Counting isolated points outside the image of a polynomial map

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Seminar in Real and Complex Geometry

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A polynomial map \( f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2 \) is said to be **dominant** if \( f(\mathbb{C}^2) \) is an open dense subset of \( \mathbb{C}^2 \).

**Example (A)**

Consider the map

\[
(u, v) \mapsto f(u, v) = (uv, uv + u(1 - uv)).
\]

- For a generic point \((a, b) \in \mathbb{C}^2\), we have \( f^{-1}(a, b) = \left( \frac{b-a}{1-a}, \frac{a(1-a)}{b-a} \right) \).
- For \( b \neq 0, 1 \), we have \( f^{-1}(1, b) = \emptyset \), and \( f^{-1}(b, b) = \emptyset \).
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The **degree** $\deg f$ of $f$ is the maximum of the degrees of $f_1$, and $f_2$. 

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**Problem**

Given $d \in \mathbb{N}^*$. Characterize the invariants of $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ (e.g. number of components, singularities, etc..) for maps above having degree $d$. 

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**Fact**

The set \( \mathbb{C}^2 \setminus f(\mathbb{C}^2) \) is a (possibly empty) finite collection of algebraic curves, and isolated points.

**Example (B)**

Consider the map

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- For a generic point \((a, b) \in \mathbb{C}^2\), we have 
  \(|f^{-1}(a, b)| = 2\), but
- we also have \(f^{-1}(-1, 0) = \emptyset\)
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**Question**

Given \( d \in \mathbb{N}^* \). How many isolated points can \( \mathbb{C}^2 \setminus f(\mathbb{C}^2) \) have for any map \( f \) above of degree \( d \)?
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- Polynomial maps $F : X \rightarrow Y$ appear as models for many real-life problems:
  - In **Multiview Geometry**: Given a fixed number of cameras, we have
    $X \rightsquigarrow$ Space of camera configurations + the data determining the 3D object $O$.
    $Y \rightsquigarrow$ Space of sets of images taken from the cameras.
    If $y$ is a set of images, then $F^{-1}(y)$ is the camera positions + object reconstruction.
Motivation

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  - In **Rigidity Theory**: Given a fixed number of edges in a **Laman** graph $G$, we have
    $X \rightsquigarrow$ Space of coordinates of vertices of $G$.
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    If $y$ is a set of edge-lengths, then $F^{-1}(y)$ is the set of all possible corresponding Laman graphs.
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In general, if $y \in Y \setminus F(X)$, then the problem has no solutions.
Main result

For any map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, let $I_f^0$ denote the set of isolated points in $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$.
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<th>Theorem (Jelonek, 1999)</th>
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Theorem (Jelonek, 1999)

A dominant polynomial map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of degree $d$ satisfies $|I_f^0| \leq (d - 1)^2$.

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**Theorem (E, 2019)**

1. There exists a large family* of dominant polynomial maps \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) of degree \( d \), for which we have \( |\mathcal{I}_f^0| < 5d \), and

2. for any \( n \in \mathbb{N}^* \), there exists a map \( f \) in this family of degree \( 2n + 2 \) such that \( |\mathcal{I}_f^0| = 2n \).
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Example (C)

Let $p, q \in \mathbb{C}[t]$ be any two univariate complex polynomials of the same degree $n$, such that $p(0) \neq 0$, $q(0) \neq 0$, and $\gcd(p, q) = 1$.

Let $f_1 := uv$, and $f_2 := uv + v \cdot p(uv) + v^2 \cdot q(uv)$.

Then, we have

$$\mathcal{I}_f^0 = \{(a, a)\}_{p(a) = 0} \cup \{(b, b)\}_{q(b) = 0} \subset (\mathbb{C}^*)^2.$$
Given a dominant polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^2$ of degree $d$, the set $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ consists of finitely-many algebraic curves, and at most $(d - 1)^2$ isolated points.

**Definition**

**Jelonek’s set** $S_f$ of $f$ is defined as

$$\left\{ y \in \mathbb{C}^2 \mid \exists \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{C}^2; |x_k| \to \infty, f(x_k) \to y \right\}.$$
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\]

**Example (A, ctnd)**

For a generic point $(a, b) \in \mathbb{C}^2$, we have $f^{-1}(a, b) = \left( \frac{b-a}{1-a}, \frac{a(1-a)}{b-a} \right)$.

Then, we have $S_f = \{ y \in \mathbb{C}^2 \mid y_1 = 1, \text{ or } y_1 = y_2 \}$. 

![Diagram of Jelonek’s set](image.png)
The Topology of Polynomial Maps

The topology of a polynomial map $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ can be characterized by the projection of its graph $(x_1, x_2, f(x_1, x_2)) \subset \mathbb{C}^4$, onto the last two coordinates.

$p : \mathbb{C} \to \mathbb{C}$

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Example (B, ctnd)

The map \( (u, v) \mapsto (uv, u(1 + uv) - v) \) satisfies

\[
\begin{align*}
    f^{-1}(-1, 0) &= \mathbb{C}^2 \setminus f(\mathbb{C}^2), \text{ and} \\
    S_f &= \{y \in \mathbb{C}^2 \mid y_1 = -1\}
\end{align*}
\]
Supports of polynomials

The **support** $\text{supp } P$ of a bivariate polynomial

$$P = \sum_{w \in \mathbb{N}^2} c_w x^w$$

is the set $\{w \in \mathbb{N}^2 | c_w \neq 0\}$ (here, $x^w = x_1^{w_1} x_2^{w_2}$).

For any $S \subset \mathbb{N}^2$, let $\mathbb{C}[S] \cong \mathbb{C}^{|S|}$ denote the space of all complex polynomials $P$ above such that $\text{supp } P \subset S$.

Let $A_1, A_2$ be two finite subsets in $\mathbb{N}^2$, and let $\mathbb{C}[A_1, A_2] := \mathbb{C}[A_1] \oplus \mathbb{C}[A_2]$ denote the space of all polynomial couples $(f_1, f_2)$ such that $\text{supp } f_1 \subset A_1$, and $\text{supp } f_2 \subset A_2$.

**Example (B, cntd, cntd)**

If $A_1 = \{(1, 1)\}$, and $A_2 = \{(1, 0), (2, 1), (0, 1)\}$, then

- $f = (f_1, f_2) : (u, v) \mapsto (uv, u(1 + uv) - v)$ satisfies $\text{supp } f_1 = A_1$, $\text{supp } f_2 = A_2$, and
- $f \leftrightarrow (1, 1, 1, -1) \in \mathbb{C}[A_1, A_2] \cong \mathbb{C}^4$. 
Theorem (E, part 1)

There exists a large family (called generically non-proper) of dominant polynomial maps \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) with given degree \( d \), satisfying \(|I^0_f| < 5d\).
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There exists a large family (called generically non-proper) of dominant polynomial maps \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) with given degree \( d \), satisfying \( |I_f^0| < 5d \).

We say that \( f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2 \) is **generically non-proper** if the polynomials \( f_1, f_2 \) are “nice enough" outside \( \mathbb{C}^2 \).

**Nice enough:** We have

\[
\forall (f_1, f_2) \cap (\mathbb{C}^*)^2 = \text{MV}(A_1, A_2) \quad \text{and} \quad \forall (f_i, |\text{Jac } f|) \cap (\mathbb{C}^*)^2 = \text{MV}(A_i, B),
\]

where \( |\text{Jac } f| \) is the determinant of \( f \)'s Jacobian matrix, \( B \subset \mathbb{N}^2 \) is its support, and \( \text{MV}(S_1, S_2) \) is the mixed volume of \( \text{conv}(S_1), \text{conv}(S_2) \).
Generically non-proper maps

Theorem (E, part 1)

There exists a large family (called generically non-proper) of dominant polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ with given degree $d$, satisfying $|\mathcal{I}_f^0| < 5d$.

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Nice enough: We have

$$\nabla(f_1, f_2) \cap (\mathbb{C}^*)^2 = \text{MV}(A_1, A_2) \quad \text{and} \quad \nabla(f_i, |\text{Jac } f|) \cap (\mathbb{C}^*)^2 = \text{MV}(A_i, B),$$

where $|\text{Jac } f|$ is the determinant of $f$'s Jacobian matrix, $B \subset \mathbb{N}^2$ is its support, and $\text{MV}(S_1, S_2)$ is the mixed volume of $\text{conv}(S_1)$, $\text{conv}(S_2)$.

For $i = 1, 2$, let $A_i \subset \mathbb{N}^2$ denote the support of $f_i$.

Proposition

Generically non-proper polynomial maps in $\mathbb{C}[A_1, A_2]$ form an open dense subset.
Let $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a generically non-proper polynomial map.

- $\mu(f) \leadsto$ the number $|f^{-1}(y)|$, for any generic $y$ in $\mathbb{C}^2$ (i.e. the topological degree).
- $\mathcal{N}_\mu(f) \leadsto$ nodes of $S_f$ having multiplicity at least $\mu(f)$
  (Jelonek’s set $S_f$ is a finite union of regular rational curves [Jelonek, 1993]).
- $\mathcal{M}_\infty(f) \leadsto$ points $y \in \mathbb{C}^2$ for $f^{-1}(y)$ has a critical point at infinity.
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- \( \mathcal{M}_\infty(f) \sim \textbf{points} \) \( y \in \mathbb{C}^2 \) for \( f^{-1}(y) \) has a \textbf{critical point at infinity}.

**Proposition**

\( A \text{ generically non-proper map } f : \mathbb{C}^2 \to \mathbb{C}^2 \text{ satisfies } \mathcal{I}_f^0 \subset \mathcal{N}_\mu(f) \cup \mathcal{M}_\infty(f). \)
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**Proposition**

A generically non-proper map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ satisfies $\mathcal{I}_f^0 \subset \mathcal{N}_{\mu}(f) \cup \mathcal{M}_{\infty}(f)$.

**Example (C, ctnd)**

The map in Example C satisfies $\mu(f) = 2$. 
For $i = 1, 2$, let $A_i$ be the support of $f_i$, and let $T_i$ be the set

$$\{X + Y = \deg f_i\} \cup \{X = 0\} \cup \{Y = 0\} \subset \mathbb{R}^2.$$ 

**Proposition**

Let $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a generically non-proper map such that $A_i = \text{supp } f_i$, $i = 1, 2$. Then, we have

$$|\mathcal{I}_f^0| \leq |\mathcal{N}_\mu(f)| + |\mathcal{M}_\infty(f)| < 5 \cdot \max_{i=1,2} |(\partial \text{conv}(A_i) \setminus T_i) \cap \mathbb{N}^2| \leq 5 \cdot \deg f.$$
Further investigate isolated missing points for:

- arbitrary dominant polynomial maps \( \mathbb{C}^2 \rightarrow \mathbb{C}^2 \),
- rational maps \( \mathbb{C}^2 \rightarrow \mathbb{C}^2 \), and
- polynomial maps \( F : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \);

Here, the set \( \mathbb{C}^3 \setminus F(\mathbb{C}^3) \) has components of pure dimensions 0, 1, and 2.
Thank you!

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