

Caporaso - Harris

for Cusps.

①

Setup

We look at degree d -curves

in $\mathbb{P}^2 \approx$ homogeneous deg d poly,
non zero, upto scaling

$$= \mathcal{D}_d = \mathbb{P}^{S_d}$$

where
$$S_d := \frac{d(d+3)}{2}$$

(2)

D fine

$$A_1 := \{ (C_d, P) \in \mathcal{D}_d \times \mathbb{P}^2 \mid$$

P is a nodal point of the curve

$$C_d \} = \left[\begin{array}{c} \text{X} \\ P \end{array} \right]$$

$$A_2 := \{ (C_d, P) \in \mathcal{D}_d \times \mathbb{P}^2 \mid$$

P is a cuspidal point of the

$$\text{curve } C_d \} = \left[\begin{array}{c} \text{P} \curvearrowright \end{array} \right]$$

$$\left. \begin{array}{l} y_d \in H^*(\mathcal{D}_d) \\ b \in H^*(\mathbb{P}^2) \end{array} \right\} \text{Respective Hypersplane Classes.}$$

(3)

Goal :- Compute all intersection
numbers involving the class

$$[A_1] \quad \& \quad [A_2].$$

Eg: $[A_1] \cdot y_d^{S_d-1}$

= # of degree d curves in
 $\mathbb{P}^2$ passing thru S_d-1 generic
points & having a node.

/// by $[A_2] \cdot y_d^{S_d-2}$

= having a cusp.

(4)

Let us focus on $d=3$
for simplicity.

Q. What is

$$\left. \begin{array}{l} [A_1] \cdot y_3^8 = ? \\ [A_2] \cdot y_3^7 = ? \end{array} \right\}$$

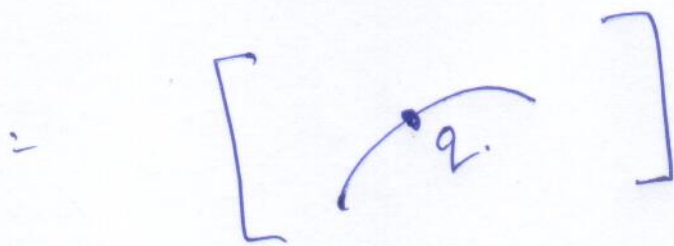
We will start with the
Computation $[A_1] \cdot y_3^8$

(5)

Recap.

Incidence Variety

$$I := \{ (\mathcal{C}_d, q) \in \mathcal{Q}_d \times \mathbb{P}^2 \mid q \in \mathcal{C}_d \}$$



Standard Fact.

Homology Class

represented by I is

$$[I] = (y_d + d b)$$

$$\in H^*(\mathcal{Q}_d \times \mathbb{P}^2).$$

(6)

The Computation of $[A_i] \cdot y_3^8$
will involve 4 Steps

(in general, the Computation of
 $[A_i] \cdot y_d^{s_d-1}$ will involve $(d+1)$
steps).

Let us now introduce the
ambient Space of Step 1
where we will do intersection
theory.

(7)

Define

$$M_1 := \underbrace{\mathcal{Q}_1 \times \mathcal{Q}_3 \times \mathbb{P}^2 \times \mathbb{P}^2_1}_{\substack{\text{Line} \quad \mathcal{C}_3 \quad p \quad q_1 \\ y_1 \quad y_3 \quad b \quad a_1 \\ \text{Hypersurface Classes.}}}$$

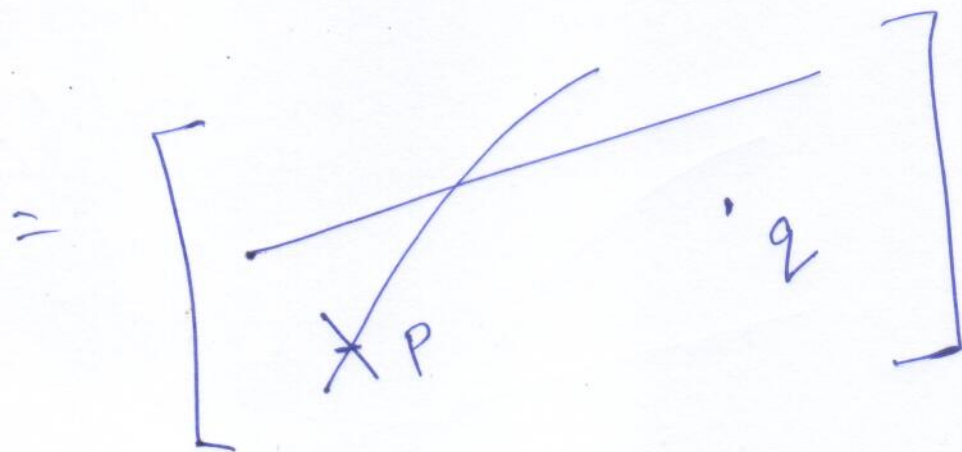
Pictorially

$$M_1 = \left[\begin{array}{c} \text{Diagram showing a line and a curve intersecting, with points } p \text{ and } q_1 \text{ marked.} \end{array} \right]$$

(8)

Let us define a Subspace γ
 M_1 called A_1^F

$$A_1^F := \left\{ (\text{line}, \mathcal{C}_3, p, q_1) \in \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2 \times \mathbb{P}_1^2 \mid \begin{array}{l} p \text{ is a nodal point of } \mathcal{C}_3 \\ q_1 \end{array} \right\}.$$



The point q is free; it
 need not lie on the line or
 the Cubic.

(9)

Note we call it A_1^F ; the

F is there to emphasize that the nodal point is "Free" (it need not lie on the line).

Remark.

The definition of A_1^F is a small lie. Actually there is one more condition: I require the nodal point p to NOT lie on the line. Let us ignore this point for now, but it will be important later (in Step 3 & 4)

(10)

Let us proceed. Let us now
impose the condition that the
point q lies on the line and
the ~~Curve~~ Curve.

Recall (Page 5)

$$[q \in \text{Line}] = (y_1 + a_1).$$

$$[q \in \text{Cubic}] = (y_3 + 3a_1).$$

(11)

Finally, we can state the first step
of Caporaso - Harris.

Step 1. On $H^*(M_1)$, the following
equality holds.

$$\left[\begin{array}{c} \text{Diagram: A line and a cubic curve intersecting at a point marked with an 'X' and labeled 'P'.} \\ \text{X P} \end{array} \cdot q_1 \right] \cdot [q \in \text{Line}] \cdot [q \in \text{Cubic}]$$

$$= \left[\begin{array}{c} \text{Diagram: A line and a cubic curve intersecting at two points. The intersection point on the line is marked with an 'X' and labeled 'P'. The intersection point on the cubic is labeled 'q_1'.} \\ \text{P X } q_1 \end{array} \right]$$

$$= \left[\begin{array}{cc} A_1^F & T_0 \end{array} \right]$$

(12)

Where

$$A_1^F T_0 = \{ (\text{line}, C_3, P, q_1) \in \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2 \times \mathbb{P}_1^2 \}.$$

P is nodal point of Cubic &

q_1 lies on line and Cubic &

$$(*) \quad P \neq q_1, \quad P \notin \text{line} \}.$$

Remark.

$$[S] = \text{Homology Class rep by Closure } \overline{S}.$$

When we take Closure,

(*) becomes irrelevant.

(13)

#3

Remark. Proof of Step 1 is

as follows.

On the set theoretic level the statement is obvious.

The non trivial part is that the intersections are transverse. That is easy to show here by writing things in local coordinates

(That is in fact the principle everywhere in the following steps — it just gets more and more complicated).

(14)

Let us now see the
numerical consequence of Step 1:

Let us intersect with a class
of complementary dimension.

Define

$$e_1 := y_1^2 \cdot y_3^7 \cdot a_1$$

(a_1 = Hyperplane Class of $\mathbb{P}_1^2$
See Page (7)).

(15)

Intersecting both sides of Step 1
with e_1 gives us

$$\left[\begin{array}{c} \text{Diagram} \\ \times_P \end{array} \cdot q \right] \cdot (y_1 + a_1)(y_3 + 3a_1) \cdot y_1^2 \cdot y_3^7 \cdot a_1$$

$$= \left[\begin{array}{c} \text{Diagram} \\ \times_P \quad a_1 \end{array} \right] \cdot y_1^2 \cdot y_3^7 \cdot a_1$$

Use that $y_1^3 = 0$, $a_1^3 = 0$

The above equation
simplifies to

(16)

$$\left[\begin{array}{c} \cdot a \\ \times p \end{array} \right] \cdot y_1^2 \cdot y_3^8 \cdot a_1^2$$

$$= \left[\begin{array}{c} \cdot a \\ \times p \end{array} \right] \cdot y_1^2 \cdot y_3^7 \cdot a_1$$

Let us now process the
left hand side.

(17)

Intersecting with y_1^2

→ Line is fixed.

Intersecting with a_1^2

→ q is fixed.

∴ The LHS is

$$\left[\cancel{A}^P \right] \cdot y_3^8$$

$$= [A_1] \cdot y_3^8 = \text{The number we wanted}$$

= # of Nodal Cubics there
8 generic points.

(18)

Let us now look at the RHS.

$$\left[\begin{array}{c} \text{Diagram of a line and a curve intersecting at point } q \text{ and } x_P \end{array} \right] \cdot y_1^2 \cdot y_3^7 \cdot a_1.$$

$y_1^2 \rightsquigarrow$ Line is fixed.

Intersecting with $a_1 \rightsquigarrow q$ is fixed.

(Intersection of the line given by y_1^2 & a_1).

$$\therefore \left[\begin{array}{c} \text{Diagram of a line and a curve intersecting at point } q \text{ and } x_P \end{array} \right] \cdot y_1^2 \cdot y_3^7 \cdot a_1$$

= # nodal Cubics. then

7 generic points & One pt
on the line.

Conclusion

of Nodal cubics thru

8 generic points

= # of Nodal Cubics thru

7 generic points &

one point on a line.

Remark. In Hindsight, an

obvious fact. But let us

proceed.

(20)

Step 2.

Define

$$M_2 := \mathcal{Q}_1 \times \mathcal{Q}_3 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2.$$

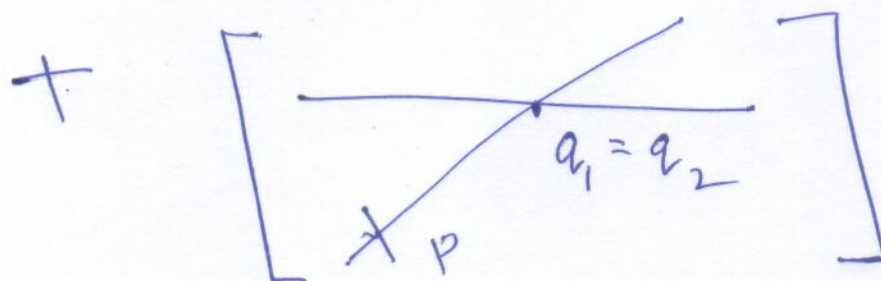
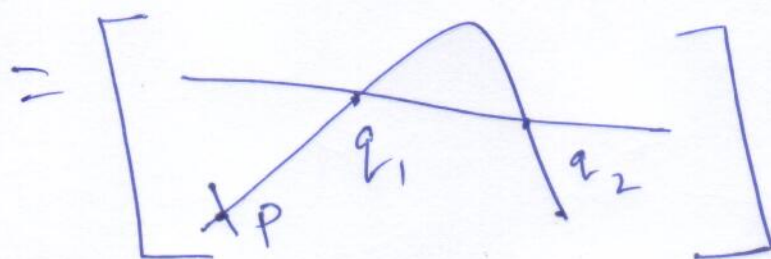
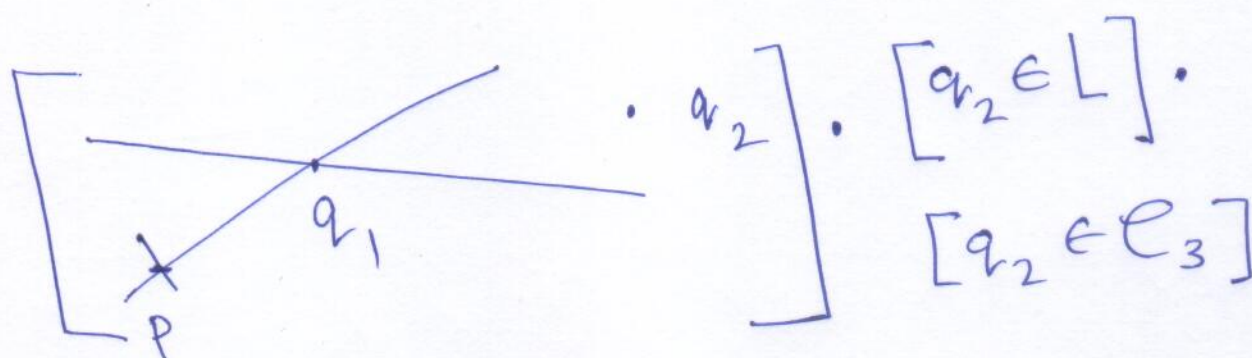
L	$\mathcal{C}_3$	P	a_1	a_2
y_1	y_3	b	a_1	a_2

⏟
Hyperplane Class.

Claim On $H^*(M_2)$

the following ~~equality~~ equality
of Homology Classes hold.

(21)



Remark. The spaces inside the $[]$ haven't been defined, but their definitions are easy to guess.

(22)

Note $[q_2 \in L] = (y_1 + a_2).$
 $[q_2 \in C_3] = (y_3 + 3a_2)$

(See Page ⑤)

Define $e_1 := y_1^2 \cdot y_3^6 \cdot a_1 a_2.$

Intersect both sides of Step 2
with $e_1.$

Simplify & what
you get is

(Also use $\Delta_{\mathbb{P}_1^2 \times \mathbb{P}_2^2} = a_1^2 + a_1 a_2 + a_2^2$)

(23)

$$\left[\begin{array}{c} \text{Diagram: A line and a curve intersecting at point } q_1. A point } p \text{ is marked on the line.} \\ p \quad q_1 \end{array} \right] \cdot a_2 \cdot y_1^2 \cdot y_3^7 \cdot a_1 \cdot a_2^2$$

$$= \left[\begin{array}{c} \text{Diagram: A line and a curve intersecting at point } q_1. A point } p \text{ is marked on the line. A point } q_2 \text{ is marked on the curve.} \\ p \quad q_1 \quad q_2 \end{array} \right] \cdot y_1^2 \cdot y_3^6 \cdot a_1 \cdot a_2$$

A little bit of thought says.

LHS = # of Nodal Cubics thru
7 generic pts & one pt on line.

RHS = # of Nodal Cubics thru
6 generic points &
2 pts on the line.

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Summary.

Initial Question

of ~~Nodal~~ Nodal Cubics then
8 generic pts.

Step 1

8 pts $\leadsto$ 7 pts + 1 pt on line

Step 2.

7 pts + 1 pt on line $\leadsto$

6 pts +

2 pts on line

(25)

Q.

of Nodal cubics thru
6 generic pts & 2 pts on line

? # of Nodal cubics thru
5 generic pts &
3 pts on line?

Answer

No!!

(26)

Step 3.

Define

$$M_3 := \mathcal{O}_1 \times \mathcal{O}_3 \times \mathbb{P}^2 \times \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2$$

L	e_3	p	a_1	a_2	a_3
y_1	y_3	b	a_1	a_2	a_3

⏟
Hyperplane classes

Claim On $H^*(M_3)$,

the following equality of Homology classes hold.

(27)

$$\left[\begin{array}{c} \text{Diagram 1} \\ \cdot q_3 \end{array} \right] \cdot [q_3 \in L] \cdot [q_3 \in \mathcal{C}_3]$$

Diagram 1: A horizontal line with a curve passing through it. The curve starts below the line, crosses it at point q_1 , reaches a peak, crosses it again at point q_2 , and ends below the line. A point x_p is marked on the line to the left of q_1 .

$$= \left[\begin{array}{c} \text{Diagram 2} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 3} \end{array} \right]$$

Diagram 2: Similar to Diagram 1, but the curve crosses the line at q_1 and q_3 , and ends below the line. q_2 is marked on the line between q_1 and q_3 . x_p is to the left of q_1 .

Diagram 3: Similar to Diagram 1, but the curve starts below the line, crosses it at q_1 , reaches a peak, crosses it again at q_2 , and ends above the line. q_3 is marked above the line to the left of q_1 . x_p is to the left of q_1 .

$$+ \left[\begin{array}{c} \text{Diagram 4} \end{array} \right]$$

Diagram 4: Similar to Diagram 1, but the curve starts below the line, crosses it at q_1 , reaches a peak, crosses it again at q_2 , and ends above the line. q_3 is marked on the line to the right of q_2 . x_p is to the left of q_1 .

$$+ \left[\begin{array}{c} \text{Diagram 5} \end{array} \right]$$

Diagram 5: A horizontal line with a curve passing through it. The curve starts below the line, crosses it at point P , reaches a peak, crosses it again, and ends below the line. Points q_1 , q_2 , and q_3 are marked on the line to the left of P , between P and the second crossing, and to the right of the second crossing, respectively.

(28)

That last term = $\left[\frac{a_1}{p} \mid a_2 \mid a_3 \right]$

is new.

Take $q := y_1^2 \cdot y_3^5 \cdot a_1 a_2 a_3$

Intersect both sides and get.

(29)

$$\left[\begin{array}{c} \text{Diagram 1: A horizontal line with points } q_1, q_2, q_3 \text{ marked above it. A curve starts at } x_p \text{ (marked below the line), passes through } q_1, \text{ and ends at } q_2. \\ x_p \quad q_1 \quad q_2 \quad q_3 \end{array} \right] \cdot y_1^2 \cdot y_3^6 \cdot a_1 a_2 a_3^2$$

$$= \left[\begin{array}{c} \text{Diagram 2: A horizontal line with points } q_1, q_2, q_3 \text{ marked above it. A curve starts at } x_p \text{ (marked below the line), passes through } q_1, \text{ and ends at } q_3. \\ q_1 \quad q_2 \quad q_3 \\ x_p \end{array} \right] \cdot y_1^2 \cdot y_3^5 \cdot a_1 a_2 a_3$$

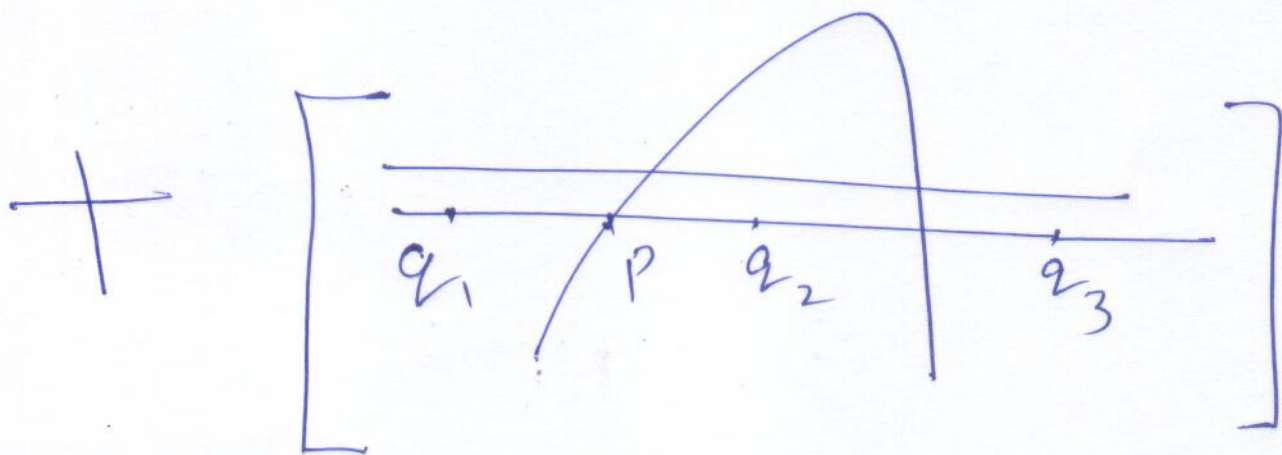
$$+ \left[\begin{array}{c} \text{Diagram 3: A horizontal line with points } q_1, p, q_2, q_3 \text{ marked below it. A curve starts at } q_1, passes through } p, \text{ and ends at } q_2. \\ q_1 \quad p \quad q_2 \quad q_3 \end{array} \right] \cdot y_1^2 \cdot y_3^5 \cdot a_1 a_2 a_3$$

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Hence, step 3 Says

of Nodal Cubics then 6 generic
pts & 2 pts on a line

= # of Nodal Cubics then 5 generic
pts & 3 pts on a line



$$y_1^2 \cdot y_3^5 \cdot q_1 \cdot q_2 \cdot q_3$$

(31)

The second term is

something involving conics

(In fact a little bit I thought
says the number is 2

Reason One conic then 5
generic pts $\times$ (2 choices for the
nodal point P')
 $= 2$) .

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Anyway our status is,

we have reduced our
question to computing

1) # of modal cubics thru 5
generic pts & 3 pts on line

2) Counts of Conics.

Let us address (1).

(33)

Step 4. (Final Step).

Define

$$M_4 := \mathcal{Q}_1 \times \mathcal{Q}_3 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

$$\begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ L & \mathcal{C}_3 & \mathbb{P} & a_1 & a_2 & a_3 & a_4 \\ & & & & & & \\ y_1 & y_3 & b & a_1 & a_2 & a_3 & a_4 \end{array}$$

Claim On $H^*(M_4)$,

the following equality of Homology classes hold.

(34)

$$\left[\begin{array}{c} \text{Diagram: A horizontal line with points } x_p, a_1, a_2, a_3. \text{ A curve passes through } a_1, a_2, a_3. \\ \cdot a_4 \end{array} \right] \cdot [a_4 \in L] \cdot [a_4 \in \mathcal{C}_3]$$

$$= \left[\begin{array}{c} \text{Diagram: A horizontal line with points } x_p, a_1, a_2, a_3. \text{ A curve passes through } a_1, a_2, a_3. \text{ A point } a_4 \text{ is marked above } a_1. \\ \cdot a_4 \end{array} \right] \neq \left[a_2 = a_4 \right] \\ + \left[a_3 = a_4 \right]$$

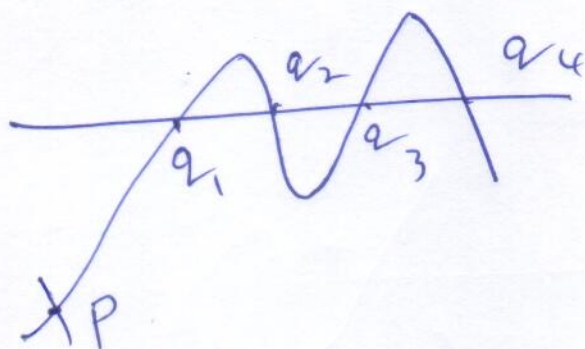
$$+ \left[\begin{array}{c} \text{Diagram: A horizontal line with points } a_2, p, a_3, a_4. \text{ A curve passes through } p. \text{ A point } a_1 \text{ is marked above } p. \\ \cdot a_1 \end{array} \right]$$

$$+ \left[a_2 = p \right] + \left[a_3 = p \right]$$

$$+ 2 \left[\begin{array}{c} \text{Diagram: A horizontal line with points } a_1, a_2, p, a_3, a_4. \text{ A curve passes through } p. \\ \cdot p \end{array} \right] + \left[\begin{array}{c} \text{Diagram: A horizontal line with points } a_1, a_2, a_3, a_4. \text{ A curve passes through } a_1, a_2, a_3, a_4. \\ \cdot x_p \end{array} \right]$$

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Notice



= Empty
Set.

(Cubic won't intersect
a line at 4 pts).

Anyway, define

$$ef := y_1^2 \cdot y_3^4 \cdot a_1 a_2 a_3 a_4$$

intersect both sides with ef
and get that.

(36)

of Nodal Cubics passing thru
5 generic points & 3 pts on the line

$$= \left[\begin{array}{c} \text{Diagram 1: A cubic curve passing through points } a_1, a_2, a_3, a_4 \text{ on a line, with a node at } P. \end{array} \right] \cdot y_1^2 y_3^4 \cdot a_1 a_2 a_3 a_4$$

+ 2 Similar terms

$$+ 2 \left[\begin{array}{c} \text{Diagram 2: A cubic curve passing through points } a_1, a_2, a_3, a_4 \text{ on a line, with a node at } P. \end{array} \right] \cdot y_1^2 y_3^4 \cdot a_1 a_2 a_3 a_4$$

$$y_1^2 y_3^4 \cdot a_1 a_2 a_3 a_4$$

$$+ \left[\begin{array}{c} \text{Diagram 3: A cubic curve passing through points } a_1, a_2, a_3, a_4 \text{ on a line, with a node at } P. \end{array} \right] \cdot y_1^2 y_3^4 \cdot a_1 a_2 a_3 a_4$$

(37)

$$= 3 \times (1)$$

$$+ 2 \left(\# \text{ of Conics then } 4 \text{ generic pts tangent to a line} \right)$$

$$+ \left(\# \text{ of Nodal Conics then } 4 \text{ generic pts} \right).$$

Conclusion $\#$ of Nodal Cubics

then 8 generic pts

→ Converted to a Question

about Counting Conics

(possibly tangent to the line)

(or possibly nodal conics)

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In general.

$$[A_1] \cdot y_d^{S_d-1}$$

= # of Nodal Cubic thru

S_d-1 generic pts

= $(d+1)$ - Step process.

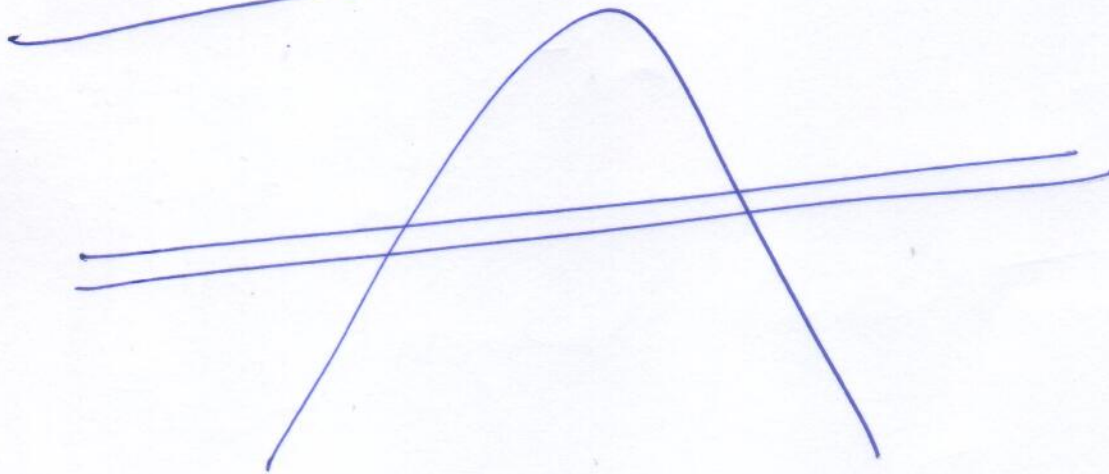
Till $(d-1)$ Step more pts

safely to the line.

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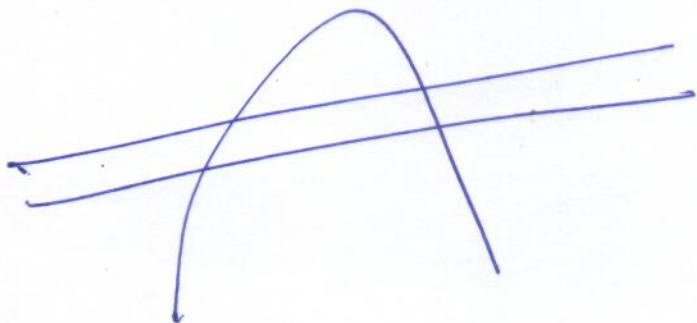
Step (d),

We get .



Step (d+1)

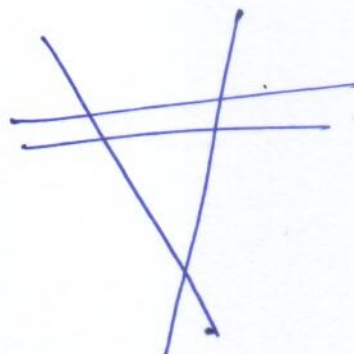
We get .



&



&



(40)

Let us now explain

how to compute

$$[A_2] \cdot y_d^{S_d-2}$$

= # of cuspidal deg d
curves then S_d-2 generic
pts.

Idea is similar.

There are $(d+1)$ -steps.

(41)

For simplicity, let us focus on $d=3$

i.e.

$$[A_2] \cdot y_3^7 = ?$$

(Later
we will
take $d=6$
For a
new thing)

There are 4 Steps.

Step 1.

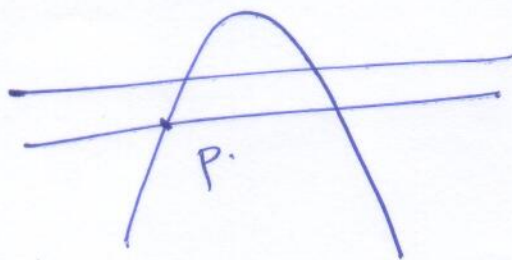
7 pts $\rightarrow$ 6pt + 1 pt on line ✓

(4 2)

Step 2. Same, but requires
a bit of care.

7 pts $\leadsto$ 5 pt + 2 pt on line.

But 5 pt $\Rightarrow$ Dimensionally,



is possible.

(Conic goes thru 5 pts).

But Cusp $\leadsto$ Node
= I impossible.



Can not occur in
Step 2.

(43)

Cuspidal Cubics thru

7 pts = # Cuspidal Cubics
thru 5 pts 2
2 pts on line.

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Step 3

This is more

non trivial

$$\left[\begin{array}{c} \text{Diagram: A curve } p \text{ with points } q_1, q_2, q_3 \text{ on it. } q_1 \text{ is a local minimum, } q_2 \text{ is a local maximum, and } q_3 \text{ is on the downward slope.} \\ p, q_1, q_2, q_3 \end{array} \right] \cdot q_3, [q_3 \in L], [q_3 \in \mathcal{C}_3]$$

$$= \left[\begin{array}{c} \text{Diagram: A curve } p \text{ with points } q_1, q_2, q_3 \text{ on it. } q_1 \text{ is a local minimum, } q_2 \text{ is a local maximum, and } q_3 \text{ is on the downward slope.} \\ p, q_1, q_2, q_3 \end{array} \right] + (q_1 = q_3) + (q_2 = q_3)$$

$$+ 3 \left[\begin{array}{c} \text{Diagram: A curve } p \text{ with points } q_1, q_2, q_3 \text{ on it. } q_1 \text{ and } q_2 \text{ are on the upward slope, } q_3 \text{ is on the downward slope, and } p \text{ is the peak.} \\ q_1, q_2, p, q_3 \end{array} \right]$$

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Define $\mathcal{C} := Y_1^2 \cdot Y_3^4 \cdot a_1 a_2 a_3$

Intersect with $\mathcal{C}$ and get.

Cuspidal Cubics thru 5 pts
& 2 pt on line

= # Cusp cubics thru 4 pts
& 3 pts on line

+ 3 (# Conics tang to
a given line, passing thru
4 pts)

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Finally, Step 4.

$$\left[\begin{array}{c} \text{Diagram: A line with points } p, q_1, q_2, q_3 \text{ and a curve above it.} \\ \cdot q_4 \end{array} \right] \cdot [q_4 \in L] \cdot [q_4 \in \mathcal{C}_3]$$

$$= (q_1 = q_4) + (q_2 = q_4) + (q_3 = q_4)$$

$$+ 3 \left[\begin{array}{c} \text{Diagram: A line with points } q_2, p, q_3, q_4 \text{ and a curve above it.} \\ \cdot q_1 \end{array} \right]$$

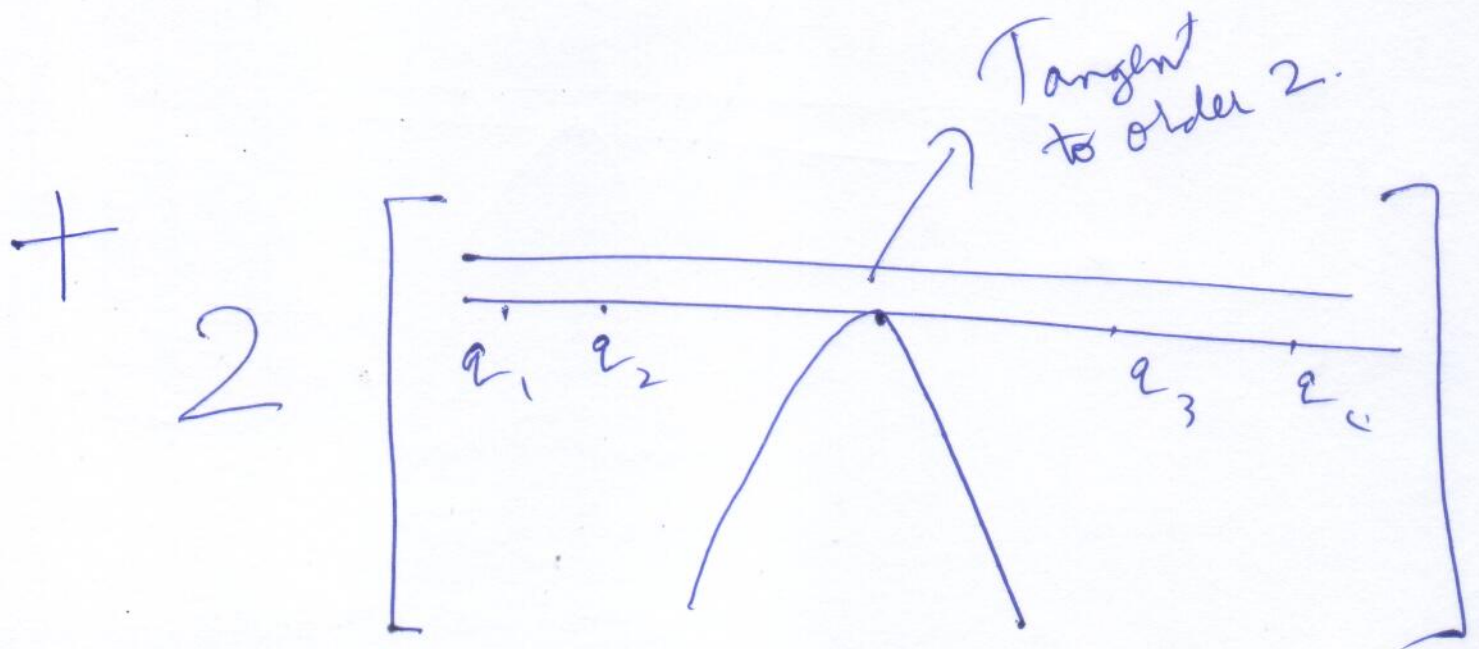
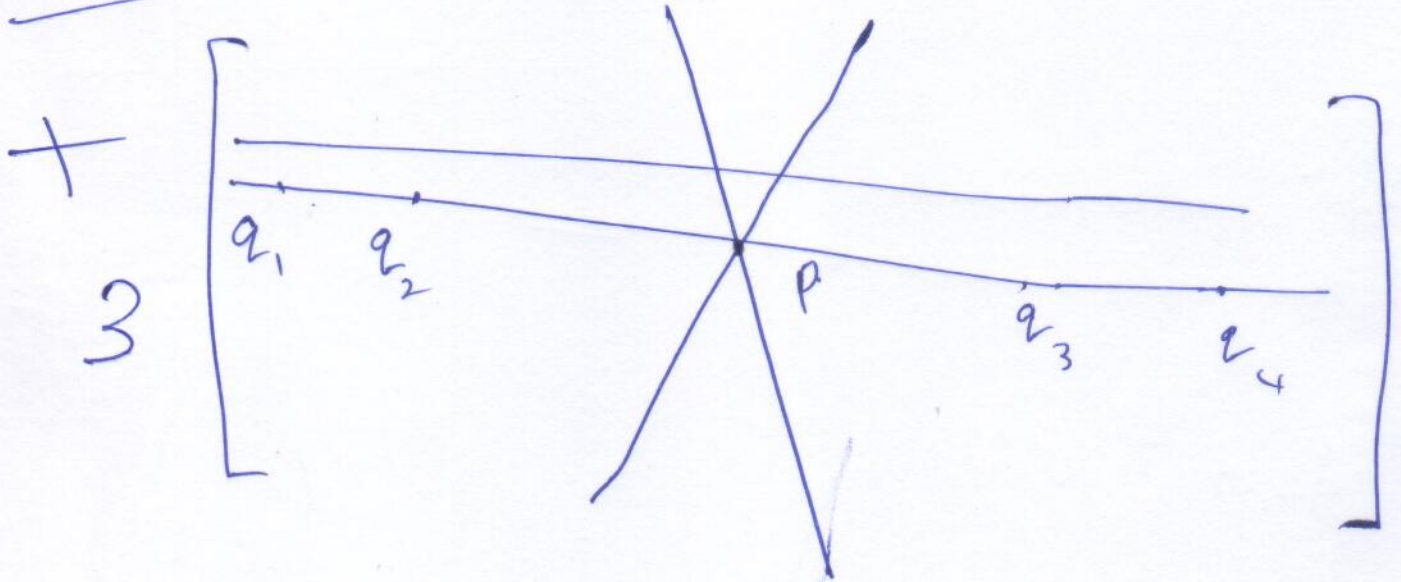
+ 2 more similar terms $(q_2 = p)$
 $\text{and } (q_3 = p).$

$$+ \left[\begin{array}{c} \text{Diagram: A line with points } q_1, q_2, p, q_3, q_4 \text{ and a curve above it.} \end{array} \right] \rightarrow \text{Empty}$$

+ Turn next Page.

(47)

Continued

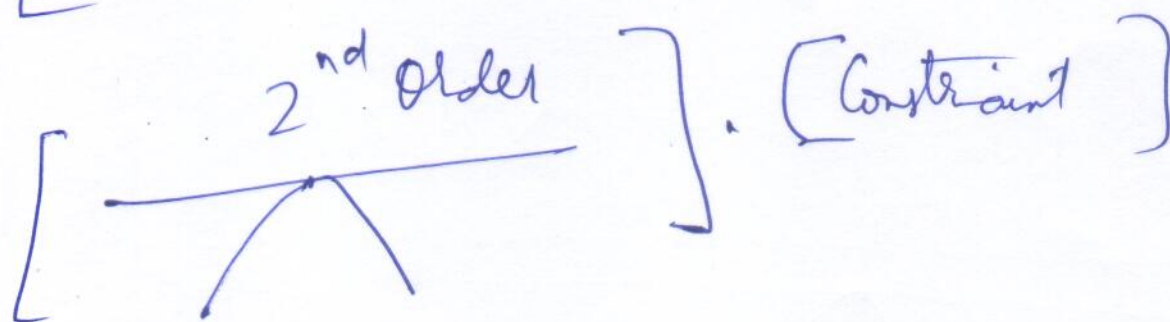
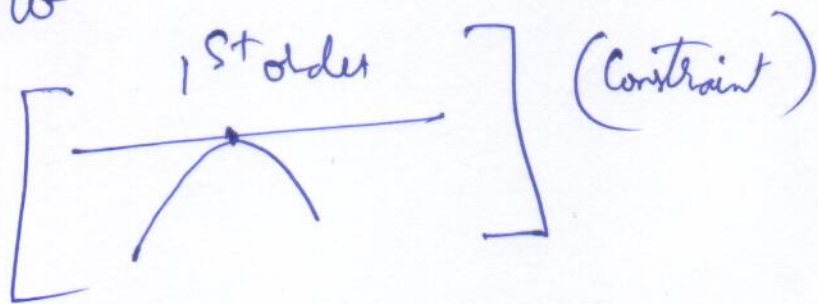


Note Last term is empty since $d-1=2$.

(48)

Intersect with $q = y_1^2 \cdot y_3^3 \cdot a_1 a_2 a_3 a_4$

Hence our task boils down to



AND.



= # of Nodal Cubics thru 7 generic pt & node on a line.

49

How to compute the last number?

Review the Computation

$$[A_1] \cdot y_1^2 \cdot y_3^8$$

At each step, intersect with

$$y_1^2 \cdot y_3^6 \cdot b_1 \cdot a_1$$

Step 1

$$y_1^2 \cdot y_3^5 \cdot b_1 \cdot a_1 a_2$$

Step 2

$$y_1^2 \cdot y_3^4 \cdot b_1 a_1 a_2 a_3$$

Step 3

$$y_1^2 \cdot y_3^3 \cdot b_1 a_1 a_2 a_3 a_4$$

Step 4.

This allows you to compute

$$[A_1] \cdot y_1^2 \cdot y_3^7 \cdot b_1$$

= which is what we want.

Remarks.

$$[A_1] \cdot y_1^2 \cdot y_3^7 \cdot b_1$$

Can not immediately be computed
from the Caporaso Harris formula as
given in their paper.

(51)

In general, the Computation

of $[A_2] \cdot y_d^{S_d-2}$ is as

follows: It is a $(d+1)$ -Step

process. At each step, more a

point to the line.

For $(d-1)$ Steps the number

is unchanged.

(52)

At the d^{th} Step, there is a Contribution from.

$$\left[\text{Diagram of a parabola with a point on its vertex} \right] \cdot \left[S_{d-2} - d \right]_{\text{pts}}$$

with Mult 3

Next $S_{d-2} - d = S_{d-1} - 1.$

Anyway at d^{th} Step

we get a Contribution from.

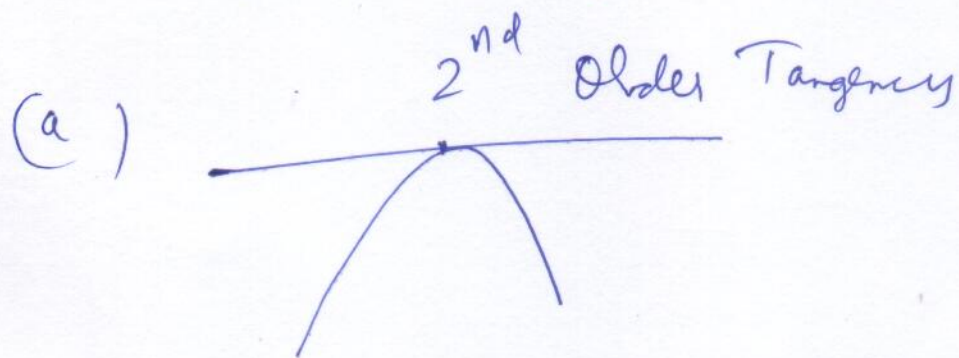
$$\left[\text{Diagram of a parabola with a point on its vertex and a tangent line labeled '1st order Tangent'} \right]$$

with Mult 3

53

Finally, at ~~d^{th}~~ $(d+1)^{\text{th}}$

Step, we get Contribution from



$$\text{Mult} = 2.$$



Node on a
Line

$$\text{Mult} = 3.$$



Cuspidal Curves
of deg $(d-1)$

(54)

In Conclusion, $[A_2] \cdot Y_d^{S_d-2}$

Can be computed recursively.

Final Remark. CH can also

be extended to a family version.

Q. How many nodal cubics are there in $\mathbb{P}^3$, whose image lies in $\mathbb{P}^2$ that ~~pass~~ intersect 11 generic lines?

A = ~~12969~~ 12960

Can be computed using the method developed here (i.e. suitably interpreting CH on the level of cycles).

55

Thank you for
your attention