Counting isolated points outside the image of a polynomial map

Boulos El Hilany

Seminar in Real and Complex Geometry

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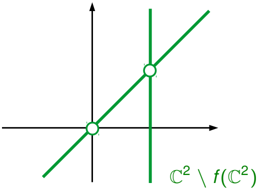
A polynomial map $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ is said to be **dominant** if $f(\mathbb{C}^2)$ is an open dense subset of $\mathbb{C}^2$.

Example (A)

Consider the map

$$(u, v) \mapsto f(u, v) = (uv,\ uv + u(1 - uv)).$$

- For a generic point $(a, b) \in \mathbb{C}^2$, we have $f^{-1}(a, b) = \left(\frac{b-a}{1-a}, \frac{a(1-a)}{b-a}\right)$
- For $b \neq 0, 1$, we have $f^{-1}(1, b) = \emptyset$, and $f^{-1}(b, b) = \emptyset$.



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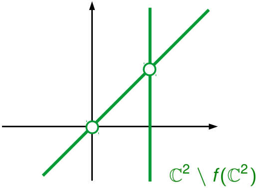

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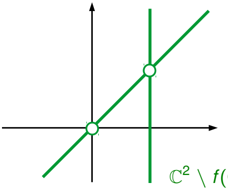

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Problem

Given $d \in \mathbb{N}^*$. Characterize the invariants of $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ (e.g. number of components, singularities, etc..) for maps above having degree d.

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Fact

The set $\overline{\mathbb{C}^2 \setminus f(\mathbb{C}^2)}$ is a (possibly empty) finite collection of algebraic curves, and isolated points.

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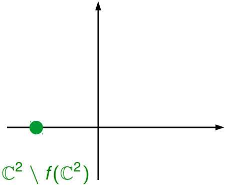
- For a generic point $(a, b) \in \mathbb{C}^2$, we have $|f^{-1}(a, b)| = 2$, but
- we also have $f^{-1}(-1, 0) = \emptyset$



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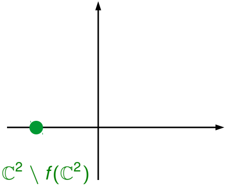

Question

Given $d \in \mathbb{N}^*$. How many isolated points can $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ have for any map f above of degree d?

Motivation

- The largeness of $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ measures the “extremeness" of a maps’ topology.

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  - In **Multiview Geometry**: Given a fixed number of cameras, we have
    $X \rightsquigarrow$ Space of camera configurations $+$ the data determining the 3D object O.
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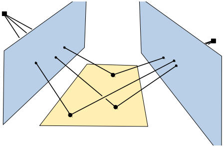
    If y is a set of images, then $F^{-1}(y)$ is the camera positions $+$ object reconstruction.



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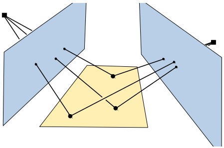

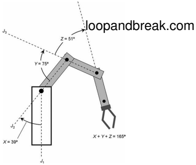

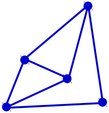

  - In **Rigidity Theory**: Given a fixed number of edges in a **Laman** graph $\mathcal{G}$, we have
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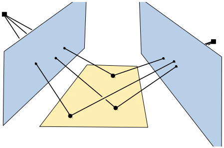

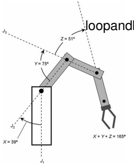

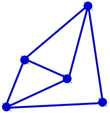

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In general, if $y \in Y \setminus F(X)$, then the problem has no solutions.

Main result

For any map $f : \mathbb{C}^2 \to \mathbb{C}^2$, let $\mathcal{I}_f^{\emptyset}$ denote the set of isolated points in $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$.

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Theorem (E, 2019)

1. *There exists a large family* of dominant polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of degree d, for which we have $|\mathcal{I}_f^{\emptyset}| < 5d$, and*
2. *for any $n \in \mathbb{N}^*$, there exists a map f in this family of degree $2n + 2$ such that $|\mathcal{I}_f^{\emptyset}| = 2n$.*

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Example (C)

Let $p, q \in \mathbb{C}[t]$ be any two univariate complex polynomials of the same degree n, such that $p(0) \neq 0$, $q(0) \neq 0$, and $\gcd(p, q) = 1$.

Let $f_1 := uv$, and $f_2 := uv + v \cdot p(uv) + v^2 \cdot q(uv)$.

Then, we have

$$\mathcal{I}_f^{\emptyset} = \{(a,a)\}_{p(a)=0} \cup \{(b,b)\}_{q(b)=0} \subset (\mathbb{C}^*)^2.$$

Jelonek's set

Given a dominant polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^2$ of degree d, the set $\overline{\mathbb{C}^2 \setminus f(\mathbb{C}^2)}$ consists of finitely-many algebraic curves, and at most $(d-1)^2$ isolated points.

Definition

Jelonek's set S_f of f is defined as

$$\left\{ y \in \mathbb{C}^2 \mid \exists \, \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{C}^2; \ |x_k| \to \infty, \ f(x_k) \to y \right\}.$$

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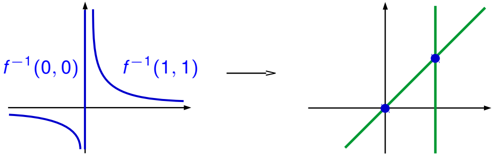
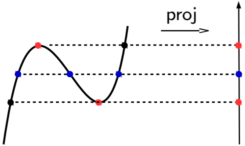
Then, we have $S_f = \{y \in \mathbb{C}^2 | \ y_1 = 1, \ \text{or } y_1 = y_2\}$.



The Topology of Polynomial Maps

The topology of a polynomial map $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ can be characterized by the projection of its **graph** $(x_1, x_2, f(x_1, x_2)) \subset \mathbb{C}^4$, onto the last two coordinates.

$p : \mathbb{C} \to \mathbb{C}$



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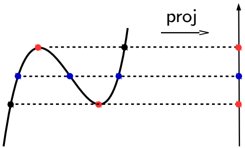

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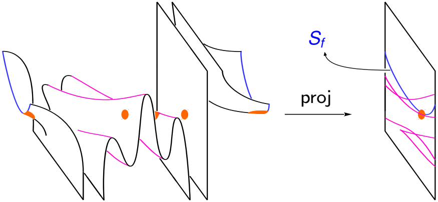

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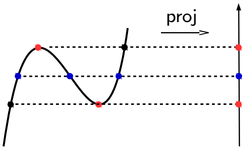

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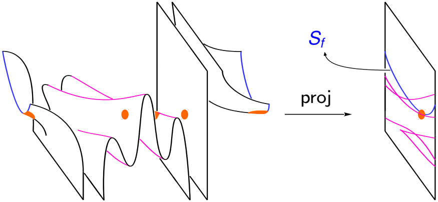

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Example (B, ctnd)

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$f^{-1}(-1, 0) = \mathbb{C}^2 \setminus f(\mathbb{C}^2)$, and
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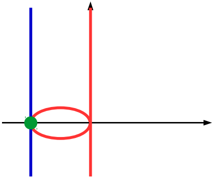

Supports of polynomials

The **support** $\operatorname{supp} P$ of a bivariate polynomial

$$P = \sum_{w \in \mathbb{N}^2} c_w x^w$$

is the set $\{w \in \mathbb{N}^2 | \ c_w \neq 0\}$ (here, $x^w = x_1^{w_1} x_2^{w_2}$).

For any $S \subset \mathbb{N}^2$, let $\mathbb{C}[S] \cong \mathbb{C}^{|S|}$ denote the space of all complex polynomials P above such that $\operatorname{supp} P \subset S$.

Let A_1, A_2 be two finite subsets in $\mathbb{N}^2$, and let $\mathbb{C}[A_1, A_2] := \mathbb{C}[A_1] \oplus \mathbb{C}[A_2]$ denote the space of all polynomial couples (f_1, f_2) such that $\operatorname{supp} f_1 \subset A_1$, and $\operatorname{supp} f_2 \subset A_2$.

Example (B, cntd, cntd)

If $A_1 = \{(1, 1)\}$, and $A_2 = \{(1, 0), (2, 1), (0, 1)\}$, then

- $f = (f_1, f_2) : (u, v) \mapsto (uv, u(1 + uv) - v)$ satisfies $\operatorname{supp} f_1 = A_1$, $\operatorname{supp} f_2 = A_2$, and
- $f \leftrightarrow (1, 1, 1, -1) \in \mathbb{C}[A_1, A_2] \cong \mathbb{C}^4$.

Generically non-proper maps

Theorem (E, part 1)

There exists a large family (called generically non-proper) of dominant polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ with given degree d, satisfying $|\mathcal{I}_f^{\emptyset}| < 5d$.

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We say that $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ is **generically non-proper** if the polynomials f_1, f_2 are "nice enough" outside $\mathbb{C}^2$.

Nice enough: We have

$$\mathbb{V}(f_1, f_2) \cap (\mathbb{C}^*)^2 = \mathrm{MV}(A_1, A_2) \quad \text{and} \quad \mathbb{V}(f_i, |\operatorname{Jac} f|) \cap (\mathbb{C}^*)^2 = \mathrm{MV}(A_i, B),$$

where $|\operatorname{Jac} f|$ is the determinant of f's Jacobian matrix, $B \subset \mathbb{N}^2$ is its support, and $\mathrm{MV}(S_1, S_2)$ is the mixed volume of $\mathrm{conv}(S_1), \mathrm{conv}(S_2)$.

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For $i = 1, 2$, let $A_i \subset \mathbb{N}^2$ denote the support of f_i .

Proposition

Generically non-proper polynomial maps in $\mathbb{C}[A_1, A_2]$ form an open dense subset.

Geometric part

Let $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a generically non-proper polynomial map.

- $\mu(f) \rightsquigarrow$ the number $|f^{-1}(y)|$, for any generic y in $\mathbb{C}^2$ (i.e. the **topological degree**).
- $\mathcal{N}_\mu(f) \rightsquigarrow$ **nodes** of S_f having multiplicity at least $\mu(f)$
  (Jelonek's set S_f is a finite union of regular rational curves [Jelonek, 1993]).
- $\mathcal{M}_\infty(f) \rightsquigarrow$ points $y \in \mathbb{C}^2$ for $f^{-1}(y)$ has a **critical point at infinity**.

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A generically non-proper map $f : \mathbb{C}^2 \to \mathbb{C}^2$ *satisfies* $\mathcal{I}_f^\emptyset \subset \mathcal{N}_\mu(f) \cup \mathcal{M}_\infty(f)$.

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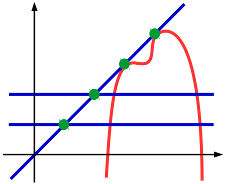
Example (C, ctnd)

The map in Example C satisfies $\mu(f) = 2$.



Combinatorial part

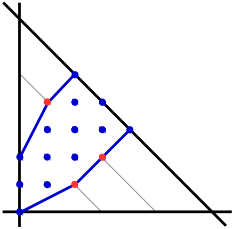
For $i = 1, 2$, let A_i be the support of f_i, and let T_i be the set

$$\{X + Y = \deg f_i\} \cup \{X = 0\} \cup \{Y = 0\} \subset \mathbb{R}^2.$$

Proposition

Let $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a generically non-proper map such that $A_i = \operatorname{supp} f_i$, $i = 1, 2$. Then, we have

$$|\mathcal{I}_f^{\emptyset}| \ \leq \ |\mathcal{N}_\mu(f)| + |\mathcal{M}_\infty(f)| \ < \ 5 \cdot \max_{i=1,2} |(\partial \operatorname{conv}(A_i) \setminus T_i) \cap \mathbb{N}^2| \leq 5 \cdot \deg f.$$



Future directions

Further investigate isolated missing points for:

- arbitrary dominant polynomial maps $\mathbb{C}^2 \to \mathbb{C}^2$,
- rational maps $\mathbb{C}^2 \to \mathbb{C}^2$, and
- polynomial maps $F : \mathbb{C}^3 \to \mathbb{C}^3$;

  Here, the set $\overline{\mathbb{C}^3 \setminus F(\mathbb{C}^3)}$ has components of pure dimensions 0, 1, and 2.

Thank you!

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