Part I: Basic definitions
Let $X$ be a smooth variety. A divisor $D$ is called a **simple normal crossing divisor** if

$$D = \bigcup_{i=1}^{n} D_i,$$

where

- $D_i$’s are all smooth divisors and
- every $D_i$ and $D_j$ intersect transversally for every $i \neq j$. 
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A divisor $D$ is called normal crossing divisor if etale locally the divisor is a simple normal crossing divisor i.e., there exists a etale surjective map from a variety $\pi : U \to X$ such that $\pi^{-1}(D)$ is a simple normal crossing divisor in $U$. 
Examples of Vector bundles: Log-Cotangent bundle

- Let $X$ be a smooth variety and $D$ be a normal crossing divisor.
Examples of Vector bundles: Log-Cotangent bundle

- Let $X$ be a smooth variety and $D$ be a normal crossing divisor. Let us denote $U := X \setminus D$ and $\tau : U \to X$ is the inclusion.

$\Omega^n_X(\log D)$ denotes the subsheaf of $\Omega^n_X(*D) := \tau_*\Omega^n_U$ of differential forms with logarithmic poles along $D$, i.e., if $V \subseteq X$ is open, then

$$\Gamma(V, \Omega^n_X(\log D)) = \{ \alpha \in \Gamma(V, \Omega^n_X(*D)) : \alpha \text{ and } d\alpha \text{ have simple poles along } D \}$$
Consider the particular case

\[ X = \mathbb{C}^n \text{ and } D = \bigcup_{i=1}^{r} D_i, \]

where \( D_1, \ldots, D_r \) are the first \( r \) coordinate hyperplanes, where \( 1 \leq r \leq n \).
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where \( D_1, \ldots, D_r \) are the first \( r \) coordinate hyperplanes, where \( 1 \leq r \leq n \).

Then \( \Omega^1_X(\log D) \) is freely generated as \( \mathcal{O}_X \)-module by

\[ \frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n. \]
A semistable degeneration is a flat morphism $p : X \to S$ varieties such that

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- generic fibre is smooth and
- the special fibre $D$ is a normal crossing divisor in $X$. 
Examples of Vector bundles: Relative Log-Cotangent bundle

Given a semistable degeneration we define

\[ \Omega^1_{X/S}(\log D) = p^*\Omega^1_S(\log 0) \]

where "0" denotes the closed point of the d.v.r \( S \).

The restriction of the bundle \( \Omega^1_{X/S}(\log D) \) to the divisor \( D \) is called the log cotangent bundle of \( D \). We denote this by \( \Omega^1_D(\log \partial D) \).

Dualizing sheaf of \( D \):

\[ \omega_D = \det \Omega^1_D(\log \partial D) \]
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  \[ \omega_D := \det \Omega^1_D(\log \partial D) \]
A nodal curve $\mathcal{C}$ is a curve with finitely many nodes, i.e., points $\{x_1, \ldots, x_n\}$ such that the analytic local ring

$$\hat{O}_{\mathcal{C}, x_i} \cong \frac{\mathbb{C}[[t_1, t_2]]}{t_1 \cdot t_2}, \forall i = 1, \ldots, n$$

(1)

A Higgs bundle on a prestable (i.e., connected+ smooth/ nodal) curve $\mathcal{C}$ is a pair $(\mathcal{E}, \phi)$, where
A nodal curve $C$ is a curve with finitely many nodes, i.e., points \( \{ x_1, \cdots, x_n \} \) such that the analytic local ring

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A Higgs bundle on a prestable (i.e., connected+ smooth/ nodal) curve $C$ is a pair $(\mathcal{E}, \phi)$, where

- $\mathcal{E}$ is a vector bundle on $C$, and
- $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C$ is any bundle homomorphism. Here $\omega_C$ denotes the dualizing sheaf of $C$.
Part II. Moduli functors and corresponding moduli spaces for smooth curves
Moduli of vector bundles and Higgs bundles on smooth curves

\[ F_{VB} : \text{Sch} \to \text{Sets}, \quad (F_{HB}) \]
\[ T \mapsto \{ \text{Isomorphism classes of vector (Higgs) bundles on } X \times T \text{ of rank } r \text{ and degree } d \} \]
Moduli of vector bundles and Higgs bundles on smooth curves

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\[ T \mapsto \{ \text{Isomorphism classes of vector (Higgs) bundles on } X \times T \text{ of rank } r \text{ and degree } d \} \]

- There is an obvious forgetful map \( F : F_{HB} \to F_{VB} \) and a map (by adding the "0"-Higgs field) \( Z : F_{VB} \to F_{HB} \) and \( F \circ Z \) is the identity transformation on \( F_{VB} \).
A Higgs bundle \((\mathcal{E}, \phi)\) on a smooth/irreducible nodal curve is called stable (semi-stable) if for any non zero and \(\phi\)-invariant subsheaf \(\mathcal{F}\) (i.e., \(\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \omega_C\)) we have

\[
\frac{\text{deg } \mathcal{F}}{\text{rank } \mathcal{F}} < \frac{\text{deg } \mathcal{E}}{\text{rank } \mathcal{E}} \quad (\leq)
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Stability and representability of the functors

A Higgs bundle \((\mathcal{E}, \phi)\) on a smooth/irreducible nodal curve is called stable (semi-stable) if for any non-zero and \(\phi\)-invariant subsheaf \(\mathcal{F}\) (i.e., \(\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \omega_C\)) we have

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- The subfunctors \(F_{HB^{st}}\) and \(F_{HB^{ss}}\) of \(F_{HB}\) consisting of all the stable and semistable Higgs bundles always have coarse moduli spaces. The moduli functor \(F_{VB^{ss}}\) are always proper.
A Higgs bundle \((E, \phi)\) on a smooth/ irreducible nodal curve is called stable (semi-stable) if for any non zero and \(\phi\)-invariant subsheaf \(F\) (i.e., \(\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \omega_C\)) we have

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- The subfunctors \(F_{HB}^{st}\) and \(F_{HB}^{ss}\) of \(F_{HB}\) consisting of all the stable and semistable Higgs bundles always have coarse moduli spaces. The moduli functor \(F_{VB}^{ss}\) are always proper.
- If the rank and degree are coprime, then the functors \(F_{HB}^{st}\) and \(F_{VB}^{st}\) are representable by smooth varieties \(M_{HB}\) and \(M_{VB}\), respectively. Moreover, \(M_{VB}\) is a proper variety.
The tangent space of the functor $F_{VB}$ at a point $[\mathcal{E}]$ is

$$F_{VB}(\text{Spec}k[\epsilon]) \cong H^1(\text{End}\mathcal{E}) \otimes H^0(\text{End}\mathcal{E} \otimes \omega_C).$$

Therefore, informally speaking the forgetful functor $F: F_{HB} \to F_{VB}$ is actually like the cotangent bundle map and $Z$ is like the zero section.
Moduli of vector bundles and Higgs bundles on smooth curves

- The tangent space of the functor $F_{\text{VB}}$ at a point $[\mathcal{E}]$ is
  \[ F_{\text{VB}}(\text{Speck}[\epsilon]) \cong H^1(\text{End}\mathcal{E}). \]

- Therefore the cotangent space of the functor $F_{\text{VB}}$ at a point $[\mathcal{E}]$ is isomorphic to
  \[ H^1(\text{End}\mathcal{E})^\vee \cong H^0(\text{End}\mathcal{E} \otimes \omega_C) = \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_C), \]
i.e., the space of all Higgs fields on the vector bundle $[\mathcal{E}]$. 

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Let $X$ be any smooth variety. Then

$$
\Omega_X \times_X \Omega_X \xrightarrow{\cong} \pi^* \Omega_X \rightarrow \Omega_Y
$$

The section $\lambda: Y \rightarrow \Omega_Y$ is called the Liouville $1$-form.

The $2$-form $\omega := -d\lambda$ is a symplectic form on $Y$.

(symplectic:=non-degenerate and skew-symmetric 2-form)

Therefore $M_{HB}$ has a symplectic form.
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- Therefore $\mathcal{M}_{HB}$ has a symplectic form.
Log-symplectic form

Let $X$ be a smooth variety and $D$ a normal crossing divisor on $X$.

A log-symplectic form on $X$ is an element $\omega \in \Omega^2_X(log D)$ which is closed and non-degenerate (on $T_X(−log D)$).

Let $\pi : X \to S$ be a semistable degeneration and $D$ denote the closed fibre.

- A relative log-symplectic form on $X$ over $S$ is an element $\omega \in \Omega^2_{X/S}(log D)$ which is closed and non-degenerate (on $T_{X/S}(−log D)$).
- A log-symplectic form on $D$ is an element $\omega \in \Omega^2_D(log \partial D)$ which is closed and non-degenerate (on $T_D(−log \partial D)$).
Relative log-symplectic form

Theorem

• Let $X$ be a smooth variety, then $\Omega^*_X$ has a natural symplectic form.

• Given a semistable degeneration $p: X \to S$ there exists a relative log-symplectic form on $\Omega^*_X/\mathcal{S}(\log D)$.

Remark: The relative log symplectic form is not unique i.e., there are other natural such forms on $X$. 
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Hitchin map and complete integrability (rank and degree coprime case)

(Hitchin87)

- The moduli $M_{HB}$ has a natural symplectic form $\omega$, 
- We have a natural morphism $h: M_{HB} \to \bigoplus_{i=0}^{n-1} H^0(X, \omega^i)$ which maps $(E, \phi)$ to the coefficients of the characteristic polynomial of the Higgs field $\phi$. This is known as the Hitchin map.
- The map is proper.
- The triple $(M_{HB}, \omega, h)$ is an example of an algebraically complete integrable system.
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Tangent and cotangent space of the moduli of vector bundles at a point

• $T_{\mathcal{E}} \mathcal{M}_{VB} \cong \mathcal{M}_{VB}(\text{Spec } k[\varepsilon]) =$

\[
\begin{cases}
\text{Isomorphism classes of families of vector bundles on } X \times \text{Spec } k[\varepsilon] \\
\text{such that the closed fiber is } \mathcal{E}
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- On a suitable affine cover \( \mathcal{U} := \{ U_i \}_i \) of \( X \), the first order deformation of the bundle is given by
  \[ \{ A_{ij} + \epsilon B_{ij} \}, \]
  where
  - \( \{ A_{ij} \} \) is the original transition functions of \( \mathcal{E} \) and
  - \( \{ B_{ij} \} \in \prod_{i,j} \Gamma(U_{ij}, \text{End } \mathcal{E}) \) such that
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\[ T_{[\mathcal{E}]} \mathcal{M}_{VB} \cong H^1(X, \mathcal{E} \text{nd}\mathcal{E}) \]
Tangent and cotangent space of the moduli of Higgs bundles at a point (Hitchin, Biswas-Ramanan, Bottacin)

- \( T_{[(\epsilon, \phi)]} \mathcal{M}_{HB} \cong \mathcal{M}_{HB}(k[\epsilon]). \)
Tangent and cotangent space of the moduli of Higgs bundles at a point (Hitchin, Biswas-Ramanan, Bottacin)

- \( T_{[(\mathcal{E}, \phi)]} \mathcal{M}_{HB} \cong \mathcal{M}_{HB}(k[\epsilon]) \).
- Intuitively, it should have two parts, deformation of the bundle and deformation of the Higgs field, and they should be compatible in some sense.
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- On a suitable affine cover $\mathcal{U} := \{U_i\}_I$, the first order deformation of the bundle is given by $\{A_{ij} + \epsilon B_{ij}\}$, where
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  - $\{A_{ij}\}$ is the original transition functions of $\mathcal{E}$ and
  - $\{B_{ij}\} \in H^1(\mathcal{U}, \text{End}\mathcal{E})$

- The deformation of the Higgs field is given by $\{\phi|_{U_i} + \epsilon s_i\}$, where $\{s_i\} \in \prod_{i \in I} H^0(U_i, \text{End}\mathcal{E} \otimes \omega_C)$ should satisfy the following condition

  $$s_i - s_j = [A_{ij}, \phi]$$
• Giving the data \( \{ (\mathcal{U}, B_{ij}, s_i) \} \) is equivalent to giving an element of

\[ H^1(\mathcal{C}_\bullet), \]

where \( \mathcal{C}_\bullet \) is the following complex of vector bundles

\[ \mathcal{E}nd\mathcal{E} \xrightarrow{[\bullet, \phi]} \mathcal{E}nd\mathcal{E} \otimes \omega_\mathcal{C} \]
• Giving the data \( \{(U, B_{ij}, s_i)\} \) is equivalent to giving an element of

\[
\mathbb{H}^1(C_{\bullet}),
\]

where \( C_{\bullet} \) is the following complex of vector bundles

\[
\mathcal{E}nd\mathcal{E} \xrightarrow{[\bullet, \phi]} \mathcal{E}nd\mathcal{E} \otimes \omega_C
\]

• Therefore, the tangent space at \((\mathcal{E}, \phi)\) is isomorphic to \( \mathbb{H}^1(C_{\bullet}) \).
Description of the Liouville form on moduli of Higgs bundles (Hitchin, Biswas-Ramanan, Bottacin)

The Liouville form is given by the composition

$$H^1(C^\bullet) \to H^1(\text{End}E) \xrightarrow{\text{evaluation at } \phi \phi} H^1(X, \omega_X)$$

$$\{ (B_{ij}, s_i) \} \mapsto \{ B_{ij} \} \mapsto \{ \text{Trace} (\phi \circ B_{ij}) \}$$

Notice $\phi \in \text{Hom}(E, E \otimes \omega_X)$
The Liouville form is given by the composition

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\[\{(B_{ij}, s_i)\} \mapsto \{B_{ij}\} \mapsto \{\text{Trace}(\phi \circ B_{ij})\}\]

Notice $\phi \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X) \cong H^1(\text{End} \mathcal{E})^\vee$
Description of the symplectic form on the moduli of Higgs bundles (Hitchin, Biswas-Ramanan, Bottacin)

The symplectic pairing is given by

$$H^1(C^\bullet) \otimes H^1(C^\bullet) \rightarrow H^2(C^\bullet \otimes C^\bullet) \rightarrow H^2(\omega_X[-1])$$

$$\left( s_{ij}, t_i \right) \mapsto s_{ij} \otimes t_j' - t_i \otimes s_{ij}' \mapsto \text{Trace} (s_{ij} \circ t_j' - t_i \circ s_{ij}')$$

By Serre duality,

$$H^2(\omega_X[-1]) \cong H^1(X, \omega_X^*) \cong H^0(X, \mathcal{O}_X^*)$$
Description of the symplectic form on the moduli of Higgs bundles (Hitchin, Biswas-Ramanan, Bottacin)

The symplectic pairing is given by

\[ \mathbb{H}^1(C_\bullet) \otimes \mathbb{H}^1(C_\bullet) \to \mathbb{H}^2(C_\bullet \otimes C_\bullet) \to \mathbb{H}^2(\omega_X[-1]) \cong \mathbb{C} \]
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\[ H^1(C_\bullet) \otimes H^1(C_\bullet) \to H^2(C_\bullet \otimes C_\bullet) \to H^2(\omega_X[-1]) \cong \mathbb{C} \]

\[ ((s_{ij}, t_i), (s'_{ij}, t'_i)) \mapsto s_{ij} \otimes t'_j - t_i \otimes s'_{ij} \mapsto \text{Trace}(s_{ij} \circ t'_j - t_i \circ s'_{ij}) \]
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\[ H^1(C_\bullet) \otimes H^1(C_\bullet) \rightarrow H^2(C_\bullet \otimes C_\bullet) \rightarrow H^2(\omega_X[-1]) \cong \mathbb{C} \]

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By Serre duality, \[ H^2(\omega_X[-1]) \cong H^1(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C} \]
Description of the symplectic form on the moduli of Higgs bundles (Hitchin, Biswas-Ramanan, Bottacin)
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The map $\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet \to \omega_X[-1]$ is given by the following commutative diagram.
Description of the symplectic form on the moduli of Higgs bundles (Hitchin, Biswas-Ramanan, Bottacin)

The map $\mathcal{C}_* \otimes \mathcal{C}_* \to \omega_X[-1]$ is given by the following commutative diagram

Remember $\mathcal{C}_*$ is the following complex of vector bundles

$$\mathcal{E} \operatorname{nd} \mathcal{E} \xrightarrow{[\cdot, \phi]} \mathcal{E} \operatorname{nd} \mathcal{E} \otimes \omega_C$$
Part III. Degeneration of moduli of vector (Higgs) bundles
Set up for a degeneration of the moduli of vector (Higgs) bundles on a smooth curve

- Fix a family of curves $X$ over a d.v.r $S$ such that
  - $X$ is regular over $\mathbb{C}$,
  - the generic fiber is a smooth curve,
  - the closed fiber $X_0$ is a nodal curve with a single node.
Set up for a degeneration of the moduli of vector (Higgs) bundles on a smooth curve

- Fix a flat family of curves $\mathcal{X}$ over a d.v.r $S$ such that
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Moduli of vector (Higgs) bundles on a nodal curve (a single node)
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Fix $\mathcal{X}/S$, $r \geq 2$, $d$, $(r, d) = 1$. 
Moduli of vector (Higgs) bundles on a nodal curve (a single node)

Fix $\mathcal{X} / S, r \geq 2, d, (r, d) = 1$. Define

$$F_{VB/S} : \text{Sch} / S \to \text{Sets}, (F_{HB/S})$$
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- The moduli functors are represented by a smooth variety.
- It gives a flat degeneration of the moduli of vector (Higgs) bundles on smooth curves.

Cons.
- $F_{VB/S}$ is not proper.
- The Hitchin map is not proper.
So basically we want to compactify the moduli $\mathcal{M}_{\mathcal{V}/\mathcal{S}}$ and the Hitchin map on the moduli $\mathcal{M}_{\mathcal{H}/\mathcal{S}}$. So we will have to introduce new objects in the moduli.

Let $X_0$ be a nodal curve with only one node. A Gieseker-Higgs bundle of rank $n$ and degree $d$ is a triple $(\pi_r: X_r \to X_0, E, \phi)$, where

- $\pi_r: X_r \to X_0$ is a semistable model of $X_0$, i.e., $X_r$ is obtained by replacing the node with a chain of $\mathbb{P}^1$'s of length $r$.
- We also call it modification of the nodal curve $X_0$. The map $\pi_r$ is the contraction of the rational chain.
Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

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Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

• $E$ is a vector bundle of rank $r$ with the following special properties
  • $E|_{R_i} = O_i \oplus a_i \oplus O_i(1) \oplus b_i$, $b_i > 0$, and
  • $(\pi_r)^* E$ is a torsion-free sheaf on the nodal curve $X_0$.

We call them Gieseker vector bundle/Admissible vector bundle

• $\varphi: E \to E \otimes (\pi_r)^* \omega_{X_0}$

Notice $\omega_{X_0}(\pi_r)^* \omega_{X_0}$
Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

Gieseker-Higgs bundle

- $E$ is a vector bundle of rank $r$ with the following special properties
  - $E|_{R_i} \cong O \oplus a_i \oplus O(1) \oplus b_i$, $b_i > 0$
  - $\left(\pi^r\right)^* E$ is a torsion-free sheaf on the nodal curve $X_0$.

We call them Gieseker vector bundle/Admissible vector bundle

$\phi: E \to E \otimes \left(\pi^r\right)^* \omega_{X_0}$

Notice $\omega_{X^r} \left(\pi^r\right)^* \omega_{X_0}$
Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

Gieseker-Higgs bundle

- $\mathcal{E}$ is a vector bundle of rank $r$ with the following special properties

  - $\mathcal{E}|_{R_i} \cong \mathcal{O} \oplus a_i \oplus \mathcal{O}(1) \oplus b_i, b_i > 0$, and
Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

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- $\mathcal{E}$ is a vector bundle of rank $r$ with the following special properties
  - $\mathcal{E}|_{R_i} \cong \mathcal{O}^{\oplus a_i} \oplus \mathcal{O}(1)^{\oplus b_i}$, $b_j > 0$, and
  - $(\pi_r)_* \mathcal{E}$ is a torsion-free sheaf on the nodal curve $X_0$. 

Notice $\omega_{X_0} \cong (\pi_r)_* \omega_X$. 

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Gieseker’s Moduli of Higgs bundles on a nodal curve (a single node)

Gieseker-Higgs bundle

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- $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes (\pi_r)^* \omega_{X_0}$

Notice $\omega_{X_r} \cong (\pi_r)^* \omega_{X_0}$
Modifications/ Gieseker curves

\[ X_\sigma = X_0 \cup \left( \bigcup_{i=1}^{n} R_i \right) \]

\[ R_i \cong \mathbb{P}^1 \]

\[ \exists \beta_1, \ldots, \beta_{n+3} \text{ modes.} \]
Modifications/ Gieseker curves

Admissible Gieseker vector bundle \( \mathfrak{g} \) (\( x_0, \mathcal{E} \))

\[ \mathcal{E} \begin{align*}
\mathfrak{g} &= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(1) \\
&\oplus b_i \\
b_i &> 0
\end{align*} \]

\( \pi_{x_0} \)

a torsion free sheaf
Functor of Gieseker-Higgs bundles

We define the functor $\mathcal{F}_{\text{GHB}}$: $\text{Sch} \to \text{Sets}$ as

$\{\text{Equivalence classes of Gieseker-Higgs bundles over } T\}$

Two Gieseker-Higgs bundles $(X_r, E_1, \phi_1)$ and $(X_r, E_2, \phi_2)$ are said to be equivalent if there exists an isomorphism $\sigma \in \text{Aut}(X_r/X_0)$ such that

• $\sigma^* E_1 = E_2$, and
• $\sigma^* \phi_1 = \phi_2$.

Notice that the equivalence class is stronger than the usual isomorphism class.
We define Functor of Gieseker-Higgs bundles

\[ F_{\text{GH}} : \text{Sch} \rightarrow \text{Sets} \]
We define Functor of Gieseker-Higgs bundles

\[ F_{\text{GHB}/S} : \text{Sch} \to \text{Sets} \]

\[ T \mapsto \{ \text{Equivalence classes of Gieseker-Higgs bundles over } T \} \]
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\[ F_{\text{GHB}/S} : \text{Sch} \to \text{Sets} \]

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**Equivalence classes** Two Gieseker-Higgs bundles \((X_r, \mathcal{E}_1, \phi_1)\) and \((X_r, \mathcal{E}_2, \phi_2)\) are said to be *equivalent* if there exists an isomorphism \(\sigma \in \text{Aut}(X_r/X_0)\) such that

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Functor of Gieseker-Higgs bundles

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Semistability and Representability of the functor

We define the degree and rank of the Gieseker vector bundle \((\pi_r: X_r \to X_0, E)\) to be the same as the degree and rank of the torsion-free sheaf \((\pi_r)^*E)\).

From here onwards we assume that the rank and degree are coprime.

We call a Gieseker-Higgs bundle \((\pi_r: X_r \to X_0, E)\) stable if the torsion-free Higgs pair \(((\pi_r)^*E, (\pi_r)^*\phi))\) is stable.
We define the **degree** and **rank** of the Gieseker vector bundle \((\pi_r : X_r \to X_0, \mathcal{E})\) to be the same as the **degree** and **rank** of the torsion-free sheaf \((\pi_r)_* \mathcal{E}\).
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We call a Gieseker-Higgs bundle \((\pi_r : X_r \to X_0, E)\) **stable** if the torsion-free Higgs pair \(((\pi_r)_* E, (\pi_r)_* \phi)\) is **stable**.
Let $S$ be a d.v.r and $X \to S$ be a /unifat family of curves whose

- $X$ is a smooth surface,
- generic fibre is smooth projective curve,
- closed fibre is a nodal curve with a single node.

Theorem (Gieseker, Balaji-Barik-Nagaraj)

There exists a /unifat family of varieties $M_{GHB}/S$ over $S$ which represents the functor $\text{Func}_{st}^{GHB}/S$. 
Let $S$ be a d.v.r and $\mathcal{X} \to S$ be a flat family of curves whose...
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**Theorem**

*(Gieseker, Balaji-Barik-Nagaraj)* There exists a flat family of varieties $\mathcal{M}_{GHB/S}$ over $S$ which represents the functor $\text{Func}^{\text{st}}_{GHB/S}$. 
Functor of Gieseker curves and its versal family

\[ \text{De/uni} \text{FB}_0 \text{ine} \]

\[ \mathcal{F}_{GC} / S : \text{Art-Sch} / S \to \text{Sets} (T \to S) \mapsto \{ \text{Isom. classes of Gieseker curves } X \text{mod } T \to X \times S T \text{ such that the closed } \text{uni} \text{ber is } X \}_n \]

Theorem (Gieseker, Nagaraj-Seshadri)

- It has a versal family \[ \mathcal{C}_{\text{ver}} \to V = \text{Spec} \mathbb{C} \left[ |z_1, \ldots, z_n + 1| \right] \to S = \text{Spec} \mathbb{C} |t| \]

of deformations of the curve \[ X \]_n. In particular, \[ \mathcal{F}_{GC} / S \] is formally smooth over \[ S \].

- The restriction \[ \mathcal{C}_{\text{ver}} \big|_{H_i} \] on \[ i \text{-th hyperplane } H_i \] is the smoothing of the \[ i \text{-th node of } X \]_n. In other words, the equation of the \[ i \text{-th node is } z_i = 0 \].
Functor of Gieseker curves and its versal family

Define \( F_{GC/S} : \text{Art-Sch}/S \to \text{Sets} \)

\[(T \to S) \mapsto \{ \text{Isom. classes of Gieseker curves } \chi^\text{mod}_T \to \chi \times_S T \text{ such that the closed fiber is } X_n \} \]
Define \( F_{GC/S} : \text{Art-Sch}/S \to \text{Sets} \)

\[(T \to S) \mapsto \{ \text{Isom. classes of Gieseker curves } \mathcal{X}_T^{\text{mod}} \to \mathcal{X} \times_S T \text{ such that the closed fiber is } X_n \} \]

**Theorem**

*(Gieseker, Nagaraj-Seshadri)*

- It has a versal family
  \[ \mathcal{C}_{\text{ver}} \to \mathcal{V} := \text{Spec } \mathbb{C}[|z_1, \ldots, z_{n+1}|] \to S := \text{Spec } \mathbb{C}[|t|] \]
  of deformations of the curve \( X_n \).
Functor of Gieseker curves and its versal family

Define \( F_{GC/S} : \text{Art-Sch}/S \to \text{Sets} \)

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**Theorem**

*(Gieseker, Nagaraj-Seshadri)*

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  \[
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  of deformations of the curve \( X_n \). **In particular**, \( F_{GC/S} \) is formally smooth over \( S \),
Functor of Gieseker curves and its versal family

Define \( F_{GC/S} : \text{Art-Sch}/S \to \text{Sets} \)

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(T \to S) \mapsto \{ \text{Isom. classes of Gieseker curves } \mathcal{X}_T^{mod} \to \mathcal{X} \times_S T \text{ such that the closed fiber is } X_n \}
\]

Theorem

(Gieseker, Nagaraj-Seshadri)

- It has a versal family

\[
C_{\text{ver}} \to \mathcal{V} := \text{Spec } \mathbb{C}[|z_1, \cdots, z_{n+1}|] \to S := \text{Spec } \mathbb{C}[|t|]
\]

of deformations of the curve \( X_n \). In particular, \( F_{GC/S} \) is formally smooth over \( S \),

- the restriction \( C_{\text{ver}}|_{H_i} \) on \( i \)-th hyperplane \( H_i \) is the smoothing of the \( i \)-th node of \( X_n \).
Functor of Gieseker curves and its versal family

Define $F_{GC/S} : \text{Art-Sch}/S \to \text{Sets}$

$(T \to S) \mapsto \{\text{Isom. classes of Gieseker curves } \mathcal{X}_T^{\text{mod}} \to \mathcal{X} \times_S T \text{ such that the closed fiber is } X_n \}$

**Theorem**

*(Gieseker, Nagaraj-Seshadri)*

- It has a versal family

  $\mathcal{C}_{\text{ver}} \to \mathcal{V} := \text{Spec } \mathbb{C}[|z_1, \cdots, z_{n+1}|] \to S := \text{Spec } \mathbb{C}[|t|]$ of deformations of the curve $X_n$. **In particular, $F_{GC/S}$ is formally smooth over $S$,**

- the restriction $\mathcal{C}_{\text{ver}}|_{H_i}$ on $i$-th hyperplane $H_i$ is the smoothing of the $i$-th node of $X_n$. **In other words, the equation of the $i$-th node is** $z_i = 0$. 

Semistability of the Gieseker type degenerations

Theorem

• (Gieseker, Nagaraj-Seshadri) The forgetful morphism $F_{GVB/S} \to F_{GC/S}$ is formally smooth. The special fibre $M_{GVB}$ is a normal crossing divisor and hence it gives a semistable degeneration of the moduli of vector bundles.

• (Balaji-Barik-Nagaraj) The forgetful morphism $F_{GHB/S} \to F_{GC/S}$ is formally smooth. The special fibre $M_{GHB}$ is a normal crossing divisor and hence it gives a semistable degeneration of the moduli of Higgs bundles.
Theorem

- **(Gieseker, Nagaraj-Seshadri)** The forgetful morphism $F_{GVB/S} \rightarrow F_{GC/S}$ is formally smooth. The special fibre $\mathcal{M}_{GVB}$ is a normal crossing divisor and hence it gives a semistable degeneration of the moduli of vector bundles.
Semistability of the Gieseker type degenerations

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Part IV: Log schemes and log structures on the moduli of Gieseker-Higgs bundles
Log Schemes

A log scheme is a triple $(X, M, \alpha)$ where

- $X$ is a scheme,
- $M$ is a etale sheaf of monoids, and
- $\alpha: M \to \mathcal{O}_X$ is a morphism of sheaves of monoids such that

\[ \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^* \]

is an isomorphism.

A chart for a log scheme $(X, M)$ is a monoid $P$ with a map of sheaves of monoids $P_X \to M$ which induces isomorphism between the associated log structures.
A log scheme is a triple \((X, \mathcal{M}, \alpha)\) where

- \(X\) is a scheme,
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is an isomorphism.
A log scheme is a triple \((X, M, \alpha)\) where

- \(X\) is a scheme,
- \(M\) is an etale sheaf of monoids, and
- \(\alpha : M \to O_X\) is a morphism of sheaves of monoids such that
  \[
  \alpha^{-1}(O_X^*) \to O_X^*
  \]
  is an isomorphism.

A chart for a log scheme \((X, M)\) is a monoid \(P\) with a map of sheaves of monoids \(P_X \to M\) which induces isomorphism between the associated log structures.
Morphism of log schemes

A log morphism between two log schemes \((X, M_X, \alpha_X)\) and \((Y, M_Y, \alpha_Y)\) is a pair \((f, f^\sharp, f^\flat)\) where

- \(f: X \rightarrow Y\) and \(f^\sharp: f^{-1}O_Y \rightarrow O_X\) define a morphism of schemes,
- \(f^\flat: f^{-1}M_Y \rightarrow M_X\) is a morphism of sheaves of monoids which is compatible with \(f^{-1}\alpha_Y, \alpha_X\) and \(f^\sharp\).

A chart for a morphism of log schemes \(f: (X, M_X) \rightarrow (Y, N_Y)\) consists of a pair of charts for the two log schemes which is compatible with the log morphism \(f\).
A log **morphism** between two log schemes \((X, \mathcal{M}_X, \alpha_X)\) and 
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A chart for a morphism of log schemes \(f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})\) consists of a pair of charts for the two log schemes which is compatible with the log morphism \(f\).
Example of Log Schemes

Example 1: Let $X$ be a smooth variety and $D$ be a normal crossing divisor. Define $U = X \setminus D$. Then for any étale open set $V \subseteq X$ we define $M(V) = \{ f \in \Gamma(V, O_X) | f|_{U \cap V} \in \Gamma(V \cap U, O_X^*) \}$. This is a sheaf of monoids. The map $\alpha$ is the obvious inclusion map. Then $(X, M, \alpha)$ is a log scheme.

If $x$ belongs to exactly $r$ number of local components of $D$, then the chart is given by $N^r \to C[x_1, \ldots, x_n]$ sending $e_i \mapsto x_i$ for $i = 1, \ldots, r$. 
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Example of Log Schemes.

Example 2:
**Theorem. (S. Mochizuki, F. Kato)** Let $f : X \to S$ be flat family of prestable curves. Then there exists a natural log structure on $X$, a log structure on $S$ and a log morphism from $X$ to $S$ such that the underlying morphism of ordinary schemes is the same as $f$. 
Brief outline of the log structure:

Let $s \in S$ be a point such that $X_s$ is a nodal curve with nodes $\{p_1, \ldots, p_{n+1}\}$. 

The Henselian local ring of $X$ at $p_i$ is $O_{X,P_i}$ for some $t_i \in m_{A[x,y]}$. The log structure of $S$ at $s$ is $N = N_1 \oplus O^* \oplus \cdots \oplus O^* N_{n+1}$ where $N_i$ is induced by $N \to A$ which sends $e_i \mapsto t_i$.

($N_i$ can be thought as the log structure due to the node $p_i$)
Example of Log Schemes

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\mathcal{O}^h_{X,p_i} \cong \frac{A[x,y]}{xy - t_i}
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for some \( t_i \in m_{A,s} \).
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The log structure of \( S \) at \( s \) is \( \mathcal{N} := \mathcal{N}_1 \oplus \mathcal{O}_S^* \cdots \oplus \mathcal{O}_S^* \mathcal{N}_{n+1} \) where \( \mathcal{N}_i \) is induced by

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(\( \mathcal{N}_i \) can be thought as the log structure due to the node \( p_i \))
Two Log structures on the moduli of Higgs bundles

There are two log structures on the relative moduli spaces $M_{GHB}/S$ and $M_{GVB}/S$.

• The universal curve $C_{\text{univ}}$ defines log structures on itself and on $M_{GHB}/S$ and $M_{GVB}/S$.

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Equivalence of the two log structures

Outline of the proof.

Recall that we have the relative picture $C^{\text{univ}} \to M_{\text{GHB}}/S \to S$ where $S$ is a d.v.r with parameter $t$. (e.g. $\text{spec } C[[t]]$)

One can show that at a point $(X^n, E, \phi) \in M_{\text{GHB}},$

- there are exactly $n+1$ components that intersect each other transversally.

To prove that the two log structures are the same it is enough to show that the local equations of nodes and the local equations of the components are the same.
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One can show that at a point \((X_n, \mathcal{E}, \phi) \in \mathcal{M}_{\text{GHB}}\),

- there are exactly \( n + 1 \) components that intersect each other transversally.

Let \( z_1, \ldots, z_{n+1} \) are the local equations of the components passing through the point \((X_n, \mathcal{E}, \phi)\).

To prove that the two log structures are the same it is enough to show that the local equations of nodes and the local equations of the components are the same.
Equivalence of the two log structures

This is because, etale locally, the curve $C_{\text{univ}}$ for some canonical local map $f: M_{\text{GHB}}/S \to V$. So it is enough to check it for the versal family, which follows from its construction. Therefore the two log structures are the same. ■

Theorem (Gieseker, Nagaraj-Seshadri)

- It has a versal family $C_{\text{ver}} \to V = \text{Spec} C[z_1, \ldots, z_n+1] \to S = \text{Spec} C[t]$ of deformations of the curve $X_n$.
- In particular, $F_{GC}/S$ is formally smooth over $S$.
- The restriction $C_{\text{ver}}|_{H_i}$ on $H_i$-th hyperplane $H_i$ is the smoothing of the $i$-th node of $X_n$.
- In other words, the equation of the $i$-th node is $z_i = 0$. 


Equivalence of the two log structures

This is because, etale locally, the curve \( C^{\text{univ}} \cong f^*C_{\text{ver}} \) for some canonical local map \( f : \mathcal{M}_{\text{GHB}}/S \to \mathcal{V} \).
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*(Gieseker, Nagaraj-Seshadri)*

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of deformations of the curve $X_n$. 
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of deformations of the curve \( X_n \). In particular, \( F_{\text{GC}/S} \) is formally smooth over \( S \),

- the restriction \( C_{\text{ver}}|_{H_i} \) on \( i \)-th hyperplane \( H_i \) is the smoothing of the \( i \)-th node of \( X_n \). In other words, the equation of the \( i \)-th node is \( z_i = 0 \).
Part V: Log deformations and log-tangent space of the moduli of Gieseker-Higgs bundles
Log tangent space

Let $(X, M, \alpha)$ be a log scheme and $x \in X$ be any point. The inclusion $x : \text{Spec} \ k \rightarrow X$ induces a log structure on the point by pulling back the sheaf of monoids. We denote this pulled back log scheme by $(x, M_x, \alpha_x)$. 
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Log tangent space

The inclusion $i : \text{Spec } k \hookrightarrow \text{Spec } k[\epsilon]$ induces a log structure on $\text{Spec } k[\epsilon]$ by composition $x^*\mathcal{M} \rightarrow k \rightarrow k[\epsilon]$. 
The inclusion \( i : \text{Spec } k \hookrightarrow \text{Spec } k[\epsilon] \) induces a log structure on \( \text{Spec } k[\epsilon] \) by composition \( x^* \mathcal{M} \to k \to k[\epsilon] \). Let us denote this log scheme by \( (\text{Spec } k[\epsilon], \mathcal{M}_\epsilon, \alpha_\epsilon) \).
Log tangent space

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Then the log tangent space is defined to be the space of log morphisms

\[
T_x^{\log} \coloneqq \text{Hom}_{(x, \mathcal{M}_x, \alpha_x)}((\text{Spec } k[\epsilon], \mathcal{M}_\epsilon, \alpha_\epsilon), (X, \mathcal{M}, \alpha)),
\]

which are also (log)-liftings of the log-morphism \( x \hookrightarrow X \).
Olsson’s Log stack

Let \((S, L)\) be a \(\text{finite-type and integral monoid,}\) \(M\) is integral if the map \(M \to M_{\text{gp}}\) is injective.

Define a \(\text{finite category,}\) \(\text{Log}\left((S, L)\right) \to (\text{Sch}/S)\) as follows.

- The objects of \(\text{Log}\left((S, L)\right)\) are morphisms of \(\text{finite log schemes}\) \((X, M) \to (S, L)\), and
- a morphism \(h: (X', M') \to (X, M)\) in \(\text{Log}\left((S, L)\right)\) is a morphism of \((S, L)\)-log-schemes for which \(h^\#: h^* M \to M'\) is an isomorphism.
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Let $(S, \mathcal{L})$ be a fine log scheme. (fine:= finite type and integral monoid, $M$ is integral if the map $M \to M^{gp}$ is injective)
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Define a fibered category

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Olsson’s Log stack

Theorem (Olsson) \( \text{Log} \left( S, L \right) \) is an algebraic stack locally of finite presentation over \( S \).

Useful properties:

• Giving a log morphism of finite log schemes \( f: \left( X, M \right) \to \left( S, L \right) \) is equivalent to giving a morphism of stacks \( \text{Log}: X \to \text{Log} \left( S, L \right) \).

• \( \Omega_{\log} f = \Omega f \).

• The relative log tangent space for \( f \) at a point is isomorphic to ordinary relative tangent space of \( \text{Log} \) at the point.
Olsson’s Log stack

Theorem

(Olsson) \( \text{Log}_{(S, \mathcal{L})} \) is an algebraic stack locally of finite presentation over \( S \).

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- $\Omega^\text{log}_f \cong \Omega^\text{fLog}_f$. 
Olsson’s Log stack

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Useful properties

- Giving a log morphism of fine log schemes \( f : (X, \mathcal{M}) \to (S, \mathcal{L}) \) is equivalent to giving a morphism of stacks \( f_{\mathcal{L}og} : X \to \mathcal{L}og_{(S,\mathcal{L})} \).

- \( \Omega_{f}^{log} \cong \Omega_{f_{\mathcal{L}og}} \).

- The relative log tangent space for \( f \) at a point is isomorphic to ordinary relative tangent space of \( f_{\mathcal{L}og} \) at the point.
Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over $S$
Relative log tangent space of $M_{GHB}/S$ and $M_{GVB}/S$ over $S$

Computation of the relative log tangent space:

- First of all
  \[ T_{M_{GHB}/S}(-\log M_{GHB}) \cong T_{M_{GHB}/S / \log(S, L)} \]
  the relative log-tangent space at a point of $M_{GHB}/S \to S$ is isomorphic to the rel. tangent space of $M_{GHB}/S \to \log(S, L)$, where $L$ is the log structure on $S$ induced by its closed point.

- Moreover,
  \[ T_{M_{GHB}}(-\log \partial M_{GHB}) \cong T_{M_{GHB} / \log(s, s^* L)} \]
  here $s : speck \hookrightarrow S$ is the closed point and $s^* L$ denotes the pull back log structure.
Relative log tangent space of $\mathcal{M}_{\text{GH}/S} \rightarrow S$
Relative log tangent space of $\mathcal{M}_{\text{GHB}/S} \to S$

Consider the fibre product
Consider the fibre product

\[
\begin{array}{c}
\mathcal{M}_{GHB}(k[\varepsilon]) \leftarrow \mathcal{M}_{GHB}(k[\varepsilon]) \times_{\text{Log}(s,s^*\mathcal{L})(k[\varepsilon])} \text{Log}(s,s^*\mathcal{L})(k) \\
\downarrow \\
\text{Log}(s,s^*\mathcal{L})(k[\varepsilon]) \leftarrow \text{Log}(s,s^*\mathcal{L})(k)
\end{array}
\]

where

- the left vertical arrow is the differential of the map 
  \( \mathcal{M}_{GHB} \to \text{Log}(k,\mathbb{N}) \) at the point \((X_r, \mathcal{E}, \phi)\),
- the lower horizontal map is given by the log structure on 
  \( k[\varepsilon] \) induced from a log structure on \( k \) via the map 
  \( k \hookrightarrow k[\varepsilon] \).
Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over $S$

\[ \mathcal{M}_{GHB}(k[\epsilon]) \leftarrow \mathcal{M}_{GHB}(k[\epsilon]) \times_{\log_{(s,s^*L)}(k[\epsilon])} \log_{(s,s^*L)}(k) \]

\[ \log_{(s,s^*L)}(k[\epsilon]) \leftarrow \log_{(s,s^*L)}(k) \]

- The relative log tangent space of $\mathcal{M}_{GHB/S}$ over $S$ at a point $(X_r, \mathcal{E}, \phi)$ is isomorphic to the fibre of the right vertical map over the point of $\log_{(s,s^*L)}(k)$ given by $(X_r, \mathcal{E}, \phi)$. 

Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over $S$

- **Step 1.** What is the log structure $\mathcal{L}$?
  
  \[ \mathbb{N} \rightarrow k[[t]] \]
  
  $e \mapsto t$
Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over $S$

- Step 1. What is the log structure $\mathcal{L}$?
  \[
  \mathbb{N} \to k[|t|] \\
  e \mapsto t
  \]

- Step 2. What is the log structure $s^* \mathcal{L}$?
  \[
  \mathbb{N} \to k \\
  e \mapsto 0
  \]
Relative log tangent space of $\mathcal{M}_{GHB/S}$ and $\mathcal{M}_{GVB/S}$ over $S$

- Step 1. What is the log structure $\mathcal{L}$?
  $$\mathbb{N} \to k[[t]]$$
  $$e \mapsto t$$

- Step 2. What is the log structure $s^*\mathcal{L}$?
  $$\mathbb{N} \to k$$
  $$e \mapsto 0$$

- Step 3. What is the point of $\text{Log}_{(s,s^*\mathcal{L})}(k)$ given by $(X_r, E, \phi)$?

The Gieseker-Higgs bundle $p := (X_r, E, \phi)$ gives an element $(p, p^*\mathcal{M})$ of $\text{Log}_{(s,s^*\mathcal{L})}(k)$, where $\mathcal{M}$ denotes the log structure on $\mathcal{M}_{GHB/S}$. 
Since there are $r + 1$ number of nodes in $X_r$ therefore this element is given by the log morphism

\[
\begin{array}{c}
\sum e_i \\
\end{array}
\begin{array}{c}
\rightarrow \\
\end{array}

\begin{array}{c}
\mathbb{N}^{r+1} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
k \\
\end{array}
\begin{array}{c}
e \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
\begin{array}{c}
e \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
\begin{array}{c}
(p, p^* \mathcal{M}) \\
(\mathbb{N}, s^* \mathcal{L}) \\
\end{array}
Step 4. What is the image \((p, p^*\mathcal{M})\) under the lower horizontal map \(\text{Log}_{(s,s^*\mathcal{L})}(k[\epsilon]) \leftarrow \text{Log}_{(s,s^*\mathcal{L})}(k)\)?

It is given by

\[
\begin{align*}
(p[\epsilon], (p^*\mathcal{M})_\epsilon) \quad & \sum e_i \quad \mathbb{N}^{r+1} \quad \rightarrow \quad k \quad \rightarrow \quad k[\epsilon] \\
\uparrow \quad \uparrow & \quad \uparrow \\
(s, s^*\mathcal{L}) \quad & e \quad \mathbb{N} \quad \rightarrow \quad k \\
\quad & e \quad \rightarrow \quad 0
\end{align*}
\]
Step 5. What is the image of \( p := (X_r, \mathcal{E}, \phi) \) under the left vertical map \( \mathcal{M}_{GHB}(k[\epsilon]) \to \mathcal{Log}_{(s,s^*L)}(k[\epsilon]) \)?

An element of \( \mathcal{M}_{GHB}(k[\epsilon]) \) is a first order infinitesimal deformation of \( p := (X_r \to X_0, \mathcal{E}, \phi) \). Let us denote the deformation by \( p_\epsilon := (X_\epsilon \to X_0 \times \text{spec } k[\epsilon], \mathcal{E}_\epsilon, \phi_\epsilon) \).

Let us write out the log structure on \( \text{spec } k[\epsilon] \) induced by the point \( (X_\epsilon, \mathcal{E}_\epsilon, \phi_\epsilon) \) i.e., by the family of curves \( X_\epsilon \) using Mochizuki’s method.
• Step 5. What is the image of $p := (X_r, \mathcal{E}, \phi)$ under the left vertical map $\mathcal{M}_{\text{GHB}}(k[\epsilon]) \to \mathcal{L}og(s, s^*\mathcal{L})(k[\epsilon])$?

An element of $\mathcal{M}_{\text{GHB}}(k[\epsilon])$ is a first order infinitesimal deformation of $p := (X_r \to X_0, \mathcal{E}, \phi)$. Let us denote the deformation by $p_\epsilon := (X_\epsilon \to X_0 \times \text{spec } k[\epsilon], \mathcal{E}_\epsilon, \phi_\epsilon)$.

Let us write out the log structure on $\text{spec } k[\epsilon]$ induced by the point $(X_\epsilon, \mathcal{E}_\epsilon, \phi_\epsilon)$ i.e., by the family of curves $X_\epsilon$ using Mochizuki’s method.

Remember that the closed fibre of $\mathcal{X}$ is $X_r$. Let us denote its nodes by $\{p_1, \ldots, p_{r+1}\}$. The henselian local ring at the node $p_i$ is the henselisation of

$$\frac{k[x_i, y_i, \epsilon]}{x_i \cdot y_i - \lambda_i \cdot \epsilon}$$

at the maximal ideal $(x_i, y_i, \epsilon)$. 
• Step 5. What is the image of $p := (X_r, \mathcal{E}, \phi)$ under the left vertical map $\mathcal{M}_{GHB}(k[\epsilon]) \to \mathcal{L}og_{(s, s^*, \mathcal{L})}(k[\epsilon])$?

An element of $\mathcal{M}_{GHB}(k[\epsilon])$ is a first order infinitesimal deformation of $p := (X_r \to X_0, \mathcal{E}, \phi)$. Let us denote the deformation by $p_\epsilon := (X_\epsilon \to X_0 \times \text{spec } k[\epsilon], \mathcal{E}_\epsilon, \phi_\epsilon)$.

Let us write out the log structure on $\text{spec } k[\epsilon]$ induced by the point $(X_\epsilon, \mathcal{E}_\epsilon, \phi_\epsilon)$ i.e., by the family of curves $X_\epsilon$ using Mochizuki’s method. Remember that the closed fibre of $X$ is $X_r$. Let us denote its nodes by $\{p_1, \ldots, p_{r+1}\}$. The henselian local ring at the node $p_i$ is the henselisation of

$$\frac{k[x_i, y_i, \epsilon]}{x_i \cdot y_i - \lambda_i \cdot \epsilon}$$

at the maximal ideal $(x_i, y_i, \epsilon)$. 
Relative log tangent space of $\mathcal{M}_{\text{GBH}}/S \rightarrow S$
The log structure on \( \text{spec } k[\epsilon] \) induced by the node \( p_i : \)

\[
\alpha_i : \mathbb{N} \rightarrow k[\epsilon] \\
e \mapsto \lambda_i \epsilon
\]

Let us denote the log structure by \( L_i. \)
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

The log structure on $\text{spec } k[\epsilon]$ induced by the node $p_i$:

\[
\alpha_i : \mathbb{N} \to k[\epsilon] \\
e \mapsto \lambda_i \epsilon
\]

Let us denote the log structure by $\mathcal{L}_i$.

Finally the induced log structure on $\text{spec } k[\epsilon]$ is the amalgamated sum

\[
\mathcal{L}_{k[\epsilon]} := \mathcal{L}_1 \oplus_{k[\epsilon]} \cdots \oplus_{k[\epsilon]} \mathcal{L}_{r+1}
\]

It is isomorphic to the log structure associated with the prelog structure

\[
\mathbb{N}^{r+1} \to k[\epsilon] \\
\epsilon_i \mapsto \lambda_i \epsilon
\]
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

The log structure on $\text{spec } k[\epsilon]$ induced by the node $p_i$:

$$\alpha_i : \mathbb{N} \to k[\epsilon]$$

$$e \mapsto \lambda_i \epsilon$$

Let us denote the log structure by $\mathcal{L}_i$.

Finally the induced log structure on $\text{spec } k[\epsilon]$ is the amalgamated sum

$$\mathcal{L}_{k[\epsilon]} := \mathcal{L}_1 \oplus_{k[\epsilon]} \cdots \oplus_{k[\epsilon]} \mathcal{L}_{r+1}$$

It is isomorphic to the log structure associated with the prelog structure

$$\mathbb{N}^{r+1} \to k[\epsilon]$$

$$e_i \mapsto \lambda_i \epsilon$$
Therefore the image of $p := (X_r, \mathcal{E}, \phi)$ under the map $\mathcal{M}_{GHB}(k[\epsilon]) \to \mathcal{L}og_{(s, s^* \mathcal{L})}(k[\epsilon])$ is given by the following log morphism

$$
e_i \quad \rightarrow \quad \epsilon \cdot \lambda_i$$

\[
\begin{array}{ccc}
(p_\epsilon, p_\epsilon^* \mathcal{M}) & \sum e_i & \mathbb{N}^{r+1} \\
\uparrow & & \uparrow \\
(s, s^* \mathcal{L}) & e & \mathbb{N} \\
\end{array}
\]

\[
\begin{array}{ccc}
e & \mathbb{N} & k \\
\uparrow & & \uparrow \\
e & 0 \\
\end{array}
\]
Relative log tangent space of $\mathcal{M}_{\text{GHB}/S} \to S$
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

- **Step 6.** Elements of the fibre product:

$$
\mathcal{M}_{GHB}(k[\epsilon]) \leftarrow \mathcal{M}_{GHB}(k[\epsilon]) \times \mathcal{L}_{\text{Log}(s,s^*\mathcal{L})(k[\epsilon])} \mathcal{L}_{\text{Log}(s,s^*\mathcal{L})(k)}
$$

Therefore, by equating the log structures in **Step 4.** and **Step 5.**, we conclude that an infinitesimal deformation $(\mathcal{X}_{\epsilon} \to X_0 \times \text{spec } k[\epsilon], \mathcal{E}_{\epsilon}, \phi_\epsilon)$ of $(X_r \to X_0, \mathcal{E}, \phi)$ is an element of the fibre product if and only if $\lambda_i = 0$ for all $i = 1, \ldots, r + 1$. 
Relative log tangent space of $\mathcal{M}_{GHBS} \to S$

Step 7.

Lemma: Such deformations are trivial i.e., $X \in X \times_{\text{spec } k} [\epsilon]$.

Proof. The space of first order infinitesimal deformations of the nodal curve $X$ is isomorphic to $\text{Ext}^1(\Omega_{X}, O_X)$.

Now we use the following short exact sequence.

Since $\lambda_i = 0$ for all $i = 1, \ldots, r+1$, it follows that the infinitesimal deformation $X \epsilon \in H^1(X, \text{Hom}(\Omega_{X}, O_X))$. 
Step 7.

Lemma: Such deformations are trivial i.e., $X_\epsilon \cong X_r \times \text{spec } k[\epsilon]$.

Proof.
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

Step 7.

**Lemma:** Such deformations are trivial i.e., $\mathcal{X}_\epsilon \cong X_r \times \text{spec } k[\epsilon]$.

**Proof.** The space of first order infinitesimal deformations of the nodal curve $X_r$ is isomorphic to

$$\text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r})$$
Relative log tangent space of $\mathcal{M}_{\text{GHB}/S} \to S$

Step 7.

**Lemma:** Such deformations are trivial i.e., $\mathcal{X}_\epsilon \cong X_r \times \text{spec } k[\epsilon]$.

**Proof.** The space of first order infinitesimal deformations of the nodal curve $X_r$ is isomorphic to

$$\text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r})$$

Now we use the following short exact sequence

$$0 \to H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to \text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r}) \to H^0(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \cong \oplus_{i=1}^{i=r+1} \text{Ext}^1(\Omega_{X_r, p_i}, \mathcal{O}_{X_r, p_i}) \to 0$$

Since $\lambda_i = 0$ for all $i = 1, \ldots, r + 1$, it follows that the infinitesimal deformation $\mathcal{X}_\epsilon \in H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$. 
Modifications/ Gieseker curves

\[ X_\sigma = X_0 \cup \left( \bigcup_{i=1}^{r} R_i \right) \]

\[ R_i \cong \mathbb{P}^1 \]

\[ \exists \beta_1, \ldots, \beta_r + 3 \text{ modes.} \]
Relative log tangent space of $\mathcal{M}_{\text{GHB}/S} \to S$

We have a natural map $\pi^\flat_r: \text{Hom}(\Omega^1_{X^r}, O_{X^r}) \to \text{Hom}(\pi_*r\Omega^1_{X^0}, O_{X^r})$.

(2)

and the induced map $H_1(\pi^\flat_r): H_1(X^r, \text{Hom}(\Omega^1_{X^r}, O_{X^r})) \to H_1(X^r, \text{Hom}(\pi_*r\Omega^1_{X^0}, O_{X^r}))$.

(3)

Moreover, given $[X^r] \in H_1(X^r, \text{Hom}(\Omega^1_{X^r}, O_{X^r}))$, $H_1(\pi^\flat_r)([X^r]) = \text{the obstruction to extending the map } X^r \to X^0 \text{ to a map } X^r \to X^0 \times_{\text{spec } k} [\mathfrak{m}]$.

But since our deformation $X^r \in F_{\text{GHB}/S}(k[\mathfrak{m}])$, by definition, comes with a morphism to $X^0 \times_{\text{spec } k} [\mathfrak{m}]$, therefore $H_1(\pi^\flat_r)([X^r]) = 0$. 
Relative log tangent space of $\mathcal{M}_{\text{GHB}/S} \rightarrow S$

We have a natural map

$$\pi^\flat_r : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \rightarrow \mathcal{H}om(\pi^*_r \Omega_{X_0}, \mathcal{O}_{X_r}).$$

(2)

and the induced map

$$H^1(\pi^\flat_r) : H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \rightarrow H^1(X_r, \mathcal{H}om(\pi^*_r \Omega_{X_0}, \mathcal{O}_{X_r}))$$
Relative log tangent space of $\mathcal{M}_{\text{GHB}}/S \to S$

We have a natural map

$$\pi_r^\flat : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r}).$$  \hspace{1cm} (2)

and the induced map

$$H^1(\pi_r^\flat) : H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to H^1(X_r, \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r})).$$ \hspace{1cm} (3)

Moreover, given $[\mathcal{X}_\epsilon'] \in H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$,

$$H^1(\pi_r^\flat)([\mathcal{X}_\epsilon']) = \text{the obstruction to extending the map } X_r \to X_0 \text{ to a map } \mathcal{X}_\epsilon' \to X_0 \times \text{spec } k[\epsilon].$$
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

We have a natural map

$$\pi_r^\flat : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r}).$$

(2)

and the induced map

$$H^1(\pi_r^\flat) : H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to H^1(X_r, \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r}))$$

(3)

Moreover, given $[\mathcal{X}'_\epsilon] \in H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$,

$$H^1(\pi_r^\flat)([\mathcal{X}'_\epsilon]) = \text{the obstruction to extending the map } X_r \to X_0 \text{ to a map } \mathcal{X}'_\epsilon \to X_0 \times \text{spec } k[\epsilon].$$

But since our deformation $\mathcal{X}_\epsilon \in F_{GHB/S}(k[\epsilon])$, by definition, comes with a morphism to $X_0 \times \text{spec } k[\epsilon]$, therefore
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

We have a natural map

$$\pi_r^\flat : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi_r^* \Omega_{X_0}, \mathcal{O}_{X_r}).$$  \hspace{1cm} (2)

and the induced map

$$H^1(\pi_r^\flat) : H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to H^1(X_r, \mathcal{H}om(\pi_r^* \Omega_{X_0}, \mathcal{O}_{X_r})).$$  \hspace{1cm} (3)

Moreover, given $[\mathcal{X}_\epsilon'] \in H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$,

$$H^1(\pi_r^\flat)([\mathcal{X}_\epsilon']) = \text{the obstruction to extending the map } X_r \to X_0 \text{ to a map } \mathcal{X}_\epsilon' \to X_0 \times \text{spec } k[\epsilon].$$

But since our deformation $\mathcal{X}_\epsilon \in F_{GHB/S}(k[\epsilon])$, by definition, comes with a morphism to $X_0 \times \text{spec } k[\epsilon]$, therefore

$$H^1(\pi_r^\flat)([\mathcal{X}_\epsilon]) = 0$$
It is not difficult to see that

\[ \text{Ker}(\pi^b_r : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi^*_r \Omega_{X_0}, \mathcal{O}_{X_r})) \cong \bigoplus_{i=1}^r \mathcal{O}_{R_i}. \]
It is not difficult to see that

\[ \text{Ker}(\pi_r^\flat : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi_r^*\Omega_{X_0}, \mathcal{O}_{X_r})) \cong \bigoplus_{i=1}^r \mathcal{O}_{R_i}. \]

Since \( H^1(R_i, \mathcal{O}_{R_i}) = 0 \), the map \( \text{Ker}(H^1(\pi_r^\flat)) = 0. \)
It is not difficult to see that
\[
\text{Ker}(\pi^b_r : \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}) \to \mathcal{H}om(\pi^*_r \Omega_{X_0}, \mathcal{O}_{X_r})) \cong \bigoplus_{i=1}^r \mathcal{O}_{R_i}.
\]
Since \( H^1(R_i, \mathcal{O}_{R_i}) = 0 \), the map \( \text{Ker}(H^1(\pi^b_r)) = 0 \). Therefore \( [\mathcal{X}_\epsilon] = 0 \) i.e., \( \mathcal{X}_\epsilon \cong X_r \times \text{spec } k[\epsilon] \).
Relative log tangent space of $\mathcal{M}_{{\text{GHB}}/S} \rightarrow S$

Theorem

• Therefore the relative log tangent space of $\mathcal{M}_{{\text{GHB}}/S} \rightarrow S$ at a point $(X_r, E)$ is isomorphic to $H^1(X_r, \text{End} E)$

• Similarly, the relative log tangent space of $\mathcal{M}_{{\text{GHB}}/S} \rightarrow S$ at a point $(X_r, E, \phi)$ is isomorphic to $H^1(C\bullet)$ where $C\bullet$ is the following complex of vector bundles on $X_r$
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

**Theorem**

- Therefore the relative log tangent space of $\mathcal{M}_{GVB/S} \to S$ at a point $(X_r, \mathcal{E})$ is isomorphic to

$$H^1(X_r, \mathcal{E} \text{nd} \mathcal{E})$$

- Similarly, the relative log tangent space of $\mathcal{M}_{GHB/S} \to S$ at a point $(X_r, \mathcal{E}, \phi)$ is isomorphic to

$$H^1(C_{\cdot}, \phi) \to \mathcal{E} \otimes \omega_{X_r} \to 0$$
Theorem

- Therefore the relative log tangent space of \( \mathcal{M}_{GVB/S} \to S \) at a point \((X_r, \mathcal{E})\) is isomorphic to
  \[
  H^1(X_r, \mathcal{E}nd \mathcal{E})
  \]

- Similarly, the relative log tangent space of \( \mathcal{M}_{GHB/S} \to S \) at a point \((X_r, \mathcal{E}, \phi)\) is isomorphic to
  \[
  H^1(C_\bullet)
  \]
Relative log tangent space of $\mathcal{M}_{GHB/S} \to S$

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>Therefore the relative log tangent space of $\mathcal{M}_{GVB/S} \to S$ at a point $(X_r, \mathcal{E})$ is isomorphic to</td>
</tr>
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<tr>
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</tr>
<tr>
<td>$0 \to \mathcal{E} \text{nd} \mathcal{E} \overset{[\cdot, \phi]}{\longrightarrow} \mathcal{E} \text{nd} \mathcal{E} \otimes \omega_{X_r} \to 0$</td>
</tr>
</tbody>
</table>
Part VI: Relative Log-Symplectic structure on the moduli of Gieseker-Higgs bundles
Relative Log-Symplectic structure on the moduli of Gieseker-Higgs bundles

We saw that the relative moduli of Gieseker-Higgs bundles \( \mathcal{M}_{\text{GHB}}/S \) is ("almost") the relative log-cotangent bundle of the relative moduli of Gieseker vector bundles \( \mathcal{M}_{\text{GVB}}/S \).

\[
T_{\log}(X_r, E) = H^1(X_r, \text{End} E) \quad \text{and so} \quad \Omega_{\log}(X_r, E) = H^1(X_r, \text{End} E)^\vee \otimes \text{Hom}(E, E \otimes \omega_{X_r}).
\]

Therefore, there is a relative log-symplectic form on the moduli \( \mathcal{M}_{\text{GHB}}/S \).

Here "almost" indicates the fact that the relative log-cotangent bundle is a dense open subset of \( \mathcal{M}_{\text{GHB}}/S \).
Relative Log-Symplectic structure on the moduli of Gieseker-Higgs bundles

We saw that the relative moduli of Gieseker-Higgs bundles $\mathcal{M}_{GHB/S}$ is ("almost") the relative log-cotangent bundle of the relative moduli of Gieseker vector bundles $\mathcal{M}_{GVB/S}$.
Relative Log-Symplectic structure on the moduli of Gieseker-Higgs bundles

We saw that the relative moduli of Gieseker-Higgs bundles $\mathcal{M}_{GHB/S}$ is ("almost") the relative log-cotangent bundle of the relative moduli of Gieseker vector bundles $\mathcal{M}_{GVB/S}$.

\[
T^{\log}_{(X_r, \mathcal{E})} \cong H^1(X_r, \mathcal{E} \text{nd}\mathcal{E})
\]

and so

\[
\Omega^{\log}_{(X_r, \mathcal{E})} \cong H^1(X_r, \mathcal{E} \text{nd}\mathcal{E})^\vee \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_{X_r}).
\]

Therefore, there is a relative log-symplectic form on the moduli $\mathcal{M}_{GHB/S}$.

Here "almost" indicates the fact that the relative log-cotangent bundle is a dense open subset of $\mathcal{M}_{GHB/S}$.
Similarly as in the case of smooth curves, the relative log-symplectic pairing is given by

\[ H_1(C^\bullet) \otimes H_1(C^\bullet) \to H_2(C^\bullet \otimes C^\bullet) \to H_2(\omega_X[−1]) \]

\[ s_{ij} \otimes t'_{ij} - t_{ii} \otimes s'_{ij} \mapsto \text{Trace}(s_{ij} \circ t'_{ij} - t_{ii} \circ s'_{ij}) \]
Similarly as in the case of smooth curves, the relative log-symplectic pairing is given by

\[ H^1(C^\bullet) \otimes H^1(C^\bullet) \rightarrow H^2(C^\bullet \otimes C^\bullet) \rightarrow H^2(\omega_X[1]) \]

\[ (s_{ij}, t_i) \mapsto s_{ij} \otimes t'_{ij} - t_i \otimes s'_{ij} \mapsto \text{Trace}(s_{ij} \circ t'_{ij} - t_i \circ s'_{ij}) \]
Similarly as in the case of smooth curves, the relative log-symplectic pairing is given by

\[ H^1(C_\bullet) \otimes H^1(C_\bullet) \rightarrow H^2(C_\bullet \otimes C_\bullet) \rightarrow H^2(\omega_{X_0}[-1]) \cong \mathbb{C} \]
Similarly as in the case of smooth curves, the relative log-symplectic pairing is given by

$$\mathbb{H}^1(C_\bullet) \otimes \mathbb{H}^1(C_\bullet) \to \mathbb{H}^2(C_\bullet \otimes C_\bullet) \to \mathbb{H}^2(\omega_{\chi_0}[-1]) \cong \mathbb{C}$$

$$((s_{ij}, t_i), (s'_{ij}, t'_i)) \mapsto s_{ij} \otimes t'_j - t_i \otimes s'_{ij} \mapsto \text{Trace}(s_{ij} \circ t'_j - t_i \circ s'_{ij})$$
Stratification by Poisson ranks

- Stratification by successive singular loci:
  \[ M_{\text{GHB}} \supset \partial M_{\text{GHB}} \supset \cdots \supset \partial_n M_{\text{GHB}} \]

- Stratification by Poisson ranks:
  \[ M_{\text{GHB}} \supset \partial_{\theta} M_{\text{GHB}} \supset \cdots \supset \partial_n \theta M_{\text{GHB}} \]

**Proposition**

- \( \partial_r M_{\text{GHB}} = \partial_{r+1} \theta M_{\text{GHB}} \) for all \( 1 \leq r \leq n \)
- \( \partial_r M_{\text{GHB}} \setminus \partial_{r+1} M_{\text{GHB}} \) consists of all Gieseker-Higgs bundles \((X_r, E, \varphi)\).
- \( \partial_{n+1} M_{\text{GHB}} \) is empty.

In particular, \( \partial_n M_{\text{GHB}} \) is a smooth Poisson subvariety.
Stratification by Poisson ranks

There are two stratifications

- Stratification by successive singular loci:
  \[ \mathcal{M}_{GH} \supset \partial \mathcal{M}_{GH} \supset \cdots \supset \partial \mathcal{n \mathcal{M}_{GH}} \]

- Stratification by Poisson ranks:
  \[ \mathcal{M}_{GH} \supset \partial \theta \mathcal{M}_{GH} \supset \cdots \supset \partial \mathcal{n \theta \mathcal{M}_{GH}} \]

**Proposition**

\[ \partial r \mathcal{M}_{GH} = \partial r \theta \mathcal{M}_{GH} \text{ for all } 1 \leq r \leq n \]

\[ \partial n + 1 \mathcal{M}_{GH} \] consists of all Gieseker-Higgs bundles \( (X_r, E, \phi) \).

\[ \partial n \mathcal{M}_{GH} \] is empty. In particular, \( \partial n \mathcal{M}_{GH} \) is a smooth Poisson subvariety.
Stratification by Poisson ranks

There are two stratifications

- Stratification by successive singular loci:
  \[ \mathcal{M}_{GHB} \supset \partial \mathcal{M}_{GHB} \supset \cdots \supset \partial^n \mathcal{M}_{GHB} \]

- Stratification by Poisson ranks:
  \[ \mathcal{M}_{GHB} \supset \partial^{\theta} \mathcal{M}_{GHB} \supset \cdots \supset \partial^{\theta n} \mathcal{M}_{GHB} \]

Proposition

\[ \partial_r \mathcal{M}_{GHB} = \partial_r^{\theta} \mathcal{M}_{GHB} \] for all \( 1 \leq r \leq n \)

\[ \partial^{n+1} \mathcal{M}_{GHB} \] consists of all Gieseker-Higgs bundles \((X^r, E, \phi)\).

\[ \partial^n \mathcal{M}_{GHB} \] is empty.

In particular, \( \partial^n \mathcal{M}_{GHB} \) is a smooth Poisson subvariety.
Stratification by Poisson ranks

There are two stratifications

- Stratification by successive singular loci:
  \[ \mathcal{M}_{GHB} \supset \partial \mathcal{M}_{GHB} \supset \cdots \supset \partial^n \mathcal{M}_{GHB} \]

- Stratification by Poisson ranks:
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- \( \partial^r \mathcal{M}_{GHB} = \partial^r_{\theta} \mathcal{M}_{GHB} \) for all \( 1 \leq r \leq n \)
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- \( \partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB} \) consists of all Gieseker-Higgs bundles \((X_r, \mathcal{E}, \phi)\).
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- \( \partial^{n+1} \mathcal{M}_{GHB} \) is empty. In particular, \( \partial^n \mathcal{M}_{GHB} \) is a smooth Poisson subvariety.
Casimir functions and the symplectic leaves of a stratum

Proposition

The Casimir functions of the stratum $\partial_r M_{GHB}$ are given by

$\{\mu_1, \ldots, \mu_r\}$,

where $\mu_i$ is $\mu_i : \partial_r M_{GHB} \to \mathbb{C}$ given by

$(X_r, E, \phi) \mapsto \text{Trace}(\phi|_{E^{i,2}})$

where $E^{i,2}$ is the non-trivial direct summand of $E|_{O^{i,1}} \oplus a_i P_1 \oplus O^{i,1}$.

Remember, $b_i$’s are always non-zero positive integer.
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where \(\mathcal{E}_{i,2}\) is the non-trivial direct summand of \(\mathcal{E}|_{R_i} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus a_i} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus b_i}\) i.e., \(\mathcal{E}_{i,2} := \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus b_i}\).
Casimir functions and the symplectic leaves of a stratum

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Thank you