Curves, degenerations, and Hirota varieties

Yelena Mandelshtam

University of California, Berkeley

March 10, 2022

Works featured

- KP Solitons from Tropical Limits with Daniele Agostini, Claudia Fevola, and Bernd Sturmfels (arxiv 2101.10392)
  - Code can be found at https://mathrepo.mis.mpg.de/KPSolitonsFromTropicalLimits
- The Hirota Variety of a Rational Nodal Curve with Claudia Fevola (arxiv 2203.00203)
  - Code can be found at https://mathrepo.mis.mpg.de/HirotaVarietyRationalNodalCurve

The Kadomtsev-Petviashvili equation

The KP equation is a PDE that describes the motion of water waves

$$\frac{\partial}{\partial x}\left(4p_t - 6pp_x - p_{xxx}\right) = 3p_{yy}$$

where $p = p(x, y, t)$

Taken in Nuevo Vallarta, Mexico by Mark J. Ablowitz

Connection to Algebraic Curves

We seek solutions of the form

$$p(x,y,t) \ = \ 2\frac{\partial^2}{\partial x^2}\log\tau(x,y,t)$$

where $\tau(x,y,t)$ satisfies the Hirota's differential equation

$$\tau\tau_{xxxx} - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 \ + \ 4\tau_x\tau_t - 4\tau\tau_{xt} + 3\tau\tau_{yy} - 3\tau_y^2 \ = \ 0$$

- One can construct τ-functions from an algebraic curve C of genus g

Connection to Algebraic Curves

Definition

The Riemann theta function is the complex analytic function

$$\theta = \theta(\mathbf{z} \mid B) \;=\; \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B \mathbf{c} + \mathbf{c}^T \mathbf{z}\right]$$

where $\mathbf{z} \in \mathbb{C}^g$ and B is a Riemann matrix, a $g \times g$ symmetric matrix normalized to have negative definite real part.

Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

$$p(x,y,t) \;=\; 2\frac{\partial^2}{\partial x^2}\log\theta(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t,B)$$

for certain vectors $\mathbf{u}=(u_1,\ldots,u_g), \mathbf{v}=(v_1,\ldots,v_g), \mathbf{w}=(w_1,\ldots,w_g)\in\mathbb{C}^g$.

Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

$$p(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\theta(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t, B)$$

for certain vectors $\mathbf{u} = (u_1, \dots, u_g), \mathbf{v} = (v_1, \dots, v_g), \mathbf{w} = (w_1, \dots, w_g) \in \mathbb{C}^g$.

Now, for a specific curve C of genus g with Riemann matrix B, we can look for τ of the form

$$\tau(x, y, t) = \theta(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t, B).$$

Connection to Algebraic Curves

Consider $(u_1,\ldots,u_g,v_1,\ldots,v_g,w_1,\ldots,w_g)$ as a point in $\mathbb{WP}^{3g-1}$ such that

$$\deg(u_i)=1\,,\quad \deg(v_i)=2\,,\quad \deg(w_i)=3\qquad \text{for } i=1,2,\ldots,g$$

Connection to Algebraic Curves

Consider $(u_1, \ldots, u_g, v_1, \ldots, v_g, w_1, \ldots, w_g)$ as a point in $\mathbb{WP}^{3g-1}$ such that

$$\deg(u_i) = 1, \quad \deg(v_i) = 2, \quad \deg(w_i) = 3 \qquad \text{for } i = 1, 2, \ldots, g$$

Definition (Agostini-Çelik-Sturmfels, 2020)

The Dubrovin threefold $\mathscr{D}_C$ comprises all points $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in $\mathbb{WP}^{3g-1}$ such that $\tau(x, y, t)$ satisfies the Hirota's differential equation.

Soliton Solutions

Fix $k < n$ and a vector of parameters $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and consider

$$\tau(x,y,t) \;=\; \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\Big[\, x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \,\Big]$$

Soliton Solutions

Fix $k < n$ and a vector of parameters $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and consider

$$\tau(x,y,t) \;=\; \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right]$$

Proposition (Sato)

The function τ is a solution to Hirota's differential equation if and only if the point $p = (p_I)_{I \in \binom{[n]}{k}}$ lies in the Grassmannian $\mathrm{Gr}(k,n)$.

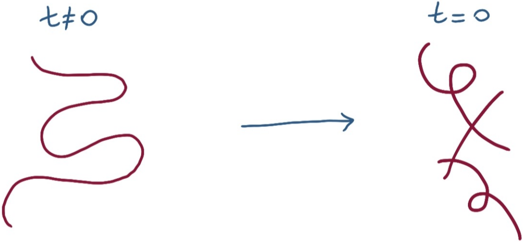
Definition

We define a (k,n)-soliton to be any function $\tau(x,y,t)$ where $\kappa \in \mathbb{R}^n$ and $p \in \mathrm{Gr}(k,n)$.

Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

A curve over $\mathbb{K}$ can be thought of as a family of curves depending on a parameter ϵ



$$\theta(\mathbf{z}) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B\mathbf{c} + \mathbf{c}^T\mathbf{z}\right] \quad \rightsquigarrow \quad \theta_{\mathscr{C}}(\mathbf{x}) = \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}]$$

Main Idea

We study solutions to the KP equation arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

For $\epsilon \to 0$

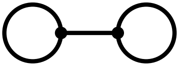
- The theta function becomes a finite sum of exponentials
- The function

  $$p(x,y,t) = 2\frac{\delta^2}{\delta x^2}\log\tau(x,y,t)$$

  becomes a soliton solution of the KP equation

Degenerations of Theta Functions

Let X be a smooth curve of genus g over $\mathbb{K}$. The metric graph is $\mathrm{Trop}(X)$.



The metric graph $\Gamma = (V, E)$ of a genus 2 hyperelliptic curve

$$H_1(\Gamma, \mathbb{Z}) = \langle \gamma_1, \dots, \gamma_g \rangle$$

is a free abelian group of rank g

- $e := |E|$
- $\Lambda := g \times e$ matrix whose i-th row records the coordinate of γ_i with respect to the standard basis of $\mathbb{Z}^e$
- $\Delta :=$ diagonal $e \times e$ matrix that records edge lengths of the metric graph.

Definition

The Riemann matrix of $\Gamma = (V, E)$ is

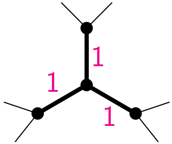
$$Q = \Lambda \Delta \Lambda^T$$

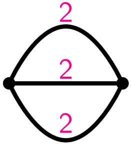
Example (g=2)

Consider $X := \{ y^2 = f(x) \}$ where

$$f(x) = (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon)$$

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2



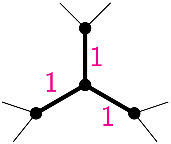


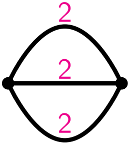
Example (g=2)

Consider $X := \{ y^2 = f(x) \}$ where

$$f(x) = (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon)$$

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2





From the graph we can read off the tropical Riemann matrix Q

$$Q = \Lambda\Delta\Lambda^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

Degenerations of Theta Functions

Consider

$$B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon)$$

Fix $\mathbf{a} \in \mathbb{R}^g$

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \;=\; \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q\mathbf{c} + \frac{1}{\epsilon}\mathbf{c}^T Q\mathbf{a}\right] \cdot \exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

Degenerations of Theta Functions

Consider

$$B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon)$$

Fix $\mathbf{a} \in \mathbb{R}^g$

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \mid B_\epsilon) \ = \ \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon} \mathbf{c}^T Q \mathbf{c} + \frac{1}{\epsilon} \mathbf{c}^T Q \mathbf{a} \right] \cdot \exp\left[\frac{1}{2} \mathbf{c}^T R(\epsilon) \mathbf{c} + \mathbf{c}^T \mathbf{z} \right]$$

Let $\epsilon \to 0$. This converges provided

$$\mathbf{c}^T Q \mathbf{c} - 2\mathbf{c}^T Q \mathbf{a} \geq 0 \text{ for all } \mathbf{c} \in \mathbb{Z}^g$$

or equivalently

$$\mathbf{a}^T Q \mathbf{a} \leq (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

Voronoi and Delaunay

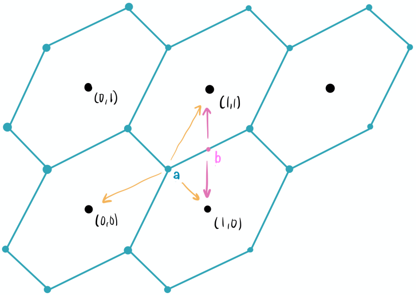
The condition

$$\mathbf{a}^T Q \mathbf{a} \leq (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

holds if and only if $\mathbf{a}$ belongs to the *Voronoi cell* for Q

For $\mathbf{a}$ in the Voronoi cell for Q,
consider the associated Delaunay set:

$$\mathscr{D}_{\mathbf{a},Q} = \left\{\mathbf{c} \in \mathbb{Z}^g : \mathbf{a}^T Q \mathbf{a} = (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c})\right\}$$



$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon) \;=\; \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q\mathbf{c}+\frac{1}{\epsilon}\mathbf{c}^T Q\mathbf{a}\right]\cdot\exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c}+\mathbf{c}^T\mathbf{z}\right]$$

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of the tropical Riemann matrix* Q*. For* $\epsilon \to 0$*, the series*

$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon)$$

converges to the theta function supported on the Delaunay set $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q}$*, namely*

$$\theta_{\mathscr{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}}\exp\left[\mathbf{c}^T\mathbf{z}\right], \quad \text{where} \quad a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$$

Example (g=2)

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of* Q *and let* $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q}$ *be the Delaunay set. As* $\epsilon \to 0$,

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \to \theta_{\mathscr{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp\left[\mathbf{c}^T \mathbf{z}\right],$$

where $a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$

Example (g=2)

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of* Q *and let* $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q}$ *be the Delaunay set. As* $\epsilon \to 0$,

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \to \theta_{\mathscr{C}}(\mathbf{x}) = \sum_{\mathbf{c} \in \mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T \mathbf{z}],$$

where $a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$

Example

For $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathscr{C} = \mathscr{D}_{\mathbf{a},Q} = \{(0,0),(1,0),(0,1),(1,1)\}$$

The associated theta function is

$$\theta_{\mathscr{C}} = a_{00} + a_{10}\exp[z_1] + a_{01}\exp[z_2] + a_{11}\exp[z_1 + z_2]$$

The Hirota Variety

Let $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\} \subset \mathbb{Z}^g$

$$\theta_{\mathscr{C}}(\mathbf{z}) \;=\; a_1 \exp[\mathbf{c}_1^T \mathbf{z}] + a_2 \exp[\mathbf{c}_2^T \mathbf{z}] + \cdots + a_m \exp[\mathbf{c}_m^T \mathbf{z}]$$

Consider

$$\tau(x, y, t) = \theta_{\mathscr{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\Big(\sum_{j=1}^{g} c_{ij} u_j\Big)\, x + \Big(\sum_{j=1}^{g} c_{ij} v_j\Big)\, y + \Big(\sum_{j=1}^{g} c_{ij} w_j\Big)\, t\Big]$$

The Hirota Variety

Let $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\} \subset \mathbb{Z}^g$

$$\theta_{\mathscr{C}}(\mathbf{z}) \;=\; a_1 \exp[\,\mathbf{c}_1^T \mathbf{z}] + a_2 \exp[\,\mathbf{c}_2^T \mathbf{z}] + \cdots + a_m \exp[\,\mathbf{c}_m^T \mathbf{z}]$$

Consider

$$\tau(x,y,t) = \theta_{\mathscr{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\Big(\sum_{j=1}^{g} c_{ij} u_j\Big)\, x + \Big(\sum_{j=1}^{g} c_{ij} v_j\Big)\, y + \Big(\sum_{j=1}^{g} c_{ij} w_j\Big)\, t\Big]$$

Definition

The Hirota variety $\mathscr{H}_{\mathscr{C}}$ consists of all points $\big(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})\big)$ in $(\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}$ such that $\tau(x,y,t)$ satisfies Hirota's differential equation

Polynomials defining the Hirota Variety

Remark

Hirota's differential equation can be written via the Hirota differential operators as

$$P(D_x, D_y, D_t)\tau \bullet \tau = 0$$

where $P(x,y,t) = x^4 - 4xt + 3y^2$ gives the soliton dispersion relation

Polynomials defining the Hirota Variety

Remark

Hirota's differential equation can be written via the Hirota differential operators as

$$P(D_x, D_y, D_t)\tau \bullet \tau = 0$$

where $P(x, y, t) = x^4 - 4xt + 3y^2$ gives the soliton dispersion relation

For any two indices k, ℓ in $\{1, \ldots, m\}$

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \ := \ P\big((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u}, \, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v}, \, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w} \big)$$

defines a hypersurface in $\mathbb{WP}^{3g-1}$

Polynomials defining the Hirota Variety

The polynomials defining $\mathscr{H}_{\mathscr{C}}$ correspond to points in

$$\mathscr{C}^{[2]} = \{\mathbf{c}_k + \mathbf{c}_\ell \,:\, 1 \leq k < \ell \leq m\} \subset \mathbb{Z}^g$$

Definition

A point $\mathbf{d}$ in $\mathscr{C}^{[2]}$ is uniquely attained if there exists precisely one index pair (k, ℓ) such that $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{d}$. In that case, (k, ℓ) is a unique pair.

Polynomials defining the Hirota Variety

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

The Hirota variety $\mathcal{H}_{\mathcal{C}}$ is defined by the quartics

$$P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w}) \;:=\; P\big(\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{u},\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{v},\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{w}\big)$$

for all unique pairs (k,ℓ) and by the polynomials

$$\sum_{\substack{1\le k<\ell\le m\\ \mathbf{c}_k+\mathbf{c}_\ell=\mathbf{d}}} P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})\,a_k a_\ell$$

for all non-uniquely attained points $\mathbf{d}\in\mathcal{C}^{[2]}$.
If all points in $\mathcal{C}^{[2]}$ are uniquely attained then $\mathcal{H}_{\mathcal{C}}$ is defined by the $\binom{m}{2}$ quartics $P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})$.

Example (The Square)

Let $g = 2$ and $\mathscr{C} = \{0,1\}^2$

$$\mathscr{C}^{[2]} = \{(0,1),(1,0),(1,1),(1,2),(2,1)\}$$

There are four unique pairs (k,ℓ)

$$P_{13} = P_{24} = u_1^4 - 4u_1w_1 + 3v_1^2$$

$$P_{12} = P_{34} = u_2^4 - 4u_2w_2 + 3v_2^2$$

The point $\mathbf{d} = (1,1)$ is not uniquely attained in $\mathscr{C}^{[2]}$

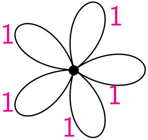
$$P(u_1+u_2, v_1+v_2, w_1+w_2)\, a_{00}a_{11} \;+\; P(u_1-u_2, v_1-v_2, w_1-w_2)\, a_{01}a_{10}$$

For any point in $\mathscr{H}_{\mathscr{C}} \subset (\mathbb{K}^*)^4 \times \mathbb{WP}^5$, we can write $\tau(x,y,t)$ as a $(2,4)$-soliton

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

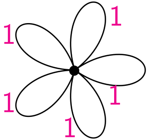
The g-Cube

- Irreducible rational nodal curve with g nodes $\longrightarrow$ metric graph is a vertex with g loops $\longrightarrow$ Tropical Riemann matrix is I_g.



The g-Cube

- Irreducible rational nodal curve with g nodes $\longrightarrow$ metric graph is a vertex with g loops $\longrightarrow$ Tropical Riemann matrix is I_g.

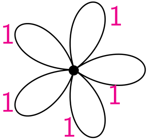


- fix the point $\mathbf{a} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{R}^g \longrightarrow$ corresponding Delaunay set $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q} = \{0,1\}^g$

$$\begin{aligned} \theta_{\mathscr{C}} \;=\; & a_{00\ldots0} + a_{10\ldots0}\exp[z_1] + a_{010\ldots0}\exp[z_2] + \cdots + a_{0\ldots01}\exp[z_g] + \\ & + a_{110\ldots0}\exp[z_1+z_2] + a_{1010\ldots0}\exp[z_1+z_3] + a_{0\ldots011}\exp[z_{g-1}+z_g] + \\ & + \cdots + a_{11\ldots1}\exp[z_1+z_2+\cdots+z_g]. \end{aligned}$$

The g-Cube

- Irreducible rational nodal curve with g nodes $\longrightarrow$ metric graph is a vertex with g loops $\longrightarrow$ Tropical Riemann matrix is I_g.



- fix the point $\mathbf{a} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^g \longrightarrow$ corresponding Delaunay set $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q} = \{0,1\}^g$

$$\begin{aligned}\theta_{\mathscr{C}} \ = \ & a_{00\dots0} + a_{10\dots0}\exp[z_1] + a_{010\dots0}\exp[z_2] + \cdots + a_{0\dots01}\exp[z_g] + \\ & + a_{110\dots0}\exp[z_1+z_2] + a_{1010\dots0}\exp[z_1+z_3] + a_{0\dots011}\exp[z_{g-1}+z_g] + \\ & + \cdots + a_{11\dots1}\exp[z_1+z_2+\cdots+z_g].\end{aligned}$$

For the equations cutting out the Hirota Variety, we are interested in the combinatorics of $\mathscr{C}^{[2]}$ (coming soon!)

The main component

- The Hirota variety $\mathcal{H}_{\mathcal{C}}$ lives in the space $(\mathbb{C}^*)^{2^g} \times \mathbb{WP}^{3g-1}$ with coordinate ring $\mathbb{C}[\mathbf{a}^{\pm}, \mathbf{u}, \mathbf{v}, \mathbf{w}]$, where $\deg(u_i) = 1, \deg(v_i) = 2$, and $\deg(w_i) = 3$, for $i = 1, 2, \ldots, g$.
- We investigate the subvariety denoted by $\mathcal{H}_{\mathcal{C}}^I = cl(\{(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})) \in \mathcal{H}_{\mathcal{C}} \, : \, \mathbf{u} \neq \mathbf{0}\})$
- $\mathcal{H}_{\mathcal{C}}^I$ contains an irreducible subvariety of $\mathcal{H}_{\mathcal{C}}$ which we call the main component, denoted by $\mathcal{H}_{\mathcal{C}}^M$.

The parameterization map

Consider the map

$$\begin{aligned} \phi : \mathbb{C}^{3g+1} & \dashrightarrow (\mathbb{C}^*)^{2^g} \times \mathbb{WP}^{3g-1} \\ (\lambda_0, \lambda_1, \ldots, \lambda_g, \kappa_1, \kappa_2, \ldots, \kappa_{2g}) & \longrightarrow (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \ldots, a_{\mathbf{c}_{2^g}}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \end{aligned} \tag{1}$$

where the coordinates $\mathbf{a} = (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \ldots, a_{\mathbf{c}_{2^g}})$ are indexed by the points in $\mathscr{C} = \{0,1\}^g$.

The parameterization map

Consider the map

$$\begin{aligned}\phi : \mathbb{C}^{3g+1} &\dashrightarrow (\mathbb{C}^*)^{2^g} \times \mathbb{WP}^{3g-1} \\ (\lambda_0, \lambda_1, \ldots, \lambda_g, \kappa_1, \kappa_2, \ldots, \kappa_{2g}) &\longrightarrow (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \ldots, a_{\mathbf{c}_{2^g}}, \mathbf{u}, \mathbf{v}, \mathbf{w})\end{aligned} \tag{1}$$

where the coordinates $\mathbf{a} = (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \ldots, a_{\mathbf{c}_{2^g}})$ are indexed by the points in $\mathscr{C} = \{0,1\}^g$. The image of ϕ is defined as

$$\begin{gathered} u_i = \kappa_{2i-1} - \kappa_{2i}, \quad v_i = \kappa_{2i-1}^2 - \kappa_{2i}^2, \quad w_i = \kappa_{2i-1}^3 - \kappa_{2i}^3, \qquad \text{for all } i = 1, \ldots, g, \\ a_{\mathbf{c}} = \lambda_0 \prod_{\substack{i,j \in I \\ i<j}} (\kappa_i - \kappa_j) \prod_{i : c_i = 1} \lambda_i \qquad \text{where } I = \{2i : c_i = 0\} \cup \{2i-1 : c_i = 1\}, \\ \text{for all } \mathbf{c} \in \mathscr{C}. \end{gathered} \tag{2}$$

The parameterization map

Consider the map

$$\begin{aligned} \phi : \mathbb{C}^{3g+1} &\dashrightarrow (\mathbb{C}^*)^{2^g} \times \mathbb{WP}^{3g-1} \\ (\lambda_0, \lambda_1, \dots, \lambda_g, \kappa_1, \kappa_2, \dots, \kappa_{2g}) &\longrightarrow (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \dots, a_{\mathbf{c}_{2^g}}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \end{aligned} \tag{1}$$

where the coordinates $\mathbf{a} = (a_{\mathbf{c}_1}, a_{\mathbf{c}_2}, \dots, a_{\mathbf{c}_{2^g}})$ are indexed by the points in $\mathscr{C} = \{0,1\}^g$. The image of ϕ is defined as

$$\begin{gathered} u_i = \kappa_{2i-1} - \kappa_{2i}, \quad v_i = \kappa_{2i-1}^2 - \kappa_{2i}^2, \quad w_i = \kappa_{2i-1}^3 - \kappa_{2i}^3, \qquad \text{for all } i = 1, \dots, g, \\ a_{\mathbf{c}} = \lambda_0 \prod_{\substack{i,j \in I \\ i<j}} (\kappa_i - \kappa_j) \prod_{i : c_i = 1} \lambda_i \qquad \text{where } I = \{2i : c_i = 0\} \cup \{2i-1 : c_i = 1\}, \\ \text{for all } \mathbf{c} \in \mathscr{C}. \end{gathered} \tag{2}$$

$\mathscr{H}_{\mathscr{C}}^M = \overline{(im(\phi))}.$

Motivation for the map (g=3)

- Let X be a rational nodal curve, where n_1, n_2, n_3 are its nodes. The normalization of $\nu : \tilde{X} \to X$ that separates the 3 nodes of X is given by a projective line.
- We can consider $\kappa_1, \kappa_2, \ldots, \kappa_6$ to be points on $\mathbb{P}^1$ and set $\nu^{-1}(n_i) := \{\kappa_{2i-1}, \kappa_{2i}\}$.
- a basis of canonical differentials is

$$\omega_1 = \frac{1}{y}\left(\frac{1}{1-\kappa_2 y} - \frac{1}{1-\kappa_1 y}\right) dy, \qquad \omega_2 = \frac{1}{y}\left(\frac{1}{1-\kappa_4 y} - \frac{1}{1-\kappa_3 y}\right) dy,$$

$$\omega_3 = \frac{1}{y}\left(\frac{1}{1-\kappa_6 y} - \frac{1}{1-\kappa_5 y}\right) dy.$$

when fixing $y = 1/x$ as local coordinate.

Motivation for the map (g=3)

- The canonical differentials define a map $\alpha' : (\mathbb{P}^1)^2 \dashrightarrow \mathbb{C}^3$ such that

$$(y_1, y_2) \longmapsto \left(\sum_{i=1}^{g-1} \int_0^{y_i} \omega_j \right), \quad \text{where} \quad \int_0^{y_i} \omega_j = \log\left(\frac{1 - \kappa_{2j-1} y_i}{1 - \kappa_{2j} y_i} \right).$$

Exponentiation allows to map directly in the Jacobian through the map $\mathbb{C}^3 \to (\mathbb{C}^*)^3$ given by $\big((z_1, z_2, z_3) \mapsto (\exp(z_1), \exp(z_2), \exp(z_3))\big)$. The composition gives the Abel map $\alpha : (\mathbb{P}^1)^2 \dashrightarrow (\mathbb{C}^*)^3$ given by

$$(y_1, y_2) \mapsto \left(\left(\frac{1-\kappa_1 y_1}{1-\kappa_2 y_1} \right) \cdot \left(\frac{1-\kappa_1 y_2}{1-\kappa_2 y_2} \right), \left(\frac{1-\kappa_3 y_1}{1-\kappa_4 y_1} \right) \cdot \left(\frac{1-\kappa_3 y_2}{1-\kappa_4 y_2} \right), \left(\frac{1-\kappa_5 y_1}{1-\kappa_6 y_1} \right) \cdot \left(\frac{1-\kappa_5 y_2}{1-\kappa_6 y_2} \right) \right).$$

Motivation for the map (g=3)

- The canonical differentials define a map $\alpha' : (\mathbb{P}^1)^2 \dashrightarrow \mathbb{C}^3$ such that

$$(y_1, y_2) \longmapsto \left(\sum_{i=1}^{g-1} \int_0^{y_i} \omega_j \right), \quad \text{where} \quad \int_0^{y_i} \omega_j = \log\left(\frac{1 - \kappa_{2j-1} y_i}{1 - \kappa_{2j} y_i} \right).$$

  Exponentiation allows to map directly in the Jacobian through the map $\mathbb{C}^3 \to (\mathbb{C}^*)^3$ given by $\left((z_1, z_2, z_3) \mapsto (\exp(z_1), \exp(z_2), \exp(z_3)) \right)$. The composition gives the Abel map $\alpha : (\mathbb{P}^1)^2 \dashrightarrow (\mathbb{C}^*)^3$ given by

$$(y_1, y_2) \mapsto \left(\left(\frac{1-\kappa_1 y_1}{1-\kappa_2 y_1} \right) \cdot \left(\frac{1-\kappa_1 y_2}{1-\kappa_2 y_2} \right), \left(\frac{1-\kappa_3 y_1}{1-\kappa_4 y_1} \right) \cdot \left(\frac{1-\kappa_3 y_2}{1-\kappa_4 y_2} \right), \left(\frac{1-\kappa_5 y_1}{1-\kappa_6 y_1} \right) \cdot \left(\frac{1-\kappa_5 y_2}{1-\kappa_6 y_2} \right) \right).$$

- The theta divisor of X is the image of the Abel map α up to translation.

Motivation for the map (g=3)

- The canonical differentials define a map $\alpha' : (\mathbb{P}^1)^2 \dashrightarrow \mathbb{C}^3$ such that

$$(y_1, y_2) \longmapsto \left(\sum_{i=1}^{g-1} \int_0^{y_i} \omega_j \right), \quad \text{where} \quad \int_0^{y_i} \omega_j = \log\left(\frac{1-\kappa_{2j-1} y_i}{1-\kappa_{2j} y_i} \right).$$

Exponentiation allows to map directly in the Jacobian through the map $\mathbb{C}^3 \to (\mathbb{C}^*)^3$ given by $\left((z_1, z_2, z_3) \mapsto (\exp(z_1), \exp(z_2), \exp(z_3))\right)$. The composition gives the Abel map $\alpha : (\mathbb{P}^1)^2 \dashrightarrow (\mathbb{C}^*)^3$ given by

$$(y_1, y_2) \mapsto \left(\left(\frac{1-\kappa_1 y_1}{1-\kappa_2 y_1} \right) \cdot \left(\frac{1-\kappa_1 y_2}{1-\kappa_2 y_2} \right), \left(\frac{1-\kappa_3 y_1}{1-\kappa_4 y_1} \right) \cdot \left(\frac{1-\kappa_3 y_2}{1-\kappa_4 y_2} \right), \left(\frac{1-\kappa_5 y_1}{1-\kappa_6 y_1} \right) \cdot \left(\frac{1-\kappa_5 y_2}{1-\kappa_6 y_2} \right) \right).$$

- The theta divisor of X is the image of the Abel map α up to translation.
- One can find the implicitizing equation cutting out the image of this map in `Macaulay2`. The resulting equation exactly gives the familiar theta function for $g = 3$, with the $a_{\mathbf{c}}$ parametrized by the κ_i's as in ϕ.

Motivation for the map (g=3)

- we consider the theta functions

$$\theta(\mathbf{z}) = a_{000} + a_{100}\exp(z_1) + a_{010}\exp(z_2) + a_{001}\exp(z_3) + a_{110}\exp(z_1+z_2) \\ + a_{101}\exp(z_1+z_3) + a_{011}\exp(z_2+z_3) + a_{111}\exp(z_1+z_2+z_3).$$

  and

$$\theta(\mathbf{z}+\mathbf{h}) = a_{000} + a_{100}\exp(h_1)\exp(z_1) + a_{010}\exp(h_2)\exp(z_2) + a_{001}\exp(h_3)\exp(z_3) \\ + a_{110}\exp(h_1+h_2)\exp(z_1+z_2) + a_{101}\exp(h_1+h_3)\exp(z_1+z_3) \\ + a_{011}\exp(h_2+h_3)\exp(z_2+z_3) + a_{111}\exp(h_1+h_2+h_3)\exp(z_1+z_2+z_3).$$

- Letting $\lambda_i := \exp(h_i)$, we have

$$\theta(\mathbf{z}+\mathbf{h}) = a_{000} + \lambda_1 a_{100}\exp(z_1) + \lambda_2 a_{010}\exp(z_2) + \lambda_3 a_{001}\exp(z_3) + \lambda_1\lambda_2 a_{110}\exp(z_1+z_2) \\ + \lambda_1\lambda_3 a_{101}\exp(z_1+z_3) + \lambda_2\lambda_3 a_{011}\exp(z_2+z_3) + \lambda_1\lambda_2\lambda_3 a_{111}\exp(z_1+z_2+z_3).$$

Soliton Matrix

With the parameterization given by ϕ, we can express

$$\tau(x,y,t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\Big[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \Big],$$

with the $p_I = \lambda_0 \prod_{i:c_i=1} \lambda_i$ for each I obtained from the points $\mathbf{c} \in \mathscr{C}$ by taking the set $I = \{2i : c_i = 0\} \cup \{2i-1 : c_i = 1\}$

Soliton Matrix

With the parameterization given by ϕ, we can express

$$\tau(x,y,t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right],$$

with the $p_I = \lambda_0 \prod_{i:c_i=1} \lambda_i$ for each I obtained from the points $\mathbf{c} \in \mathscr{C}$ by taking the set $I = \{2i : c_i = 0\} \cup \{2i-1 : c_i = 1\}$

The corresponding soliton matrix is the $g \times 2g$ matrix

$$\begin{pmatrix} \lambda_0\lambda_1 & \lambda_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \lambda_g & 1 \end{pmatrix}$$

The main component is an irreducible subvariety

Theorem (Fevola- M., 2022)

Consider the map ϕ as before. This is a birational map onto its image, which is an irreducible subvariety of $\mathscr{H}_{\mathscr{C}}$ and has dimension $3g$.

Proof Idea.

- Any point in the image of ϕ is a point in the Hirota variety $\mathscr{H}_{\mathscr{C}}$ since it can be expressed as a $(g, 2g)$-soliton
- The map ϕ is invertible outside the closed set where the u_i's vanish:

$$\kappa_{2i-1} = \frac{u_i^2 + v_i}{2u_i} \qquad \text{and} \qquad \kappa_{2i} = \frac{v_i - u_i^2}{2u_i}$$

- We can conclude that the map ϕ is birational. This implies that the closure of the image is irreducible and of dimension $3g$.

□

The Schottky problem

- Let $\mathscr{M}_g$ be the moduli space of curves of genus g and $\mathscr{A}_g$ the moduli space of abelian varieties of dimension g.
- Let $J : \mathscr{M}_g \to \mathscr{A}_g$ be the Torelli map, taking curves to Jacobians

The Schottky problem

- Let $\mathscr{M}_g$ be the moduli space of curves of genus g and $\mathscr{A}_g$ the moduli space of abelian varieties of dimension g.
- Let $J : \mathscr{M}_g \to \mathscr{A}_g$ be the Torelli map, taking curves to Jacobians
- The Schottky problem is to find the defining equations for the locus of Jacobians, defined as the closure of $J(\mathscr{M}_g)$ in $\mathscr{A}_g$.
- trivial in genus 3, solved in genus 4, hard for genus ≥ 5

The Schottky problem

- Let $\mathscr{M}_g$ be the moduli space of curves of genus g and $\mathscr{A}_g$ the moduli space of abelian varieties of dimension g.
- Let $J: \mathscr{M}_g \to \mathscr{A}_g$ be the Torelli map, taking curves to Jacobians
- The Schottky problem is to find the defining equations for the locus of Jacobians, defined as the closure of $J(\mathscr{M}_g)$ in $\mathscr{A}_g$.
- trivial in genus 3, solved in genus 4, hard for genus ≥ 5
- The weak Schottky problem is to find an ideal whose zero locus contains the locus of Jacobians as an irreducible component.

The Schottky problem

- Let $\mathscr{M}_g$ be the moduli space of curves of genus g and $\mathscr{A}_g$ the moduli space of abelian varieties of dimension g.
- Let $J : \mathscr{M}_g \to \mathscr{A}_g$ be the Torelli map, taking curves to Jacobians
- The Schottky problem is to find the defining equations for the locus of Jacobians, defined as the closure of $J(\mathscr{M}_g)$ in $\mathscr{A}_g$.
- trivial in genus 3, solved in genus 4, hard for genus ≥ 5
- The weak Schottky problem is to find an ideal whose zero locus contains the locus of Jacobians as an irreducible component.
- This is related to finding solutions to the KP equation because a theta function satisfies the KP equation when the corresponding abelian variety is a Jacobian of a curve

The main component is an irreducible component?

Showing that $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component is equivalent to solving the Weak Schottky Problem for rational nodal curves, which can be solved by showing that the map ϕ is dominant into $\mathscr{H}_{\mathscr{C}}^{M}$.

The main component is an irreducible component?

Showing that $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component is equivalent to solving the Weak Schottky Problem for rational nodal curves, which can be solved by showing that the map ϕ is dominant into $\mathscr{H}_{\mathscr{C}}^{M}$.

Theorem (Fevola-M., 2022)

For genus $g \leq 7$, the subvariety $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component of the Hirota variety $\mathscr{H}_{\mathscr{C}}$.

The main component is an irreducible component?

Showing that $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component is equivalent to solving the Weak Schottky Problem for rational nodal curves, which can be solved by showing that the map ϕ is dominant into $\mathscr{H}_{\mathscr{C}}^{M}$.

Theorem (Fevola-M., 2022)

For genus $g \leq 7$, the subvariety $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component of the Hirota variety $\mathscr{H}_{\mathscr{C}}$.

Conjecture (Weak Schottky Problem)

$\mathscr{H}_{\mathscr{C}}^{M}$ is a $3g$-dimensional irreducible component of $\mathscr{H}_{\mathscr{C}}$ with a parametric representation given as in the map ϕ.

The main component is an irreducible component?

Showing that $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component is equivalent to solving the Weak Schottky Problem for rational nodal curves, which can be solved by showing that the map ϕ is dominant into $\mathscr{H}_{\mathscr{C}}^{M}$.

Theorem (Fevola-M., 2022)

For genus $g \leq 7$, the subvariety $\mathscr{H}_{\mathscr{C}}^{M}$ is an irreducible component of the Hirota variety $\mathscr{H}_{\mathscr{C}}$.

Conjecture (Weak Schottky Problem)

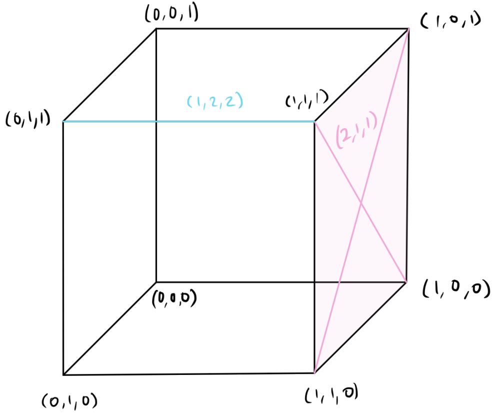
$\mathscr{H}_{\mathscr{C}}^{M}$ is a $3g$-dimensional irreducible component of $\mathscr{H}_{\mathscr{C}}$ with a parametric representation given as in the map ϕ.

Conjecture (Strong Schottky Problem)

$\mathscr{H}_{\mathscr{C}}^{M} = \mathscr{H}_{\mathscr{C}}^{I}$

The g-Cube: $\mathscr{C}^{[2]}$

One can observe that $\mathscr{C}^{[2]}$ is the set of lattice points in $2\text{conv}(\mathscr{C})$ that are not vertices, so there are $3^g - 2^g$ points.
Each d–dimensional face of conv$\mathscr{C}$ corresponds to a point that is attained 2^{d-1} times.



Combinatorics of $\mathscr{H}_{\mathscr{C}}$

Proposition (Fevola-M., 2022)

A point $\mathbf{c} = (c_1, \ldots, c_g)$ *in the set* $\mathscr{C}^{[2]}$ *is attained exactly* 2^{d-1} *times, where* $d = |\{i : c_i = 1\}|$.

Combinatorics of $\mathscr{H}_{\mathscr{C}}$

Proposition (Fevola-M., 2022)

A point $\mathbf{c} = (c_1, \ldots, c_g)$ *in the set* $\mathscr{C}^{[2]}$ *is attained exactly* 2^{d-1} *times, where* $d = |\{i : c_i = 1\}|$.

Lemma (Fevola-M., 2022)

The set $\mathscr{C}^{[2]}$ *contains* $g2^{g-1}$ *points which are uniquely attained. These contribute* g *quartics of the form* $u_j^4 - 4u_j w_j + 3v_j^2$, *for* $j = 1, \ldots, g$ *as generators of* $\mathscr{H}_{\mathscr{C}}$.

Proof.

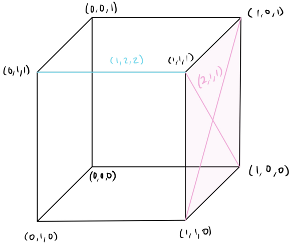
A point $\mathbf{c} \in \mathbb{C}$ is uniquely attained any time that the points $\mathbf{c}_k, \mathbf{c}_\ell \in \mathscr{C}$ such that $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{c}$ lie on same edge of the cube. Such points contribute the quartics

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \;:=\; P\big((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w}\big).$$

The difference $\mathbf{c}_k - \mathbf{c}_\ell$ corresponds to the direction of the edge. Hence out of these $g2^{g-1}$ quartics of this form, g of them are distinct. □

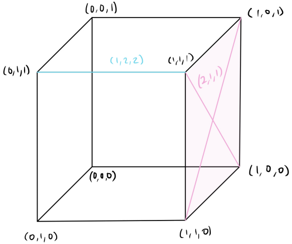
Additional Combinatorics in $\mathscr{H}_{\mathscr{C}}^{M}$

A face of the g-cube is defined by fixing $g-d$ indices of the corresponding points. Let the non-fixed indices be given by the set I. We call this set the direction of the face.



Additional Combinatorics in $\mathscr{H}_{\mathscr{C}}^{M}$

A face of the g-cube is defined by fixing $g-d$ indices of the corresponding points. Let the non-fixed indices be given by the set I. We call this set the direction of the face.



Theorem (Fevola-M., 2022)

There are $\binom{g}{d}$ face directions for each dimension d, and all faces with the same direction contribute the same quartic, up to a multiple, to the ideal defining $\mathscr{H}_{\mathscr{C}}^{M}$.

Thank you!