Higgs bundles on nodal curves

Sourav Das

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Part I: Basic definitions

Simple normal crossing divisor

Let X be a smooth variety. A divisor D is called a **simple normal crossing divisor** if

$$D = \cup_{i=1}^{n} D_i,$$

where

- D_i's are all smooth divisors and
- every D_i and D_j intersect transversally for every $i \neq j$.

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A divisor D is called **normal crossing divisor** if etale locally the divisor is a simple normal crossing divisor i.e., there exists a etale surjective map from a variety $\pi : U \to X$ such that $\pi^{-1}(D)$ is a simple normal crossing divisor in U.

Examples of Vector bundles:Log-Cotangent bundle

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- Let X be a smooth variety and D be a normal crossing divisor. Let us denote $U := X \setminus D$ and $\tau : U \to X$ is the inclusion.

$\Omega^n_X(log \;\; D)$ denotes the subsheaf of $\Omega^n_X(*D) := \tau_* \Omega^n_U$ of differential; forms with logarithmic poles along D, i.e., if $V \subseteq X$ is open, then

$$\Gamma(V, \Omega^n_X(log \;\; D)) =$$
$$\{\alpha \in \Gamma(V, \Omega^n_X(*D)) : \alpha \;\text{ and }\; d\alpha \;\text{ have simple poles along } D\}$$

Log-Cotangent bundle

Consider the particular case

$$X = \mathbb{C}^n \text{ and } D = \cup_{i=1}^r D_i,$$

where $D_1, \dots, D_r$ are the first r coordinate hyperplanes, where $1 \leq r \leq n$.

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where $D_1, \ldots, D_r$ are the first r coordinate hyperplanes, where $1 \leq r \leq n$.

Then $\Omega^1_X(log \;\; D)$ is freely generated as $\mathcal{O}_X$-module by

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n.$$

Semistable degeneration

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- generic fibre is smooth and
- the special fibre D is a normal crossing divisor in X.

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where "0" denotes the closed point of the d.v.r S.

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- Dualizing sheaf of D
$$\omega_D := det \;\; \Omega^1_D(log\ \partial D)$$

Higgs bundles on a Smooth/ Nodal curve

A nodal curve $\mathcal{C}$ is a curve with finitely many nodes, i.e., points $\{x_1, \cdots, x_n\}$ such that the analytic local ring

$$\widehat{\mathcal{O}_{\mathcal{C},x_i}} \cong \frac{\mathbb{C}[|t_1, t_2|]}{t_1 \cdot t_2}, \forall i = 1, \ldots, n \tag{1}$$

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- $\mathcal{E}$ is a vector bundle on $\mathcal{C}$, and
- $\phi : \mathcal{E} \to \mathcal{E} \otimes \omega_{\mathcal{C}}$ is any bundle homomorphism. Here $\omega_{\mathcal{C}}$ denotes the dualizing sheaf of $\mathcal{C}$

Part II. Moduli functors and corresponding moduli spaces for smooth curves

Moduli of vector bundles and Higgs bundles on smooth curves

$$F_{VB} : \mathrm{Sch} \to \mathrm{Sets}, \ (F_{HB})$$
$$T \mapsto \{\text{Isomorphism classes of vector (Higgs) bundles on } X \times T \text{ of rank } r \text{ and degree } d\}$$

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$$T \mapsto \{\text{Isomorphism classes of vector (Higgs) bundles on } X \times T \text{ of rank } r \text{ and degree } d\}$$

- There is an obvious forgetful map $F : F_{HB} \to F_{VB}$ and a map (by adding the "0"-Higgs field) $Z : F_{VB} \to F_{HB}$ and $F \circ Z$ is the identity transformation on F_{VB}.

Stability and representability of the functors

A Higgs bundle $(\mathcal{E}, \phi)$ on a smooth/ irreducible nodal curve is called stable (semi-stable) if for any non zero and ϕ-invariant subsheaf $\mathcal{F}$ (i.e., $\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \omega_{\mathcal{C}}$) we have

$$\frac{\text{deg } \mathcal{F}}{\text{rank } \mathcal{F}} \underset{(\leq)}{<} \frac{\text{deg } \mathcal{E}}{\text{rank } \mathcal{E}}$$

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- The subfunctors $F_{HB^{st}}$ and $F_{HB^{ss}}$ of F_{HB} consisting of all the stable and semistable Higgs bundles always have coarse moduli spaces. The moduli functor $F_{VB^{ss}}$ are always proper.

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- If the rank and degree are coprime, then the functors $F_{HB^{st}}$ and $F_{VB^{st}}$ are representable by smooth varieties $\mathcal{M}_{HB}$ and $\mathcal{M}_{VB}$, respectively. Moreover, $\mathcal{M}_{VB}$ is a proper variety.

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- Therefore the cotangent space of the functor F_{VB} at a point $[\mathcal{E}]$ is isomorphic to
$$H^1(\mathcal{E}nd\mathcal{E})^\vee \cong H^0(\mathcal{E}nd\mathcal{E} \otimes \omega_{\mathcal{C}}) = Hom(\mathcal{E}, \mathcal{E} \otimes \omega_{\mathcal{C}}),$$
i.e., the space of all Higgs fields on the vector bundle $[\mathcal{E}]$.

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Therefore, informally speaking the forgetful functor $F : F_{HB} \to F_{VB}$ is actually like the cotangent bundle map and Z is like the zero section.

Liouville form and the natural symplectic form on any cotangent bundle ⋆

Let X be any smooth variety. Then

$$\begin{array}{ccccc}
\Omega_X \times_X \Omega_X & \xrightarrow{\cong} & \pi^*\Omega_X & \longrightarrow & \Omega_Y \\
& \nwarrow^{\Delta} & \downarrow & & \\
& & Y := \Omega_X & & \\
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- The 2-form $\omega := -d\lambda$ is a symplectic form on Y.
  (symplectic:=non-degenerate and skew-symmetric 2-form)

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  (symplectic:=non-degenerate and skew-symmetric 2-form)
- **Therefore $\mathcal{M}_{HB}$ has a symplectic form.**

Log-symplectic form

Let X be a smooth variety and D a normal crossing divisor on X.

A log-symplectic form on X is an element $\omega \in \Omega^2_X(\mathit{log}\ D)$ which is closed and non-degenerate (on $T_X(-\mathit{log}\ D)$).

Let $\pi : X \to S$ be a semistable degeneration and D denote the closed fibre.

- A relative log-symplectic form on X over S is an element $\omega \in \Omega^2_{X/S}(\mathit{log}\ D)$ which is closed and non-degenerate (on $T_{X/S}(-\mathit{log}\ D)$).
- A log-symplectic form on D is an element $\omega \in \Omega^2_D(\mathit{log}\ \partial D)$ which is closed and non-degenerate (on $T_D(-\mathit{log}\ \partial D)$).

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- *Given a semistable degeneration $p : X \to S$ there exists a relative log-symplectic form on $\Omega_{X/S}(\log D)$.*

Remark: The relative log symplectic form is not unique i.e., there are other natural such forms on X

Hitchin map and complete integrability (rank and degree coprime case)

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$$h : \mathcal{M}_{HB} \to \oplus_{i=0}^{i=n} H^0(X, \omega_X^{\otimes i})$$

which maps $(\mathcal{E}, \phi)$ to the coefficients of the characteristic polynomial of the Higgs field ϕ. This is known as the **Hitchin map.**

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- the map is proper.
- the triple $(\mathcal{M}_{HB}, \omega, h)$ is an example of an algebraically complete integrable system.

Tangent and cotangent space of the moduli of vector bundles at a point ⋆

- $T_{[\mathcal{E}]}\mathcal{M}_{VB} \cong \mathcal{M}_{VB}(Spec\ k[\epsilon]) =$

$$\left\{\begin{array}{l}\text{Isomorphism classes of families of} \\ \text{vector bundles on } X \times Spec\ k[\epsilon] \\ \text{such that the closed fiber is } \mathcal{E}\end{array}\right\}$$

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- On a suitable affine cover $\mathcal{U} := \{U_i\}_I$ of X, the first order deformation of the bundle is given by
$$\{A_{ij} + \epsilon B_{ij}\},$$
where
  - $\{A_{ij}\}$ is the original transition functions of $\mathcal{E}$ and
  - $\{B_{ij}\} \in \prod_{i,j} \Gamma(U_{ij}, \mathcal{E}nd\mathcal{E})$ such that
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- $T_{[\mathcal{E}]}\mathcal{M}_{VB} \cong H^1(X, \mathcal{E}nd\mathcal{E})$

Tangent and cotangent space of the moduli of Higgs bundles at a point (Hitchin, Biswas-Ramanan, Bottacin)

- $T_{[(\mathcal{E},\phi)]}\mathcal{M}_{HB} \cong \mathcal{M}_{HB}(k[\epsilon])$.

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  - $\{A_{ij}\}$ is the original transition functions of $\mathcal{E}$ and
  - $\{B_{ij}\} \in H^1(\mathcal{U}, \mathcal{E}nd\mathcal{E})$
- The deformation of the Higgs field is given by $\{\phi|_{U_i} + \epsilon s_i\}$, where $\{s_i\} \in \prod_{i\in I} H^0(U_i, \mathcal{E}nd\mathcal{E} \otimes \omega_{\mathcal{C}})$ should satisfy the following condition

$$s_i - s_j = [A_{ij}, \phi]$$

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- Giving the data $\{(\mathcal{U}, B_{ij}, s_i)\}$ is equivalent to giving an element of
$$\mathbb{H}^1(\mathcal{C}_\bullet),$$
where $\mathcal{C}_\bullet$ is the following complex of vector bundles
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- Therefore, the tangent space at $(\mathcal{E}, \phi)$ is isomorphic to $\mathbb{H}^1(\mathcal{C}_\bullet)$.

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$$\{(B_{ij}, s_i)\} \mapsto \{B_{ij}\} \mapsto \{Trace(\phi \circ B_{ij})\}$$

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Notice $\phi \in Hom(\mathcal{E}, \mathcal{E} \otimes \omega_X) \cong H^1(\mathcal{E}nd\,\mathcal{E})^\vee$

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$$\mathbb{H}^1(\mathcal{C}_\bullet) \otimes \mathbb{H}^1(\mathcal{C}_\bullet) \to \mathbb{H}^2(\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet) \to \mathbb{H}^2(\omega_X[-1]) \cong \mathbb{C}$$

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$$((s_{ij}, t_i), (s'_{ij}, t'_i)) \mapsto s_{ij} \otimes t'_j - t_i \otimes s'_{ij} \mapsto \mathit{Trace}(s_{ij} \circ t'_j - t_i \circ s'_{ij})$$

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By Serre duality, $\mathbb{H}^2(\omega_X[-1]) \cong H^1(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C}$

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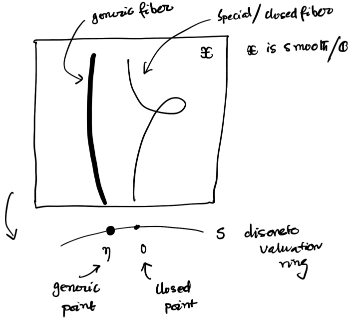
\begin{array}{ccccc}
0 & & 1 & & 2 \\
\mathcal{E}nd\mathcal{E} \otimes \mathcal{E}nd\mathcal{E} & \longrightarrow & \mathcal{E}nd\mathcal{E} \otimes (\mathcal{E}nd\mathcal{E} \otimes \omega_X) \oplus (\mathcal{E}nd\mathcal{E} \otimes \omega_X) \otimes \mathcal{E}nd\mathcal{E} & \longrightarrow & (\mathcal{E}nd\mathcal{E} \otimes \omega_X) \otimes (\mathcal{E}nd\mathcal{E} \otimes \omega_X) \\
\downarrow & & \downarrow{\scriptstyle Trace} & & \downarrow \\
0 & \longrightarrow & \omega_X & \longrightarrow & 0
\end{array}
$$

Remember $\mathcal{C}_\bullet$ is the following complex of vector bundles

$$\mathcal{E}nd\mathcal{E} \xrightarrow{[\bullet,\phi]} \mathcal{E}nd\mathcal{E} \otimes \omega_C$$

Part III. Degeneration of moduli of vector (Higgs) bundles

Set up for a degeneration of the moduli of vector (Higgs) bundles on a smooth curve



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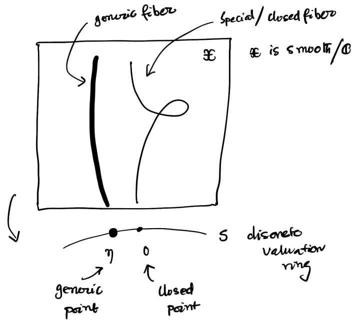
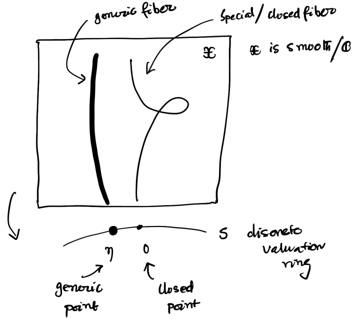


- Fix a flat family of curves $\mathcal{X}$ over a d.v.r S such that

Set up for a degeneration of the moduli of vector (Higgs) bundles on a smooth curve



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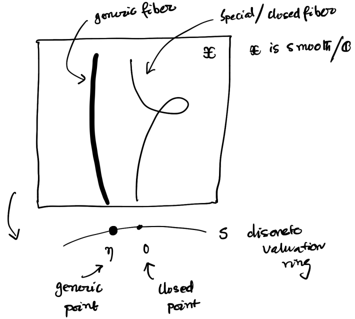

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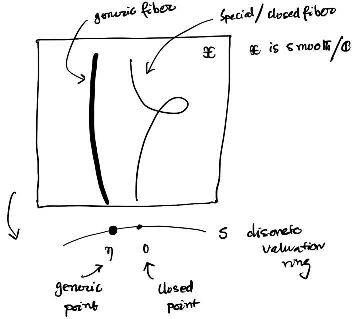

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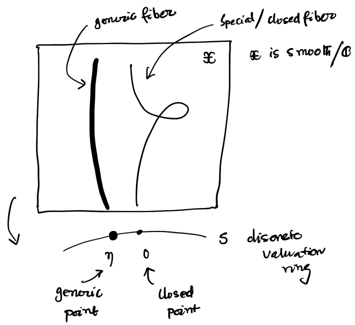

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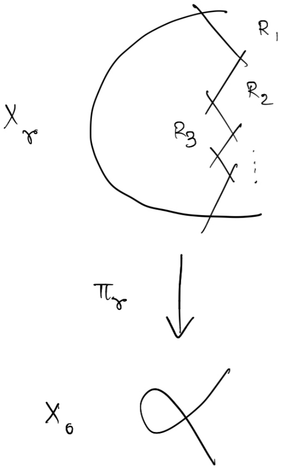
- $\phi : \mathcal{E} \to \mathcal{E} \otimes (\pi_r)^*\omega_{X_0}$

Notice $\omega_{X_r} \cong (\pi_r)^*\omega_{X_0}$

Modifications/ Gieseker curves



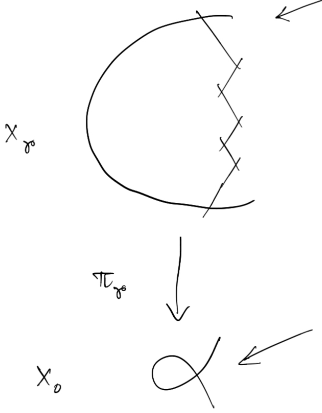
$$X_r = \tilde{X}_0 \cup \left(\bigcup_{i=1}^{r} R_i \right)$$

$$R_i \cong \mathbb{P}^1$$

$\{p_1, \dots, p_{r+1}\}$ nodes.

Modifications/ Gieseker curves

Admissible/Gieseker Vector bundle: $(X_\gamma, \mathcal{E})$



$\mathcal{E}$ vector bundle

$$\mathcal{E}|_{R_i} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus a_i} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus b_i},$$

$$b_i > 0$$

$(\pi_\gamma)_* \mathcal{E}$

a torsion free sheaf

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- $\sigma^* \mathcal{E}_1 \cong \mathcal{E}_2$, and
- $\sigma^* \phi_1 = \phi_2$

Notice that the equivalence class is stronger than the usual isomorphism class

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We call a Gieseker-Higgs bundle $(\pi_r : X_r \to X_0, \mathcal{E})$ **stable** if the torsion-free Higgs pair $((\pi_r)_*\mathcal{E}, (\pi_r)_*\phi)$ is **stable**.

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Theorem

(Gieseker, Balaji-Barik-Nagaraj) *There exists a flat family of varieties $\mathcal{M}_{GHB/S}$ over S which represents the functor $Func^{st}_{GHB/S}$.*

Functor of Gieseker curves and its versal family ⋆

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Part IV: Log schemes and log structures on the moduli of Gieseker-Higgs bundles

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A **chart** for a log scheme $(X, \mathcal{M})$ is a monoid P with a map of sheaves of monoids $P_X \to \mathcal{M}$ which induces isomorphism between the associated log structures.

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A **chart for a morphism** of log schemes $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ consists of a pair of charts for the two log schemes which is compatible with the log morphism f.

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If x belongs to exactly r number of local components of D, then the chart is given by $\mathbb{N}^r \to \mathbb{C}[x_1, \ldots, x_n]$ sending $e_i \mapsto x_i$ for $i = 1, \ldots, r$.

Example of Log Schemes.

Example 2:

Theorem. (S. Mochizuki, F. Kato) Let $f : X \to S$ be flat family of prestable curves. Then there exists a natural log structure on X, a log structure on S and a log morphism from X to S such that the underlying morphism of ordinary schemes is the same as f.

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Brief outline of the log structure :

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($\mathcal{N}_i$ can be thought as the log structure due to the node p_i) ■

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- The universal curve $\mathcal{C}^{univ}$ defines log structures on itself and on $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$.
- The special fibres $\mathcal{M}_{GHB}$ and $\mathcal{M}_{GVB}$ are normal crossing divisors on their respective relative moduli spaces. Therefore they also define log structures on the relative moduli spaces.

Equivalence of the two log structures

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Outline of the proof. Recall that we have the relative picture

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Let $z_1, \ldots, z_{n+1}$ are the local equations of the components passing through the point $(X_n, \mathcal{E}, \phi)$.

To prove that the two log structures are the same it is enough to show that the local equations of nodes and the local equations of the components are the same.

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(Gieseker, Nagaraj-Seshadri)

- *It has a versal family*

  $$\mathcal{C}_{ver} \to \mathcal{V} := Spec\ \mathbb{C}[|z_1, \cdots, z_{n+1}|] \to S := Spec\ \mathbb{C}[|t|]$$

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Part V: Log deformations and log-tangent space of the moduli of Gieseker-Higgs bundles

Log tangent space

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The inclusion $x : \operatorname{Spec} k \hookrightarrow X$ induces a log structure on the point by pulling back the sheaf of monoids. We denote this pulled back log scheme by $(x, \mathcal{M}_x, \alpha_x)$.

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Then the log tangent space is defined to be the space of log morphisms

$$T_x^{log} := Hom_{(x,\mathcal{M}_x,\alpha_x)}((Spec\ k[\epsilon], \mathcal{M}_\epsilon, \alpha_\epsilon), (X, \mathcal{M}, \alpha)),$$

which are also (log)-liftings of the log-morphism $x \hookrightarrow X$.

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Let $(S, \mathcal{L})$ be a fine log scheme. (fine:= finite type and integral monoid, M is integral if the map $M \to M^{gp}$ is injective)

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Useful properties

- Giving a log morphism of fine log schemes $f : (X, \mathcal{M}) \to (S, \mathcal{L})$ is equivalent to giving a morphism of stacks $f_{Log} : X \to \mathcal{L}og_{(S,\mathcal{L})}$.

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- $\Omega_f^{log} \cong \Omega_{f_{Log}}$.
- The relative log tangent space for f at a point is isomorphic to ordinary relative tangent space of f_{Log} at the point.

Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over S

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Computation of the relative log tangent space :

- First of all
$$T_{M_{GHB/S}/S}(-log\, M_{GHB}) \cong T_{M_{GHB/S}/\mathcal{L}og_{(S,\mathcal{L})}}$$
the relative log-tangent space at a point of $\mathcal{M}_{GHB/S} \to S$ is isomorphic to the rel. tangent space of$\mathcal{M}_{GHB/S} \to \mathcal{L}og_{(S,\mathcal{L})}$, where $\mathcal{L}$ is the log structure on S induced by its closed point.
- Moreover,
$$T_{M_{GHB}}(-log\, \partial M_{GHB}) \cong T_{M_{GHB}/\mathcal{L}og_{(s,s^*\mathcal{L})}}$$
here $s : speck \hookrightarrow S$ is the closed point and $s^*\mathcal{L}$ denotes the pull back log structure.

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$$\begin{array}{ccc} \mathcal{M}_{GHB}(k[\epsilon]) & \longleftarrow & \mathcal{M}_{GHB}(k[\epsilon]) \times_{\mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon])} \mathcal{L}og_{(s,s^*\mathcal{L})}(k) \\ \downarrow & & \downarrow \\ \mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon]) & \longleftarrow & \mathcal{L}og_{(s,s^*\mathcal{L})}(k) \end{array}$$

where

- the left vertical arrow is the differential of the map $\mathcal{M}_{GHB} \to \mathcal{L}og_{(k,\mathbb{N})}$ at the point $(X_r, \mathcal{E}, \phi)$,
- the lower horizontal map is given by the log structure on $k[\epsilon]$ induced from a log structure on k via the map $k \hookrightarrow k[\epsilon]$.

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\downarrow & & \downarrow \\
\mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon]) & \longleftarrow & \mathcal{L}og_{(s,s^*\mathcal{L})}(k)
\end{array}$$

- The relative log tangent space of $\mathcal{M}_{GHB/S}$ over S at a point $(X_r, \mathcal{E}, \phi)$ is isomorphic to the fibre of the right vertical map over the point of $\mathcal{L}og_{(s,s^*\mathcal{L})}(k)$ given by $(X_r, \mathcal{E}, \phi)$.

Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over S

- Step 1. What is the log structure $\mathcal{L}$?

$$\begin{aligned} \mathbb{N} &\to k[|t|] \\ e &\mapsto t \end{aligned}$$

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- Step 3. What is the point of $\mathcal{L}og_{(s,s^*\mathcal{L})}(k)$ given by $(X_r, \mathcal{E}, \phi)$?

  The Gieseker-Higgs bundle $p := (X_r, \mathcal{E}, \phi)$ gives an element $(p, p^*\mathcal{M})$ of $\mathcal{L}og_{(s,s^*\mathcal{L})}(k)$, where $\mathcal{M}$ denotes the log structure on $\mathcal{M}_{GHB/S}$.

Since there are $r+1$ number of nodes in X_r therefore this element is given by the log morphism

$$\begin{array}{ccccc}
 & & e_i & \longrightarrow & 0 \\
 & & & & \\
(p, p^*\mathcal{M}) & \sum e_i & \mathbb{N}^{r+1} & \longrightarrow & k \\
 & \uparrow & \uparrow & & =\uparrow \\
(s, s^*\mathcal{L}) & e & \mathbb{N} & \longrightarrow & k \\
 & & & & \\
 & & e & \longrightarrow & 0
\end{array}$$

- Step 4. What is the image $(p, p^*\mathcal{M})$ under the lower horizontal map $\mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon]) \leftarrow \mathcal{L}og_{(s,s^*\mathcal{L})}(k)$?

  It is given by

$$\begin{array}{ccccccc}
 & & e_i & \longrightarrow & 0 & \longrightarrow & 0 \\
(p[\epsilon], (p^*\mathcal{M})_\epsilon) & \sum e_i & \mathbb{N}^{r+1} & \longrightarrow & k & \longrightarrow & k[\epsilon] \\
 & \uparrow & \uparrow & & & & \uparrow \\
(s, s^*\mathcal{L}) & e & \mathbb{N} & & \longrightarrow & & k \\
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- Step 5. What is the image of $p := (X_r, \mathcal{E}, \phi)$ under the left vertical map $\mathcal{M}_{GHB}(k[\epsilon]) \to \mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon])$?

  An element of $\mathcal{M}_{GHB}(k[\epsilon])$ is a first order infinitesimal deformation of $p := (X_r \to X_0, \mathcal{E}, \phi)$. Let us denote the deformation by $p_\epsilon := (\mathcal{X}_\epsilon \to X_0 \times spec\ k[\epsilon], \mathcal{E}_\epsilon, \phi_\epsilon)$.

  Let us write out the log structure on $spec\ k[\epsilon]$ induced by the point $(\mathcal{X}_\epsilon, \mathcal{E}_\epsilon, \phi_\epsilon)$ i.e., by the family of curves $\mathcal{X}_\epsilon$ using Mochizuki's method.

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  Remember that the closed fibre of $\mathcal{X}$ is X_r. Let us denote its nodes by $\{p_1, \ldots, p_{r+1}\}$. The henselian local ring at the node p_i is the henselisation of

$$\frac{k[x_i, y_i, \epsilon]}{x_i \cdot y_i - \lambda_i \cdot \epsilon}$$

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The log structure on *spec* $k[\epsilon]$ induced by the node p_i :

$$\alpha_i : \mathbb{N} \to k[\epsilon]$$
$$e \mapsto \lambda_i \epsilon$$

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Finally the induced log structure on *spec* $k[\epsilon]$ is the amulgumated sum

$$\mathcal{L}_{k[\epsilon]} := \mathcal{L}_1 \oplus_{k[\epsilon]^*} \cdots \oplus_{k[\epsilon]^*} \mathcal{L}_{r+1}$$

It is isomorphic to the log structure associated with the prelog structure

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Therefore the image of $p := (X_r, \mathcal{E}, \phi)$ under the map $\mathcal{M}_{GHB}(k[\epsilon]) \to \mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon])$ is given by the following log morphism

$$e_i \longrightarrow \epsilon \cdot \lambda_i$$

$$\begin{array}{ccccc} (p_\epsilon, p_\epsilon^*\mathcal{M}) & \qquad \sum e_i & \mathbb{N}^{r+1} & \longrightarrow & k[\epsilon] \\ & \qquad \uparrow & \uparrow & & \uparrow \\ (s, s^*\mathcal{L}) & \qquad e & \mathbb{N} & \longrightarrow & k \end{array}$$

$$e \longrightarrow 0$$

Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

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- Step 6. Elements of the fibre product :

$$\begin{array}{ccc} \mathcal{M}_{GHB}(k[\epsilon]) & \longleftarrow & \mathcal{M}_{GHB}(k[\epsilon]) \times_{\mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon])} \mathcal{L}og_{(s,s^*\mathcal{L})}(k) \\ \downarrow & & \downarrow \\ \mathcal{L}og_{(s,s^*\mathcal{L})}(k[\epsilon]) & \longleftarrow & \mathcal{L}og_{(s,s^*\mathcal{L})}(k) \end{array}$$

Therefore, by equating the log structures in *Step4*. and *Step5*., we conclude that an infinitesimal deformation $(\mathcal{X}_\epsilon \to X_0 \times spec\ k[\epsilon], \mathcal{E}_\epsilon, \phi_\epsilon)$ of $(X_r \to X_0, \mathcal{E}, \phi)$ is an element of the fibre product if and only if $\lambda_i = 0$ for all $i = 1, \ldots, r+1$.

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Step 7.

Lemma: Such deformations are trivial i.e., $\mathcal{X}_\epsilon \cong X_r \times spec\ k[\epsilon]$.

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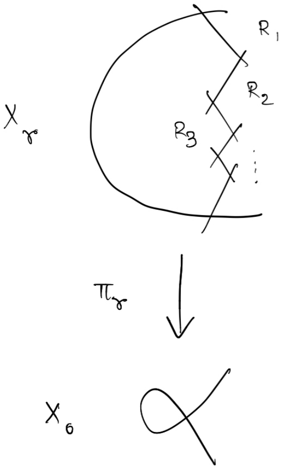
Now we use the following short exact sequence

$$0 \to H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to Ext^1(\Omega_{X_r}, \mathcal{O}_{X_r}) \to H^0(X_r, \mathcal{E}xt^1(\Omega_{X_r}, \mathcal{O}_{X_r})) \cong \oplus_{i=1}^{i=r+1} Ext^1(\Omega_{X_r,p_i}, \mathcal{O}_{X_r,p_i}) \to 0$$

Since $\lambda_i = 0$ for all $i = 1, \ldots, r+1$, it follows that the infinitesimal deformation $\mathcal{X}_\epsilon \in H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$.

Modifications/ Gieseker curves



$$X_r = \tilde{X}_0 \cup \left(\bigcup_{i=1}^{r} R_i \right)$$

$$R_i \cong \mathbb{P}^1$$

$\{ p_1, \dots, p_{r+1} \}$ nodes.

Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

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We have a natural map

$$\pi_r^\flat : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r}). \tag{2}$$

and the induced map

$$H^1(\pi_r^\flat) : H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to H^1(X_r, \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r}))$$

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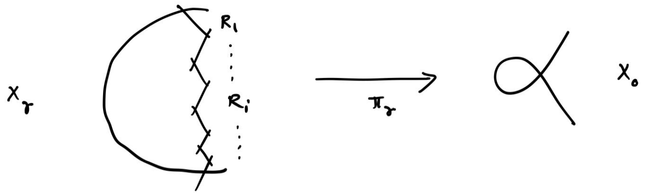

$$T_{X_r} \cong \nu_* \left(T_{\tilde{X}_0}(-\log) \bigoplus_{i=1}^{r} T_{R_i}(-\log) \right)$$

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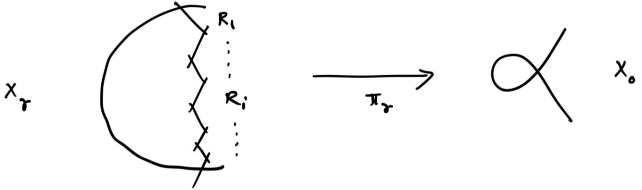

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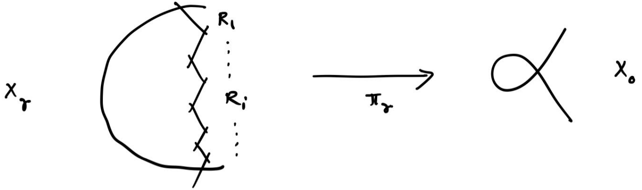

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Since $H^1(R_i, \mathcal{O}_{R_i}) = 0$, the map $Ker(H^1(\pi_r^\flat)) = 0$. Therefore $[\mathcal{X}_\epsilon] = 0$ i.e., $\mathcal{X}_\epsilon \cong X_r \times spec\, k[\epsilon]$. ∎

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Theorem

- *Therefore the relative log tangent space of* $\mathcal{M}_{GVB/S} \to S$ *at a point* $(X_r, \mathcal{E})$ *is isomorphic to*

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where $\mathcal{C}_\bullet$ is the following complex of vector bundles on X_r
$$0 \to \mathcal{E}nd\, \mathcal{E} \xrightarrow{[\bullet,\phi]} \mathcal{E}nd\, \mathcal{E} \otimes \omega_{X_r} \to 0$$

Part VI: Relative Log-Symplectic structure on the moduli of Gieseker-Higgs bundles

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and so

$$\Omega^{log}_{(X_r,\mathcal{E})} \cong H^1(X_r, \mathcal{E}nd\mathcal{E})^\vee \cong Hom(\mathcal{E}, \mathcal{E} \otimes \omega_{X_r}).$$

Therefore, there is a relative log-symplectic form on the moduli $\mathcal{M}_{GHB/S}$.

Here "almost" indicates the fact that the relative log-cotangent bundle is a dense open subset of $\mathcal{M}_{GHB/S}$

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$$((s_{ij}, t_i), (s'_{ij}, t'_i)) \mapsto s_{ij} \otimes t'_j - t_i \otimes s'_{ij} \mapsto Trace(s_{ij} \circ t'_j - t_i \circ s'_{ij})$$

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Remember, b_i's are always non-zero positive integer.

Thank you