ 

Standard Fact: Homology Class represented by I is

$$[I] = (y_d + d\, b) \in H^*(\mathcal{D}_d \times \mathbb{P}^2).$$

The computation of $[A_1] \cdot y_3^8$ will involve 4 steps

(in general, the computation of $[A_1] \cdot y_d^{\delta_d - 1}$ will involve $(d+1)$ steps).

Let us now introduce the ambient space of Step 1 where we will do intersection theory.

Define

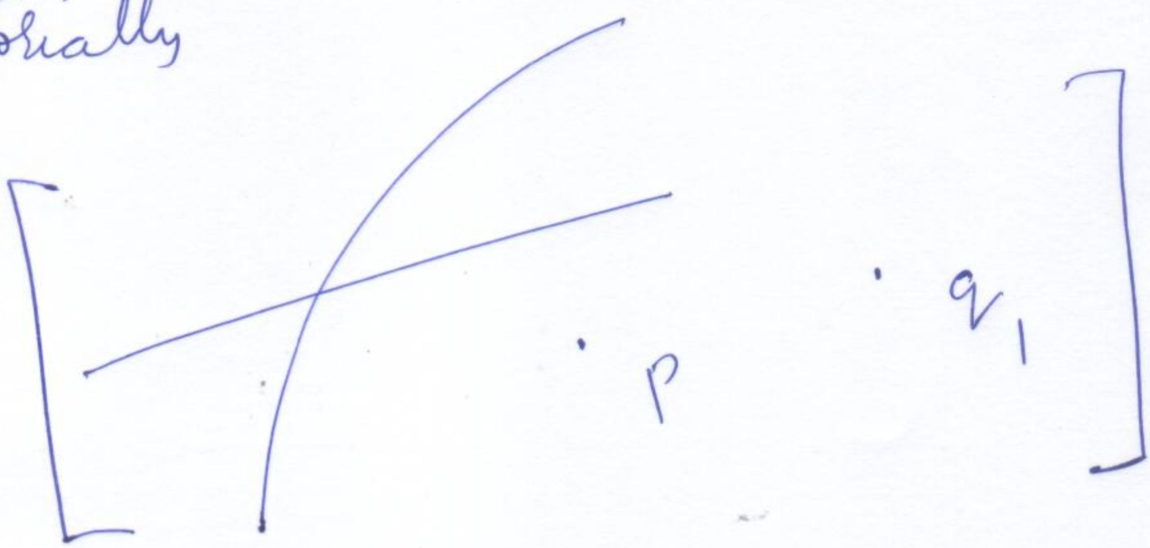
$$M_1 := \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2 \times \mathbb{P}^2_1$$

$$\begin{array}{cccc} \text{Line} & C_3 & p & q_1 \end{array}$$

$$\underbrace{y_1 \quad y_3 \quad b \quad a_1}_{\text{Hyperplane Classes.}}$$

Pictorially

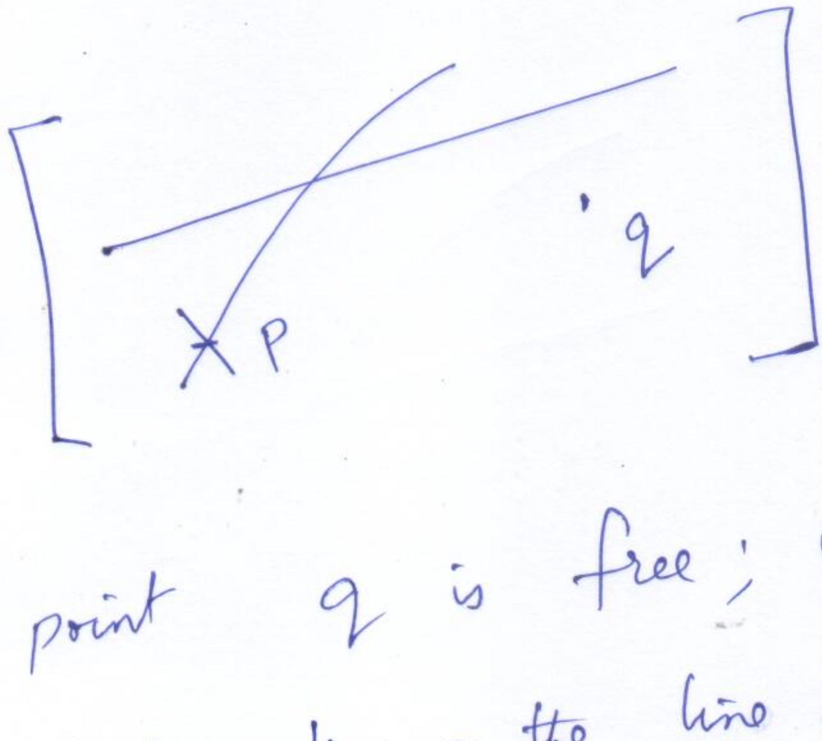
$M_1 =$



Let us define a subspace of M_1 called A_1^F

$$A_1^F := \{(\text{line}, \mathcal{C}_3, p, q_1) \in \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2 \times \mathbb{P}^2_1 \mid p \text{ is a nodal point of } \mathcal{C}_3\}.$$

$=$



The point q is free; it need not lie on the line or the cubic.

Note we call it A_1^F; the F is there to emphasize that the nodal point is "Free" (it need not lie on the line).

<u>Remark.</u> The definition of A_1^F is a small lie. Actually there is one more condition: I require the nodal point p to NOT lie on the line. Let us ignore this point for now, but it will be important later (in Step 3 & 4)

Let us proceede. Let us now impose the condition that the point q lies on the line and the ~~curve~~ curve.

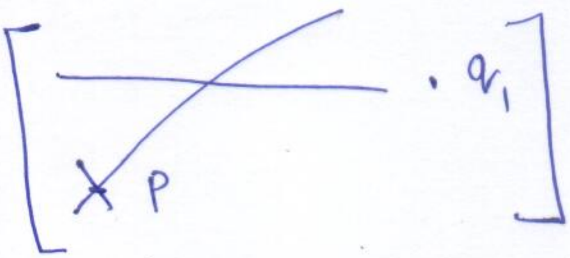
Recall (Page 5)

$$[q \in \text{Line}] = (y_1 + a_1).$$

$$[q \in \text{Cubic}] = (y_3 + 3a_1).$$

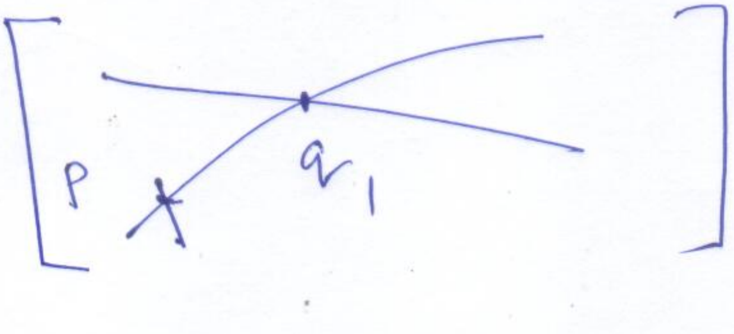
Finally, we can state the first step of Caporaso - Harris.

<u>Step 1.</u> On $H^*(M_1)$, the following equality holds.



$\cdot\,[q \in \text{Line}] \cdot [q \in \text{Cubic}]$

$=$



$$= [A_1^F \; T_0]$$

Where

$$A_1^F T_0 = \{ (\text{line}, C_3, P, q_1) \in \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2 \times \mathbb{P}_1^2 \mid$$

P is nodal point of Cubic &

q_1 lies on line and Cubic &

(*) $p \neq q_1$, $p \notin$ Line $\}$.

Remark.

$[S]$ = Homology Class rep by Closure $\overline{S}$.

∴ When we take Closure, (*) becomes irrelevant.

~~13~~

Remark: Proof of Step 1 is as follows.

On the set theoretic level the statement is obvious.

The non trivial part is that the intersections are transverse. That is easy to show here by writing things in local coordinates

(That is infact the principle everywhere in the following steps — it just gets more and more complicated).

Let us now see the **numerical** consequence of **Step 1**:

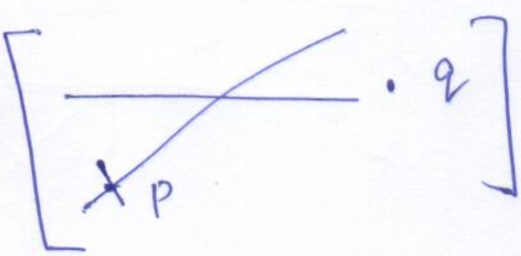
Let us intersect with a class of complementary dimension.

Define

$$\ell := y_1^2 \cdot y_3^7 \cdot a_1$$

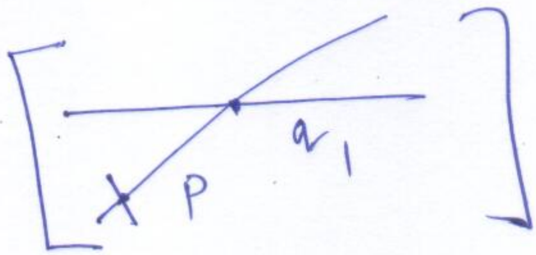
(a_1 = Hyperplane class of $\mathbb{P}_1^2$
See Page 7).

Intersecting both sides of Step 1 with ef gives us



$$\cdot (y_1 + a_1)(y_3 + 3a_1) \cdot y_1^2 \cdot y_3^7 \cdot a_1$$

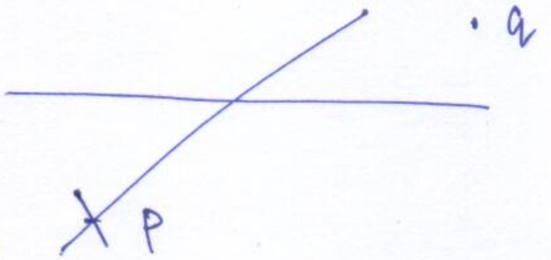
$$=$$

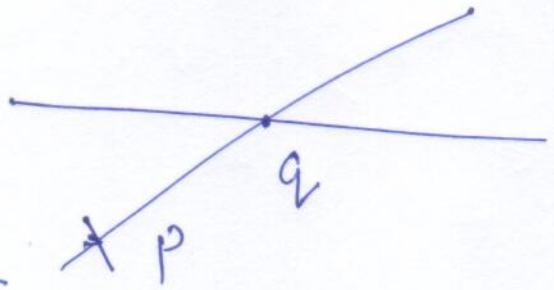


$$\cdot y_1^2 \cdot y_3^7 \cdot a_1$$

Use that $y_1^3 = 0$, $a_1^3 = 0$

The above equation simplifies to

[] $\cdot\, y_1^{2} \cdot y_3^{8} \cdot a_1^{2}$

$=$ [] $\cdot\, y_1^{2} \cdot y_3^{7} \cdot a_1$

Let us now process the Left Hand Side.

Intersecting with y_1^2

$\rightsquigarrow$ Line is fixed.

Intersecting with a_1^2

$\rightsquigarrow$ q is fixed.

The LHS is

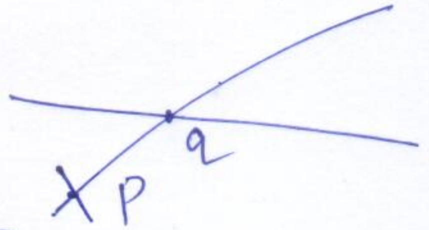
$\Big[$ 

$\Big] \cdot y_3^8$

$= [A_1] \cdot y_3^8$ = The number we wanted

= # of Nodal Cubics thru 8 generic points

Let us now look at the RHS.

[

] $\cdot\, y_1^2 \cdot y_3^7 \cdot a_1$.

$y_1^2 \rightsquigarrow$ Line is fixed.

Intersecting with $a_1 \rightsquigarrow q$ is fixed.

(Intersection) the Line given by y_1^2
& a_1).

$\therefore$ [

] $\cdot\, y_1^2 \cdot y_3^7 \cdot a_1$

= # nodal Cubics thru
7 generic points & <u>One</u> pt
on the Line.

Conclusion

\# of Nodal cubics thru
8 generic points

= \# of Nodal Cubics thru
7 generic points &
one point on a line.

Remark. In Hindsight, an obvious fact. But let us proceede.

Step 2: Define

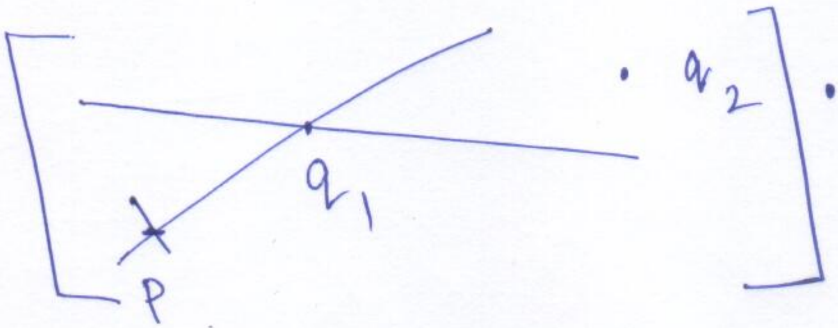
$$
\begin{array}{cccccccccc}
M_2 := & \mathcal{D}_1 & \times & \mathcal{D}_3 & \times & \mathbb{P}^2 & \times & \mathbb{P}^2_1 & \times & \mathbb{P}^2_2 . \\
 & L & & C_3 & & p & & q_1 & & q_2 \\
 & y_1 & & y_3 & & b & & a_1 & & a_2
\end{array}
$$

$$\underbrace{y_1 \quad y_3 \quad b \quad a_1 \quad a_2}_{\text{Hyperplane Class.}}$$

Claim On $H^*(M_2)$

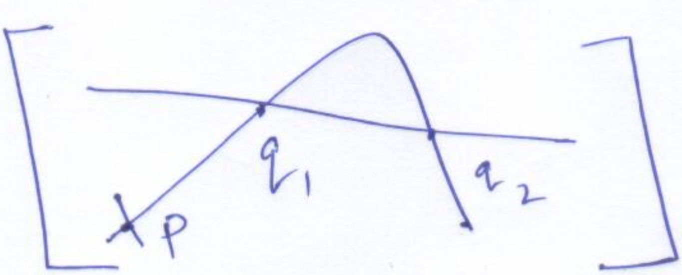
the following ~~equalitu~~ equality

of Homology Classes hold.

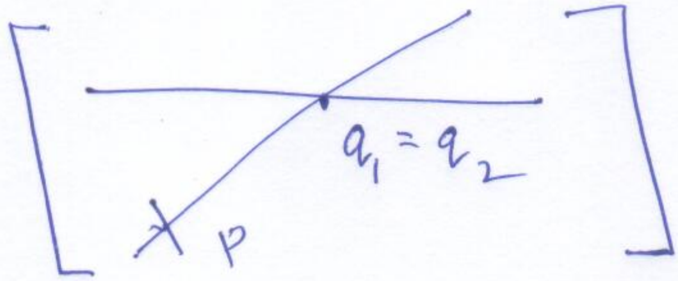


$\cdot\ [q_2 \in L] \cdot [q_2 \in C_3]$

$=$



$+$



Remark. The spaces inside the [ ] haven't been defined, but their definitions are easy to guess.

Note $[q_2 \in L] = (y_1 + a_2)$.
$[q_2 \in C_3] = (y_3 + 3a_2)$

(See Page 5).

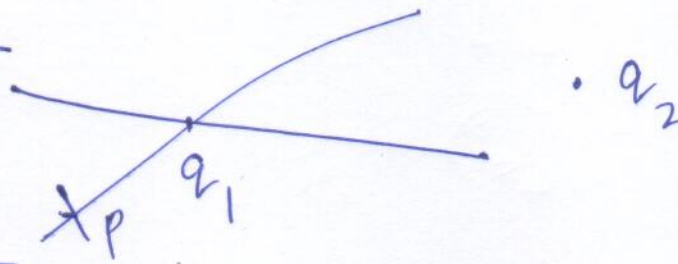
Define $\mathcal{U} := y_1^2 \cdot y_3^6 \cdot a_1 a_2$.

Intersect both sides of Step 2

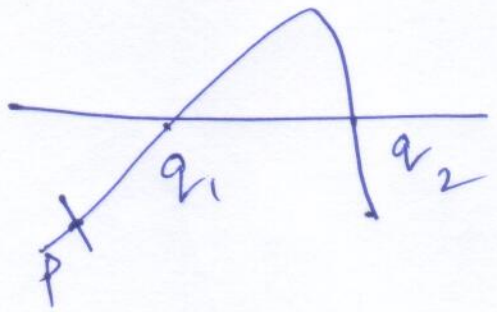
with $\mathcal{U}$.

~~Simplity~~ Simplify & what
you get is

(Also use
$\Delta_{\mathbb{P}^2_1 \times \mathbb{P}^2_2} = a_1^2 + a_1 a_2 + a_2^2$),



$$\left[\ \ \right] \cdot y_1^{2} \cdot y_3^{7} \cdot a_1 \cdot a_2^{2}$$



$$= \left[\ \ \right] \cdot y_1^{2} \cdot y_3^{6} \cdot a_1 a_2$$

A little bit of thought says.

LHS = # of Nodal Cubics thru
7 generic pts & one pt on line.

RHS = # of Nodal Cubics thru
6 generic points &
2 pts on the line.

Summary.

Initial Question

of ~~Nodal~~ Nodal Cubics thru 8 generic pts

Step 1

8 pts $\rightsquigarrow$ 7 pts + 1 pt on line

Step 2.

7 pts + 1 pt on line $\rightsquigarrow$

6 pts + 2 pts on line

25

Q.

# of Nodal cubics thru
6 generic pts & 2 pts on line

$\stackrel{?}{=}$ # of Nodal cubics thru
5 generic pts &
3 pts on line ?

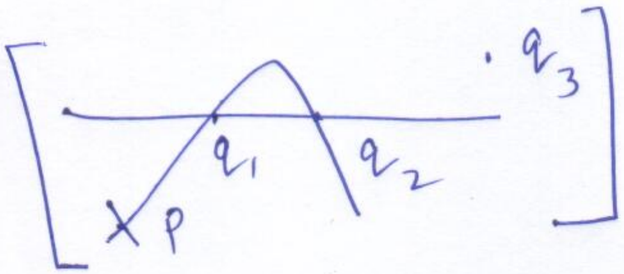
Answer

No !!

Step 3. Define

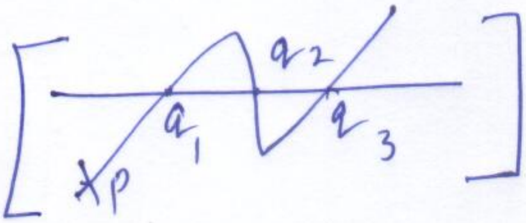
$$M_3 := \begin{array}{cccccc} \mathcal{D}_1 \times & \mathcal{D}_3 \times & \mathbb{P}^2 \times & \mathbb{P}^2_1 \times & \mathbb{P}^2_2 \times & \mathbb{P}^2_3 \\ L & \mathcal{C}_3 & p & q_1 & q_2 & q_3 \\ \multicolumn{6}{c}{\underbrace{y_1 \qquad y_3 \qquad b \qquad a_1 \qquad a_2 \qquad a_3}_{\text{Hyperplane classes}}} \end{array}$$

Claim On $H^*(M_3)$, the following equality of Homology classes hold.

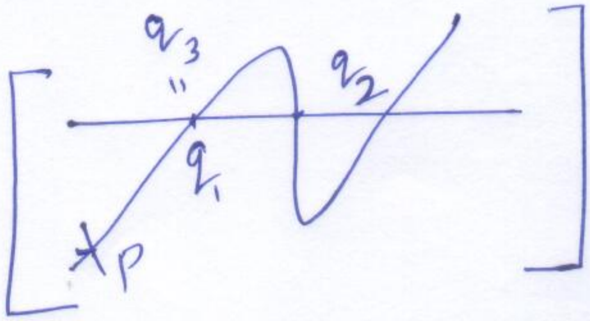


$\cdot\, [q_3 \in L] \cdot [q_3 \in C_3]$

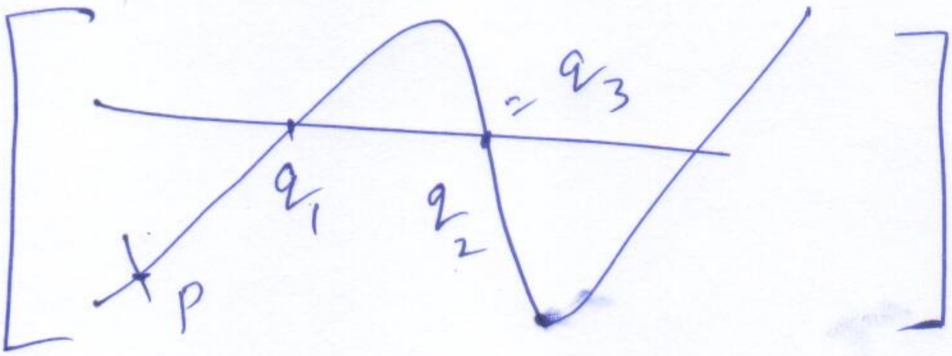
$=$



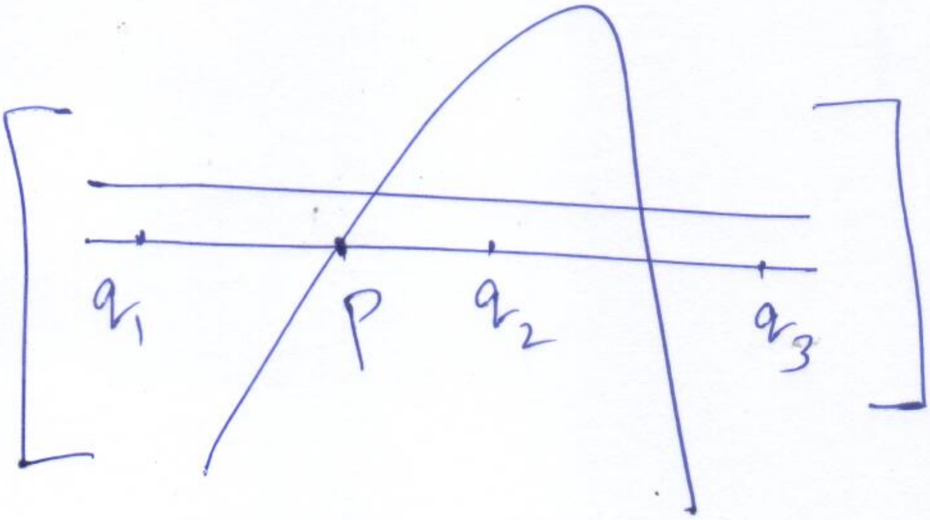
$+$

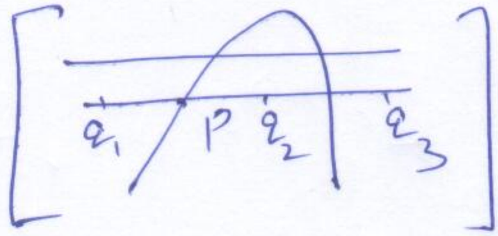


$+$



$+$

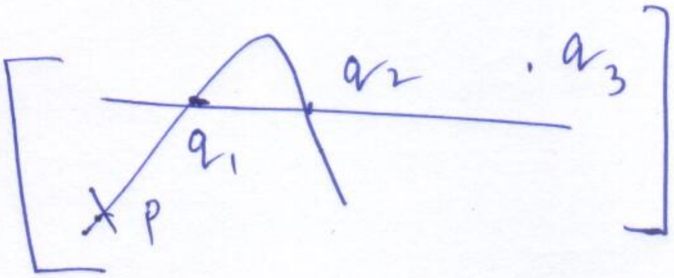


That last term = 

is new.

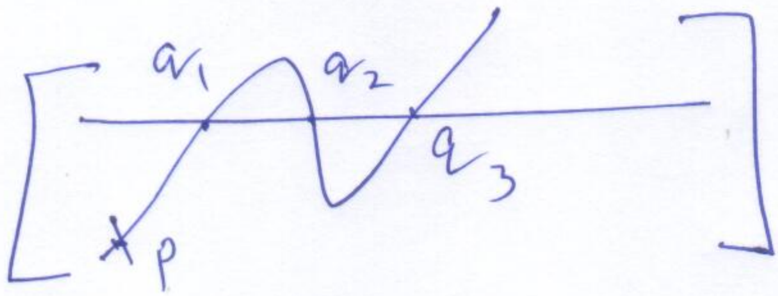
Take $\mathcal{Y} := y_1^2 \cdot y_3^5 \cdot a_1 a_2 a_3$

Intersect both sides and get.



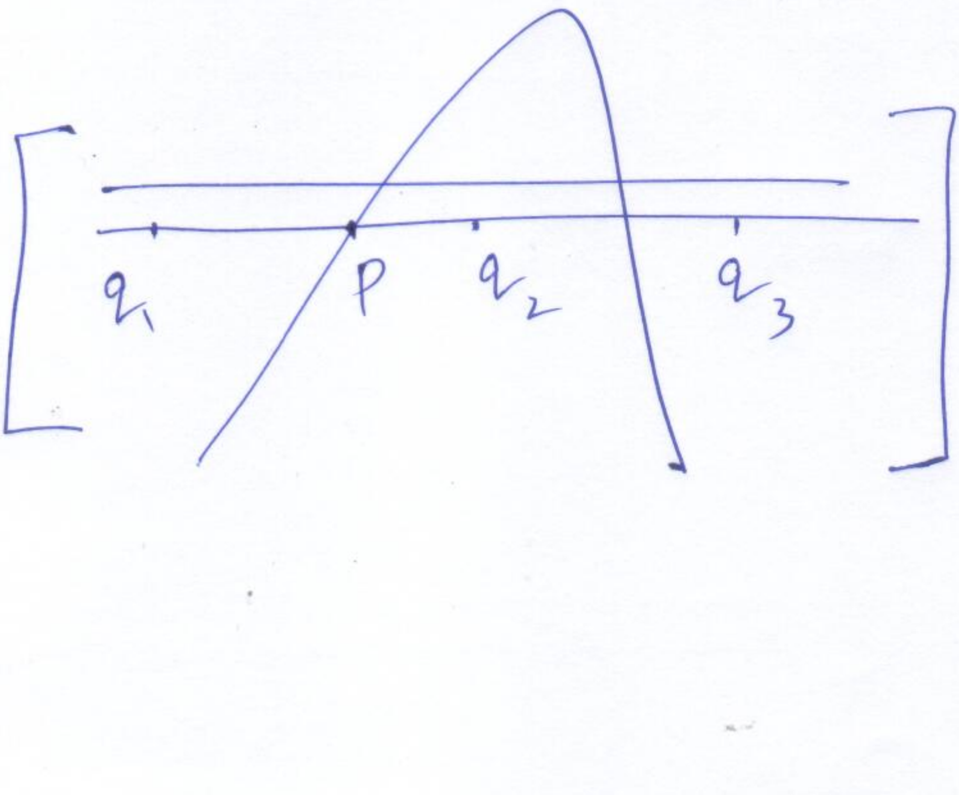
$$\cdot\, y_1^2 \cdot y_3^6 \cdot a_1\, a_2 \cdot a_3^2$$

$=$



$$\cdot\, y_1^2 \cdot y_3^5 \cdot a_1 a_2 a_3$$

$+$

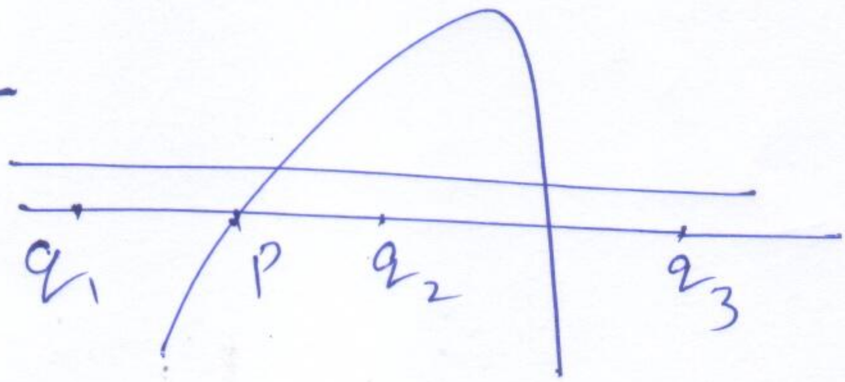


$$\cdot\, y_1^2 \cdot y_3^5 \; a_1 a_2 a_3$$

Hence, step 3 says

# of Nodal Cubics thru 6 generic pts & 2 pts on a line

= # of Nodal Cubics thru 5 generic pts & 3 pts on a line

$+$ [] $\cdot\, y_1^{\,2} \cdot y_3^{\,5}\, q_1 q_2 q_3$

The second term is

something involving conics

(Infact a little bit of thought

says the number is 2

Reason One conic thru 5

generic pts $\times$ (2 choices for the nodal point p.)

$= 2$).

Anyway our status is,

we have reduced our question to computing

1) # of nodal cubics thru 5 generic pts & 3 pts on line

2) Counts of Conics.

Let us address 1).

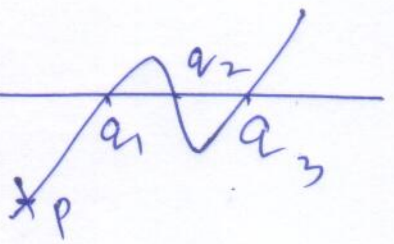
<u>Step 4.</u> (Final Step).

Define

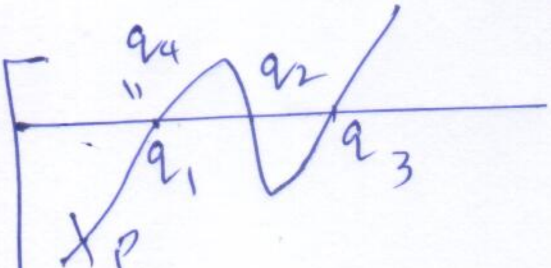
$$
\begin{array}{ccccccccc}
M_4 := & \mathcal{D}_1 \times & \mathcal{D}_3 \times & \mathbb{P}^2 \times & \mathbb{P}^2_1 \times & \mathbb{P}^2_2 \times & \mathbb{P}^2_3 \times & \mathbb{P}^2_4 \\
& L & e_3 & q_p & q_1 & q_2 & q_3 & q_4 \\
& & & & & & & \\
& y_1 & y_3 & b & a_1 & a_2 & a_3 & a_4
\end{array}
$$

<u>Claim</u> On $H^*(M_4)$, the following equality of Homology classes hold.

34

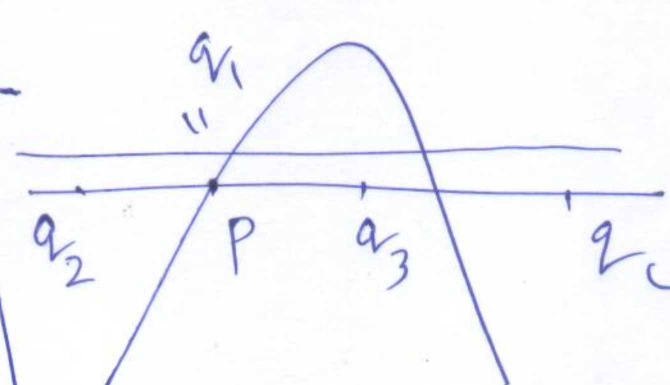
$\Bigg[$ 

$\cdot\, q_4 \Bigg] \cdot [q_4 \in L] \cdot [q_4 \in C_3]$

$= \Bigg[$ 

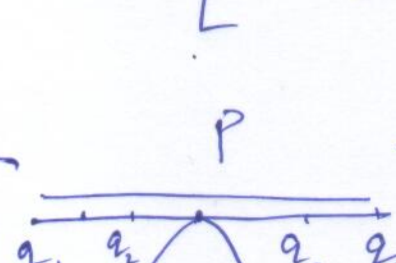
$\Bigg] + [q_2 = q_4]$

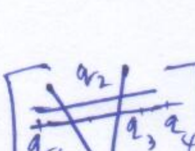
$+ [q_3 = q_4]$

$+ \Bigg[$ 

$\Bigg]$

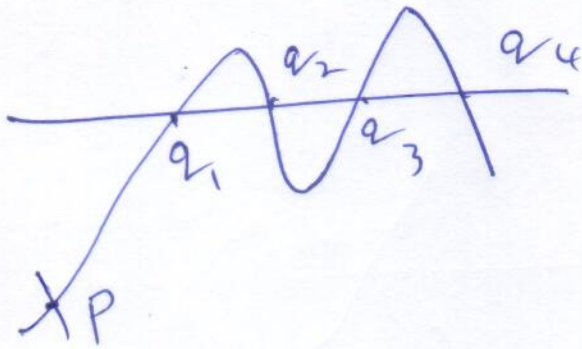
$+ [q_2 = p] + [q_3 = p]$

$+ \; 2 \Bigg[$ 

$\Bigg] + \Bigg[$ 

$\Bigg]$

Notice



= Empty Set

(Cubic won't intersect a line at 4 pts).

Anyway, define

$$e\!f := y_1^{2} \cdot y_3^{4} \cdot a_1 a_2 a_3 a_4$$

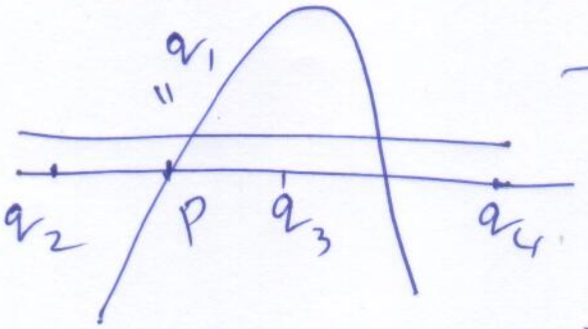
intersect both sides with $e\!f$

and get that

36

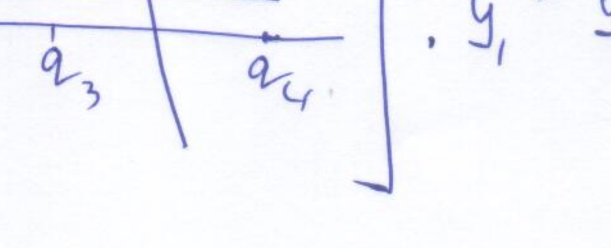
# of Nodal Cubics passing thru

5 generic points & 3 pts on the line

$= \Big[$ 

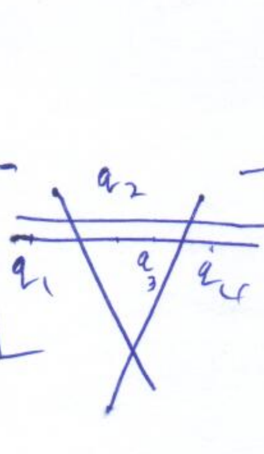
$\Big] \cdot y_1^{2}\, y_3^{4} \cdot a_1 a_2 a_3 a$

$+$ 2 Similar terms

$+ 2 \Big[$ 

$\Big] \cdot$

$y_1^{2} \cdot y_3^{4} \cdot a_1 a_2 a_3 a_4$

$+ \Big[$ 

$\Big] \cdot y_1^{2} \cdot y_3^{4} \cdot a_1 a_2 a_3 a_4$

$$= 3 \times (1)$$

$+2$ (# of Conics thru 4 generic pts tangent to a line)

$+$ (# of Nodal Conics thru 4 generic pts).

<u>Conclusion</u> # of Nodal Cubics thru 8 generic pts

$\rightsquigarrow$ Converted to a Question about Counting Conics (possibly tangent to the line) (or possibly nodal conics).

In general.

$[A_1] \cdot y_d^{S_d - 1}$

= # of Nodal Cubic thru

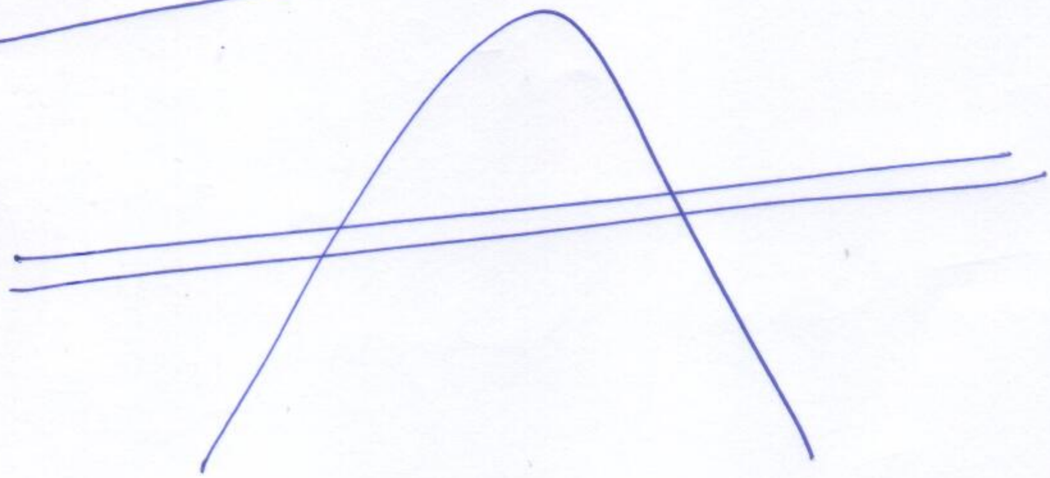
$S_d - 1$ generic pts

= $(d+1)$ - Step process.

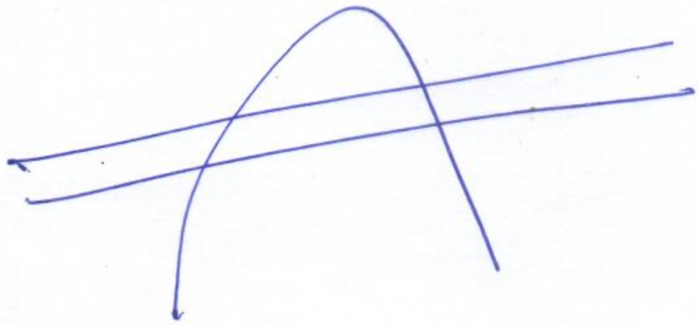
Till $(d-1)$ Step more pts

safely to the line.

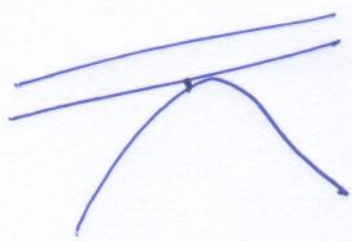
<u>Step (d).</u> We get:



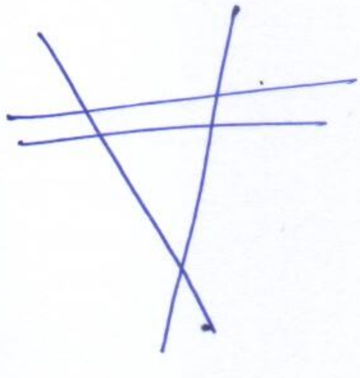
<u>Step (d+1)</u> We get:



&



&



Let us now explain

how to compute

$$[A_2] \cdot \mathcal{Y}_d^{\delta_d - 2}$$

$= \#$ of cuspidal deg d

curves thru $\delta_d - 2$ generic

pts.

Idea is similar.

There are $(d+1)$ - steps.

(41)

For simplicity, let us focus on $d = 3$

i.e.

$$[A_2] \cdot Y_3^7 = ?$$

(Later we will take $d = 4$ For a new thing)

There are 4 Steps.

<u>Step 1.</u>

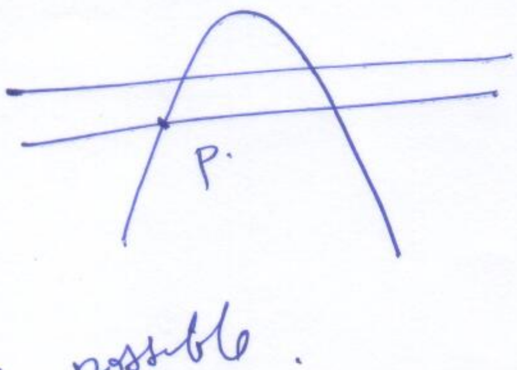
7 pts $\rightarrow$ 6 pt + 1 pt on line.

42

<u>Step 2.</u> Same, but requires a bit of care.

7 pts $\rightsquigarrow$ 5 pt + 2 pt on line.

But 5 pt $\Rightarrow$ Dimensionally

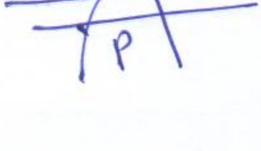


is possible.

(Conic goes thru 5 pts).

But Cusp $\rightsquigarrow$ Node

= Impossible.

$\therefore$ 

Can not occurs in Step 2.

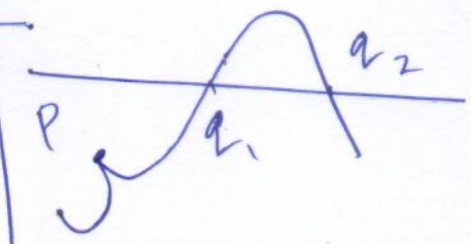
(43)

\# Cuspidal Cubics thru 7 pts = \# Cuspidal cubics thru 5 pts & 2 pts on line.

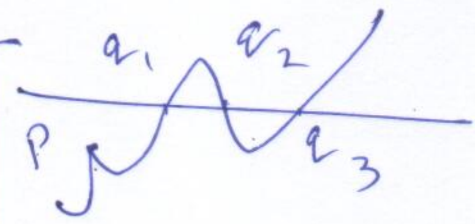
44

Step 3

This is more non ~~to~~ trivial.

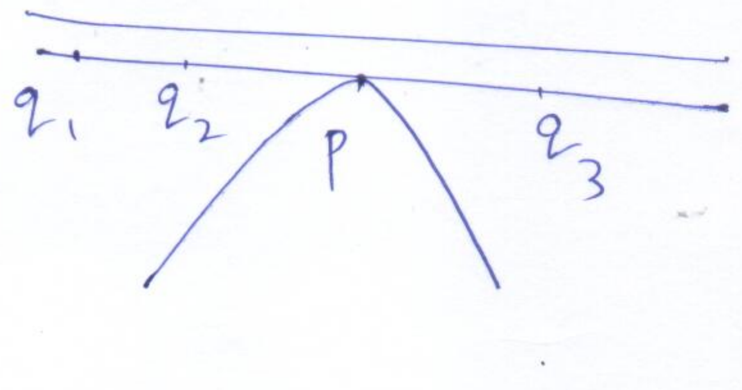
[

$\cdot\, q_3$] $\cdot\ [q_3 \in L] \cdot [q_3 \in C_3]$

= [

] $+ (q_1 = q_3)$

$+ (q_2 = q_3)$

$+\ 3$ [

]

45

Define $\mathcal{M} := y_1^2 \cdot y_3^4 \cdot a_1 a_2 a_3$

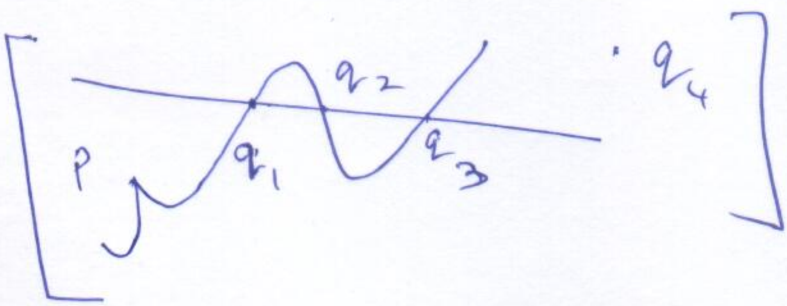
Intersect with $\mathcal{M}$ and get:

Cuspidal Cubics thru 5 pts
& 2 pt on line

$=$ # Cusp cubics thru 4 pts
& 3 pts on line

$+ 3$ (# Conics tang to a given line, passing thru 4 pts).

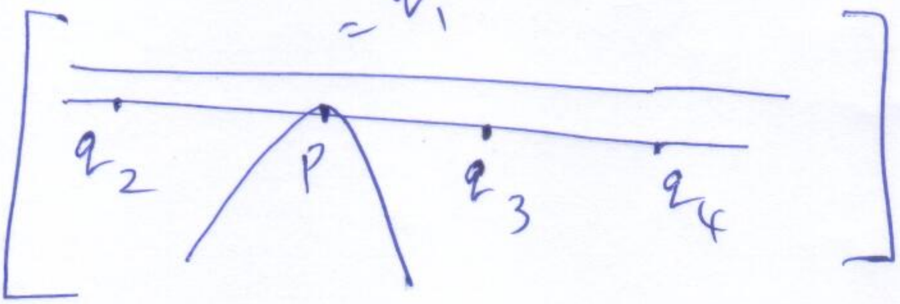
Finally, Step 4:



$\cdot\,[q_4 \in L]\cdot[q_4 \in \mathcal{C}_3]$

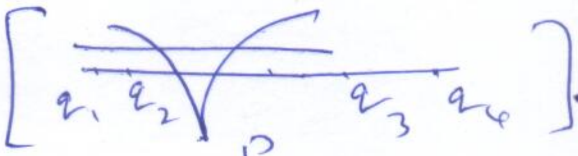
$= (q_1 = q_4) + (q_2 = q_4) + (q_3 = q_4)$

$+\,3$



$+$ 2 more similar terms $(q_2 = p)$ & $(q_3 = p)$.

$+$

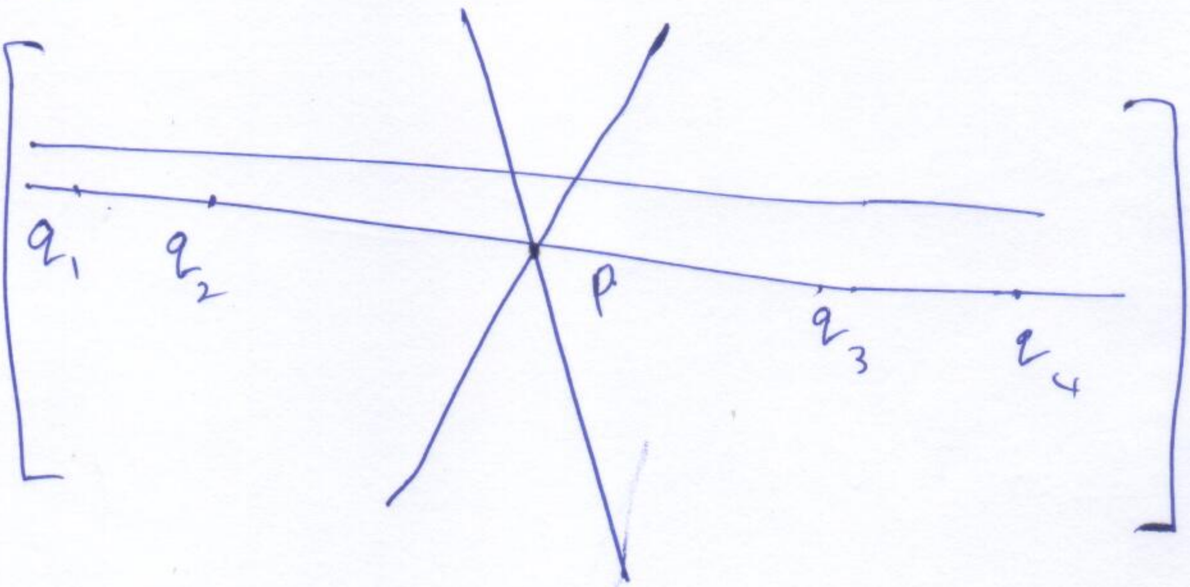


$\longrightarrow$ Empty

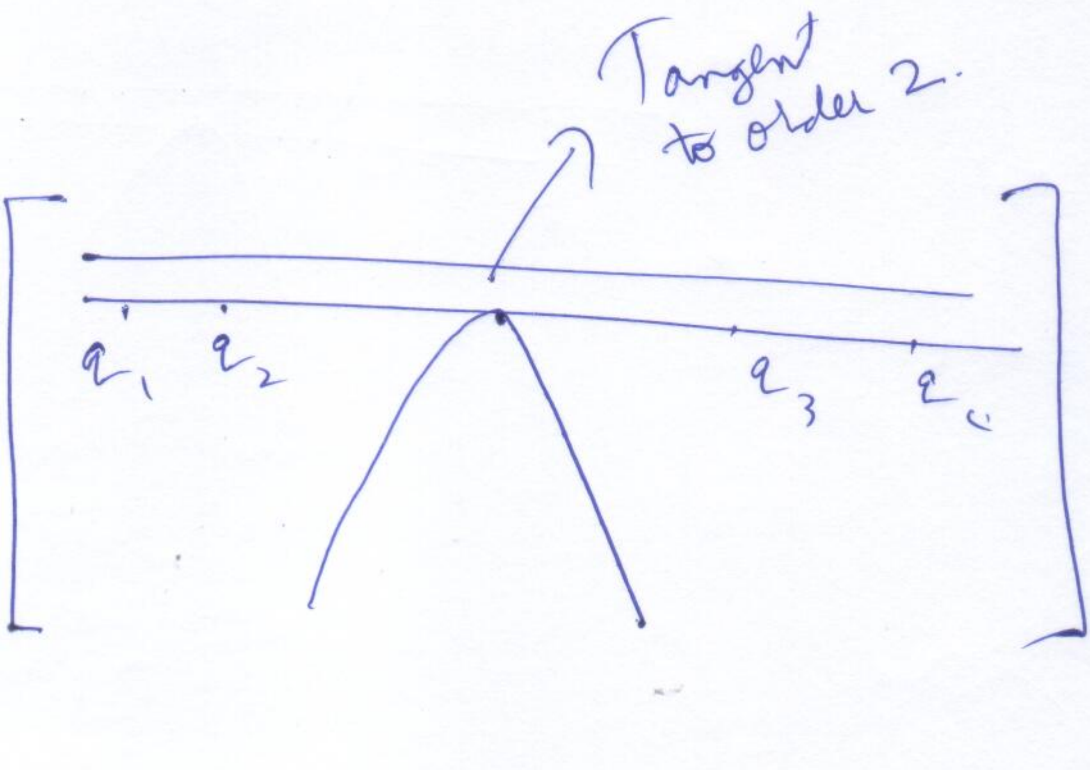
$+$ Turn next Page.

Continued

$+ \; 3$



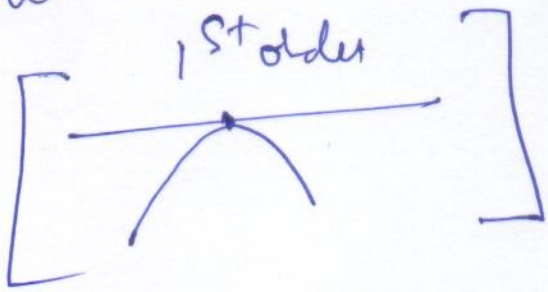
$+ \; 2$



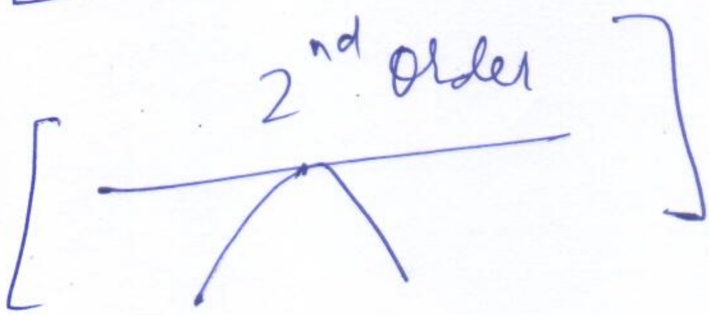
Note Last term is empty since $d - 1 = 2$.

Intersect with $\ell = y_1^2 \cdot y_3^3 \cdot a_1 a_2 a_3 a_4$

Hence our task boils down to



(Constraint)



. [Constraint]

AND.

. [ ]

= # of Nodal Cubics thru 7 generic pt & node on a line.

49

How to compute the last number?

Review the computation of

$[A_1] \cdot y_1^2 \cdot y_3^8$.

At each step, intersect with

$y_1^2 \cdot y_3^6 \cdot b_1 \cdot a_1$ Step 1

$y_1^2 \cdot y_3^5 \cdot b_1 \cdot a_1 a_2$ Step 2

$y_1^2 \cdot y_3^{4} \cdot b_1 a_1 a_2 a_3$ Step 3

$y_1^2 \cdot y_3^3 \cdot b_1 a_1 a_2 a_3 a_4$ Step 4.

This allows you to compute

$[A_1] \cdot y_1^2 \cdot y_3^7 \cdot b_1$

= which is what we want.

Remark.

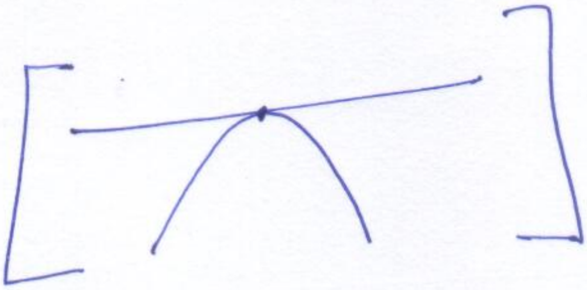
$$[A_1] \cdot y_1^2 \, y_3^7 \cdot b_1$$

Can not immediately be computed from the Caporaso Harris formula as given in their paper.

In general, the computation of $[A_2] \cdot y_d^{S_d - 2}$ is as follows: It is a $(d+1)$-step process. At each step, move a point to the line. Till $(d-1)$ steps the number is unchanged.

At the d^{th} Step, there is a

Contribution from.

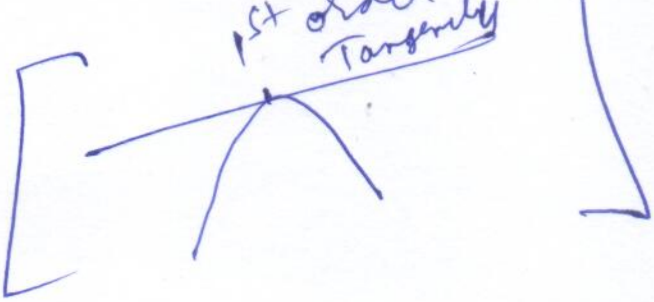
 $\cdot \left[\delta_d - 2 - d \right]$ pts

with Mult ≥ 3

Note $\delta_d - 2 - d = \delta_{d-1} - 1.$

Anyway at d^{th} Step

we get a contribution from.



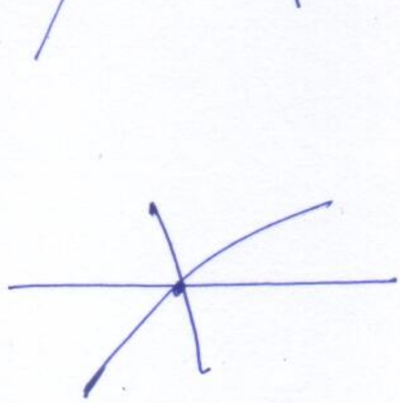
with Mult ≥ 3

Finally, at ~~d^{th}~~ $(d+1)^{th}$ Step, we get Contribution from

(a) 2^{nd} Order Tangency

Mult = 2.

(b)



Node on a Line

Mult = 3.

(c)

Cuspidal Curves of deg $(d-1)$

In Conclusion, $[A_2]\cdot y_d^{\,S_d-2}$

Can be computed recursively.

Final Remark. CH can also

be extended to a family version.

Q. How many nodal cubics are there in $\mathbb{P}^3$, whose image lies in $\mathbb{P}^2$ that ~~pass~~ intersect 11 generic lines?

A = ~~12969~~ 12960

Can be computed using the method developed here (i.e. suitably interpreting CH on the level of cycles).

Thank you for

your attention