NEWTON POLYHEDRA AND COMPONENTS OF COMPLETE INTERSECTIONS

Seminar in Real and Complex Geometry

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The amazing Bernstein-Kouchnirenko theorem inspired much activity that eventually lead to the creation of the Newton polyhedra theory, of a birationally invariant version of the intersection theory for divisors [3] and of the theory of Newton-Okounkov bodies [4,5].

Newton polyhedra

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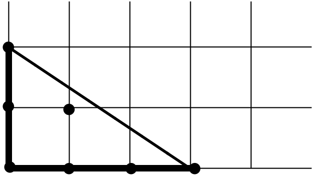
Example. Let P be $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$. Then $\Delta(P)$ is



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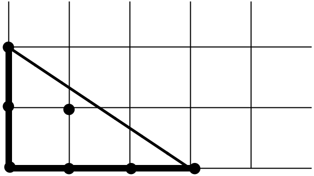

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Discrete invariants of $X \subset (\mathbb{C}^*)^n$ defined by a generic system of equations $P_1 = \cdots = P_k = 0$ with fixed support $s(P_i)$ depend only on Newton polyhedra $\Delta(P_1), \ldots, \Delta(P_k)$ of $P_1, \ldots, P_k$.

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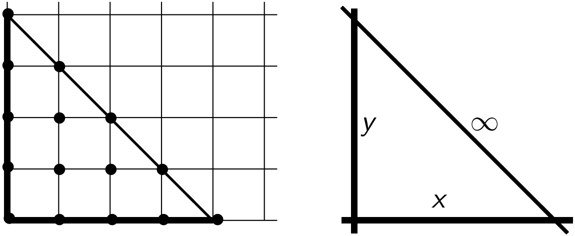
Toy geometric application. The invariants 1)-3) are related:
$\chi(\bar{X}) = \chi(X) + \#A(X) = 2 - 2g(X)$.
It implies the **Pick formula** for an integral polygon Δ:
$V(\Delta) = \#((\Delta \setminus \partial\Delta) \bigcap \mathbb{Z}^2) + 1/2\#\partial(\Delta \bigcap \mathbb{Z}^2) - 1.$

Newton polyhedra and Toric varieties



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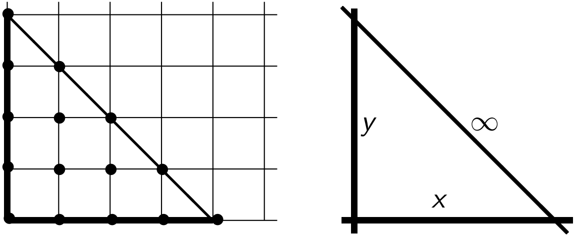

Toric variety is a normal connected n-dimensional algebraic variety M on which an $(\mathbb{C}^*)^n$ acts algebraically and has one orbit isomorphic to $(\mathbb{C}^*)^n$. Under the action of $(\mathbb{C}^*)^n$, M is broken up into a finite number of orbits isomorphic to tori of different dimensions. To every Newton polyhedron Δ we can associate a compact projective toric variety M_Δ in such a way that every k-dimensional face $\Gamma \subset \Delta$ corresponds to a complex k-dimensional orbit $O_\Gamma \subset M_\Delta$. If $\Gamma_1 \subset \Gamma_2$ then $O_{\Gamma_1} \subset \bar{O}_{\Gamma_2}$.

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Definition 1 For fixed k-tuple of convex bodies $\Delta_1, \dots, \Delta_k$ in $\mathbb{R}^n$ for any nonempty subset $J \subset \{1, \dots, k\}$ we define the *defect* $d(J)$ of J to be the number $d(J) = \dim(\Delta_J) - |J|$, where $\Delta_J = \sum_{i \in J} \Delta_i$ and $|J|$ is the number of elements in J.

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Theorem (David Bernstein, 1975) The algebraic variety $X \subset (\mathbb{C}^*)^n$ defined by a generic system of equations $P_1 = \dots = P_k = 0$ with fixed support $s(P_i)$ is nonempty if and only if the k-tuple of Newton polyhedra $\Delta_1, \dots, \Delta_k$ of $P_1, \dots, P_k$ is independent (in the sense of Definition 2).

How many solutions in $(\mathbb{C}^*)^n$ has a system of equations $P_1 = \cdots = P_n = 0$ where $P_1, \ldots, P_n$ are generic Laurent polynomials with the fixed supports $A_1, \ldots, A_n \subset \mathbb{Z}^n$?

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This result is also known as Berstein-Koushnirenko theorem and as BKK theorem. A lot of proofs of this fantastic fact are known now.

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Newton polyhedra of Laurent polynomials on the torus $(\mathbb{C}^*)^n$ belong to the space $\mathbb{R}^n$ of characters, so one can talk about the integral volume of Newton polyhedra. Exactly this volume we mean in the statement of the Bernstein-Koushnirenko theorem.

Mixed volume is a unique function $V(\Delta_1, \ldots, \Delta_n)$ on n-tuples of convex bodies in $\Delta_i \subset \mathbb{R}^n$, such that:

1. $V(\Delta, \ldots, \Delta)$ is the volume of Δ;
2. V is symmetric;
3. V is multi-linear; for example,
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4. V is nonnegative, i.e. $0 \leq V(\Delta_1, \ldots, \Delta_n)$;
5. $\Delta_1' \subseteq \Delta_1, \ldots, \Delta_n' \subseteq \Delta_n \Rightarrow V(\Delta_1', \ldots, \Delta_n') \leq V(\Delta_1, \ldots, \Delta_n)$;
6. The following **Alexandrov–Fenchel inequality** holds: $V^2(\Delta_1, \Delta_2, \ldots, \Delta_n) \geq$ $V(\Delta_1, \Delta_1, \ldots, \Delta_n) V(\Delta_2, \Delta_2, \ldots, \Delta_n)$;
7. in particular (for $n = 2$, the unite ball Δ_1 and for $\Delta = \Delta_2$) the **isoperimetric inequality** $(\frac{1}{2}$ length of $\partial\Delta)^2 \geq \pi V(\Delta)$ holds.

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If in David Bernstein theorem the number of equations k is equal to the dimension n of ambient space then this deduction is almost straightforward. The case $k < n$ can be reduced to the case $k = n$.

Sufficient condition for irreducibility

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Theorem 1 ([7]) *If for the k-tuple of Newton polyhedra $\Delta_1, \ldots, \Delta_k$ the defect $d(J)$ of each nonempty subset $J \subset \{1, \ldots, k\}$ is positive then the algebraic variety X defined by a generic system (1) is irreducible.*

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Very similar arguments allow to compute the arithmetic genus of X. For zero dimensional varieties X it implies the Bernstein-Koushnirenko theorem (see [1,2]).

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The theorem 2 could be easily reduced to the theorem 1 (see [7]).

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Corollary 1 The variety X is irreducible only in the following cases:

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The answer on this question is positive. Such classification is described in [6].

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With the rational p-dimensional space $L_J \subset \mathbb{R}^n$ one can associate the subtorus T^m of dimension $m = n - p$ in the torus $(\mathbb{C}^*)^n$, defined by the following condition: $g \in T^m$ if and only if $\chi(g) = 1$ for each character χ whose power belongs to the lattice $\mathbb{Z}^n \bigcap L_J$. The embedding $\pi : T^m \to (\mathbb{C}^*)^n$ induces the linear map $\pi^* : \mathbb{R}^n \to \mathbb{R}^m$ from the space $\mathbb{R}^n$ of characters on $(\mathbb{C}^*)^n$ to the space $\mathbb{R}^m$ of characters on T^m.

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Theorem 3([7]) *In the assumptions of the theorem 2 each irreducible component of the variety X is isomorphic to a variety $Y \subset T^m$ defined by a system $Q_{q_1} = \cdots = Q_{q_m} = 0$ where $\{q_1, \ldots, q_m\} = \{1, \ldots, k\} \setminus J$ and $Q_{q_1}, \ldots, Q_{q_m}$ is a generic m-tuple of Laurent polynomials with Newton polyhedra $\pi^*(\Delta_{q_1}), \ldots, \pi^*(\Delta_{q_m})$.*

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Our proof of the theorem 3 is based on a simple explicit construction (see [7]).

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The theorem 3 allows to compute all natural discrete invariants of each component of X (each such invariant takes the same value at all components of X). Indeed, according to the Newton polyhedra theory all natural discrete invariants of Y can be computed in terms of Newton polyhedra $\pi^*(\Delta_{q_1}), \dots, \pi^*(\Delta_{q_m})$.

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