Embedded graphs on Riemann surfaces and beyond

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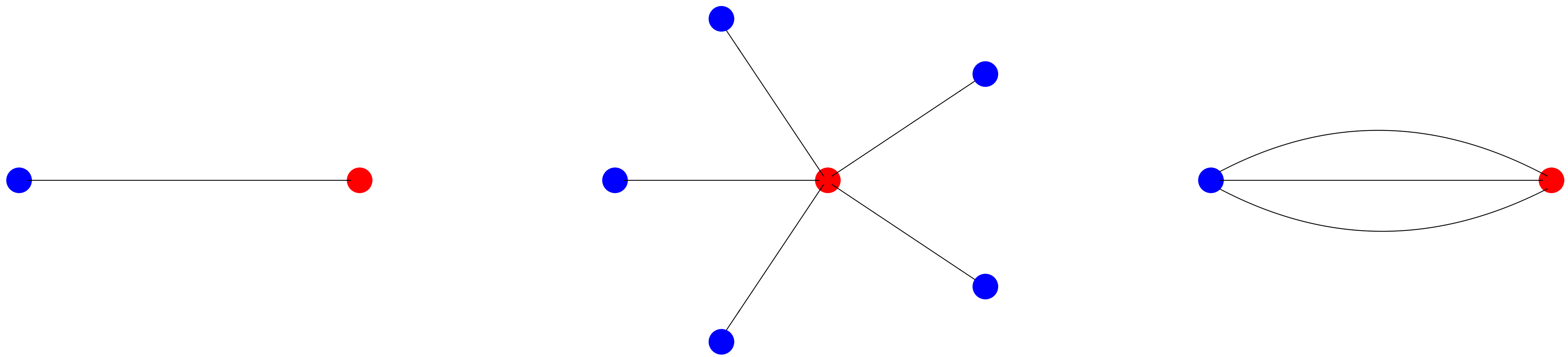
Seminar in Real and Complex Geometry

The talk is based on the joint works with Natalia Amburg and George Shabat

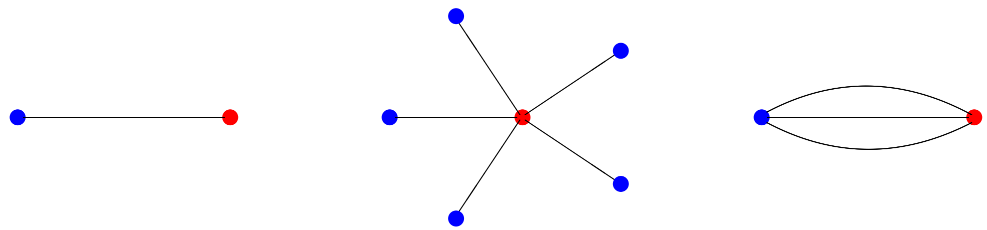
Def. A graph, $\Gamma = (V, E)$, is two sets $V = \{v_1, \ldots, v_k\}$, vertices, and $E = \{(v_{i_1}, v_{j_1}), \ldots, (v_{i_n}, v_{j_n})\} \subseteq V \times V$, edges.

Γ can be disconnected, can have multiple edges, etc.

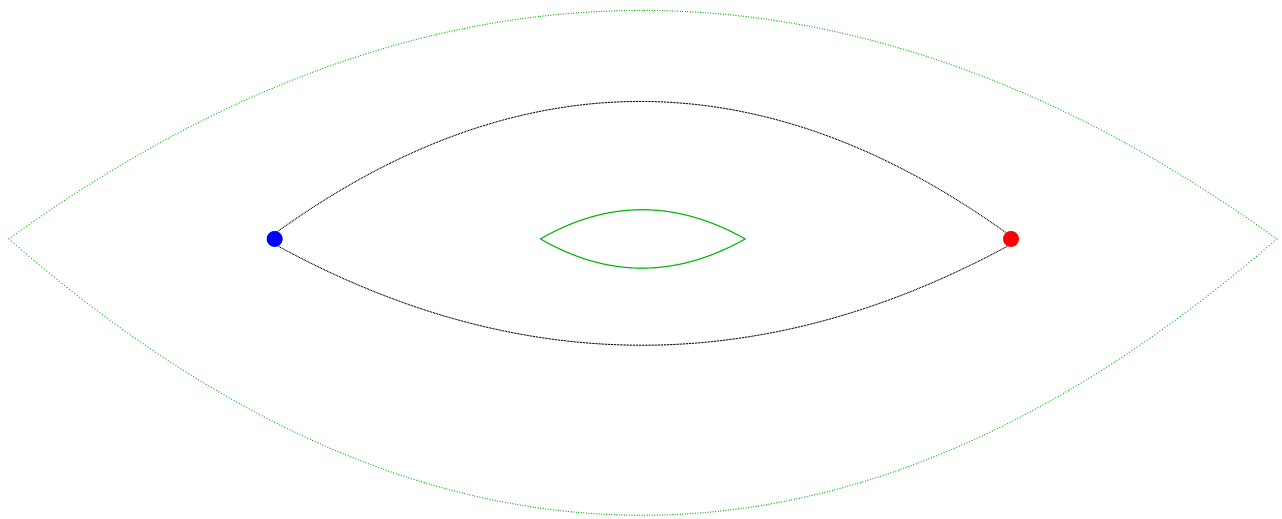
Def. A graph, Γ, is called bipartite if all v_i are colored into blue and red in such a way that each $(v_j, v_l) \in E$ has the vertices of two different colors.



Def. A dessin d'enfant D is a compact connected smooth oriented surface M together with a connected bipartite graph Γ embedded into M such that the complement $M \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. Such a disc is called a face of the dessin. The vertices and the edges of the dessin are vertices and edges of the graph.



This is not a dessin:

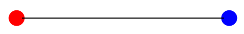


Def. A valency of a vertex v is the number of edges incident to v. A valency of a face is the $\frac{1}{2}$ of the number of edges incident to this face.

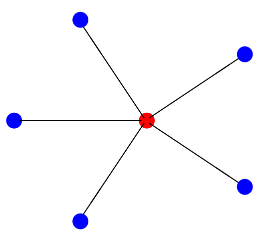
Def. A sequence of numbers

$$\langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$$

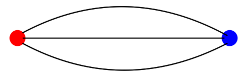
is called a combinatorial type if it is a list of valencies of all red vertices, blue vertices, and faces of a certain dessin.



$\langle 1|1|1 \rangle$



$\langle 5|1,1,1,1,1|5 \rangle$



$\langle 3|3|1,1,1 \rangle$

Def. Two dessins are said to be isomorphic if there exists an orientation preserving homeomorphism between corresponding surfaces under which one dessin is transformed into another.

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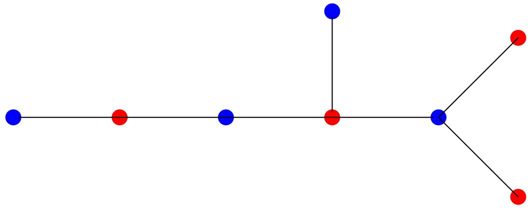
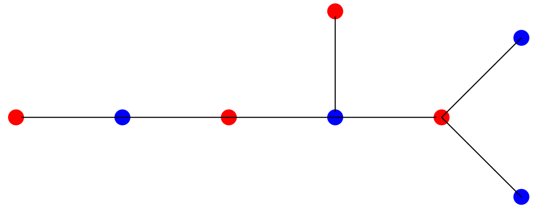
Isomorphic dessins have the same combinatorial types. However, combinatorial type does not determine a dessin up to an isomorphism, see

$$\langle 3,2,1|3,1,1,1|6\rangle.$$

Also non-isomorphic dessins:





Note that for trees

$$\sum_{i=1}^{\alpha} a_i = \sum_{j=1}^{\omega} w_j = n$$

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Lemma 1. *[Shabat, Zvonkine, 93] For any pair (λ_1, λ_2) of partitions of n with $n+1$ parts there exists a plane tree with the combinatorial type $\langle \lambda_1 | \lambda_2 | n \rangle$.*

Note that for trees

$$\sum_{i=1}^{\alpha} a_i = \sum_{j=1}^{\omega} w_j = n \quad \text{and} \quad \alpha + \omega = n + 1$$

Lemma 2. *[Shabat, Zvonkine, 93] For any pair (λ_1, λ_2) of partitions of n with $n+1$ parts there exists a plane tree with the combinatorial type $\langle \lambda_1 | \lambda_2 | n \rangle$.*

For more general dessins it is an open question: if a sequence $\langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$ satisfying $\sum_{i=1}^{\alpha} a_i = \sum_{j=1}^{\omega} w_j = \sum_{l=1}^{\gamma} c_l$ and $\alpha + \omega + \gamma = 2 - 2g + \sum_{i=1}^{\alpha} a_i$ is a combinatorial type?

Group theory approach to dessins d'enfants

M is oriented. Let's enumerate the edges, $|E| = n$. Then

$\rho_{\bullet} \in S_n$ — the rotation around red vertices, and

$\rho_{\bullet} \in S_n$ — the rotation around blue vertices,

are well-defined. Hence edge rotation group or monodromy group $\langle \rho_{\bullet}, \rho_{\bullet} \rangle \subseteq S_n$ is defined.

With a high probability (goes to 1 if n goes to ∞) $\langle \rho_{\color{red}\bullet}, \rho_{\color{blue}\bullet} \rangle$ is either A_n or S_n. However, any group can appear.

How to draw a group?

Let G be a group. Then $\exists\ n$: $G \subseteq S_n$. Then $G = \langle \rho_{\color{red}\bullet}, \rho_{\color{blue}\bullet} \rangle$ for some $\rho_{\color{red}\bullet}, \rho_{\color{blue}\bullet} \in S_n$.

The number of red vertices is the number of cycles in $\rho_{\color{red}\bullet}$.
The number of blue vertices is the number of cycles in $\rho_{\color{blue}\bullet}$.
The valency of a vertex is the length of the cycle.

Put several blue and red hedgehogs and connect them to each other...

Let $G = M_{11}$. Then $\rho_{\color{red}\bullet} = (2365)(789\,10) \in S_{11}$
and $\rho_{\bullet} = (12)(34)(67)(9\,11) \in S_{11}$.

Let $G = PSL_2(7)$. Then $\rho_{\color{red}\bullet} = (123)(475) \in S_8$
and $\rho_{\bullet} = (12)(34)(56)(78) \in S_8$.

Def. A Belyi pair $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. β is a Belyi function.

Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of β, $crit(\beta) \subseteq \{0, 1, \infty\}$.

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$(\mathbb{CP}^1, T_n(x) = \cos(n \arccos x))$,

$(\mathbb{T} : y^2 = x^3 - x, x^2)$

$(w^2 = z^5 - z, z^4)$

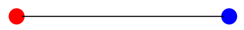
$(w^2 = (z^2 - 2)(z^4 - 2z^2 + 2), -(-zw + z^4 - 2z^2 + 1))$

Thm (G. Belyi, 1979). *Let $\mathcal{X}$ be a smooth complete irreducible algebraic curve over $\mathbb{C}$. Then the following statements are equivalent:*

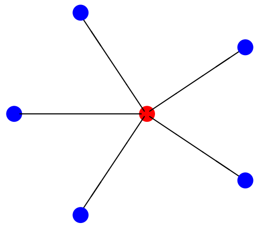
1. *$\mathcal{X}$ is isomorphic to the complexification of a curve defined over a number field (a subfield of $\bar{\mathbb{Q}}$);*

2. *There exists a Belyi function on $\mathcal{X}$.*

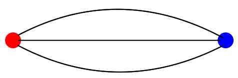
It holds, that if $(\mathcal{X}, \beta)$ is a Belyi pair, then $\beta^{-1}([0, 1])$ is a dessin d'enfant on the topological model of $\mathcal{X}$ whose edges are $\{\beta^{-1}((0, 1))\}$, red vertices are $\{\beta^{-1}(0)\}$, and blue vertices are $\{\beta^{-1}(1)\}$.



$\beta = x$



$\beta = x^5$



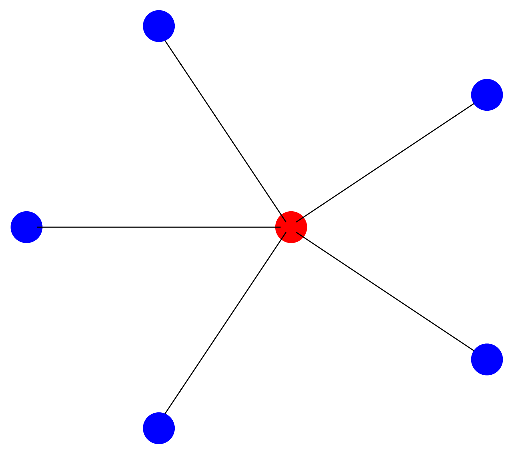
$\beta = \frac{x^3}{3x^2-6x+5{,}25}$

Moreover, the following result is true.

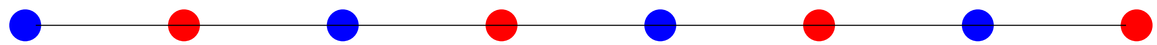
Thm (Shabat, Voevodsky, 1983). *Any dessin d'enfant can be obtained by the above construction from some Belyi pair. This pair is defined uniquely up to an isomorphism.*

P is a Shabat polynomial ?

$P(z) = z^n$:



$P(z) = T_n(z) = \cos(n \arccos z)$, $P^{-1}([-1,1])$:



Chebyshev polynomials: $\cos 2\varphi = 2\cos^2\varphi - 1$, $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$, $\ldots$,
$\cos n\varphi = T_n(\cos\varphi)$. $n-1$ critical points: $\cos\frac{k\pi}{n}$ but only 2 critical values: ± 1.

$T_0(z) = 1$, $T_1(z) = z$, $T_2(z) = 2z^2 - 1$, $T_3(z) = 4z^3 - 3z$, $\ldots$,
$T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$, $n \geq 2$.

Def. Belyi function which is a polynomial is called a Shabat polynomial.

Def. Let $P(z), Q(z)$ be Shabat polynomials, $c_0, c_1, \quad d_0, d_1$ be their critical values. The two triples $(P, c_0, c_1), \quad (Q, d_0, d_1)$ are equivalent if there exist $A, B, a, b \in \mathbb{C}$, $a, A \neq 0$ such that

$$Q(z) = AP(az + b) + B, \quad d_0 = Ac_0 + B, \quad d_1 = Ac_1 + B.$$

Thm. *There is a bijection between the set of bipartite plane trees and equivalence classes of Shabat polynomials.*

Proof. ($\Leftarrow$) 1. $P^{-1}([c_0, c_1])$ is a bipartite graph.

Its red vertices are $P^{-1}(c_0)$, blue vertices are $P^{-1}(c_1)$.
A valency of a vertex is a multiplicity of the corresponding root of $P(z) = c_i$.

2. Why do this graph has no circuits?

Assume, there is, C.

Up to a lin. transform. of w-plane, c_0, c_1 are real. Then P takes only real values on ∂C.

$Im(P(z))$ is a harmonic function in x, y $(z = x + i\,y)$. It is identically 0 on the boundary
$\Rightarrow Im(P(z)) \equiv 0$ inside — contradiction.

3. $P^{-1}([c_0, c_1])$ is connected.

Indeed, $P(z) = c_0, P(z) = c_1$ have altogether $2n$ roots with multiplicities.

$$\alpha + \omega = 2n - \sum_v (Val(v) - 1) = 2n - (n-1) = n + 1$$

number of vertices.

Thus it is a tree.

4. $(\Rightarrow)$ $c_0 = 0, c_1 = 1$, add the 3'rd, ∞ and use triangularization of both planes.

Each triangle is conformly equivalent to the half (low) plane (by orientation).
Take natural projections. Then glue them to each other following the triangularization of z-plane. This function is meromorphic by Riemann theory. It maps only ∞ to ∞. Thus it is a polynomial.

5. Uniqueness.

We obtained a covering of a Riemann sphere unramified outside $0, 1, \infty$. It is unique up to a linear-fractional transformation!

$\infty \to \infty \Rightarrow$ only linear.

Corollary 1. *Every plane tree has a unique canonical geometric form.*

Several related problems and applications:

1. ABC-conjecture.

All coprime integers a, b, c satisfying $a+b+c = 0$ are bounded by a small power of the kernel $K = K(abc) := \prod_{\{p:p|abc\}} p$. More precisely, $\forall\ \varepsilon > 0\ \exists\ C_\varepsilon$ s.t. $\max\{a, b, c\} \le C_\varepsilon K^{1+\varepsilon}$.

ABC-conjecture for polynomials.

Thm. *[Stothers, 81, Mason, 84] Let* $A, B, C \in \mathbb{C}[x]$ *be coprime, not all constant,* $A + B + C = 0$. *Denote* $n_0 := |\{\delta \in \mathbb{C} | ABC(\delta) = 0\}|$. *Then* $\max\{\deg A, \deg B, \deg C\} < n_0$.

Is it possible to improve this bound?

Characterizations of polynomials for which that bound in ABC-conjecture is tight:

Thm. *[Zanier, 95] Let* $A, B, C \in \mathbb{C}[x]$ *be coprime, not all constant,* $A+B+C = 0$. *Denote* $n_0 := |\{\delta \in \mathbb{C} | ABC(\delta) = 0\}|$. *Then* $\max\{\deg A, \deg B, \deg C\} = n_0 - 1 \Leftrightarrow f = A(x)/B(x)$ *is a Belyi function on and* $\frac{A}{B}(\infty) \in \{0, 1, \infty\}$.

2. Davenport – Stothers – Zannier bound:

Let $P, Q \in \mathbb{C}[z]$ be coprime.

What is the smallest possible degree for $P^3 - Q^2$?

The smallest means $\deg P = 2n$, $\deg Q = 3n$.

Conjecture (1965): 1. $\deg R \geq n + 1$;

2. this bound is sharp – is attained for infinitely many n.

1. – Davenport, 1965

2. – Stothers, 1981

Denote $f = \frac{P^3}{R}$. Then $f - 1 = \frac{P^3 - R}{R} = \frac{Q^2}{R}$.

Let f be a Belyi function ! What do we get?

$\deg P = 2n$, $\deg Q = 3n$, $f : \mathbb{C} \to \mathbb{C} \Rightarrow$ dessin with $2n$ red vertices of val. 3 and $3n$ blue vertices of val. 2.

For γ: Euler formula $2n - 3n + \gamma = 2 \Rightarrow \gamma = n + 2$.

Put one face to $\infty \Rightarrow \deg R = \sum$ val. of other $n + 1$ faces.

The question is now, if there is a plane graph with $3n$ edges, $2n$ vertices of valency 3 and $n + 1$ faces of valency 1?

For any n draw a tree with $2n$ vertices of valencies 1 or 3 and attach a loop to each end vertex.

3. Pell equation: $x^2 - dy^2 = 1$, d is square free.

To find solutions of Pell-Abel equation: $P^2 - D \cdot Q^2 = 1$, where $P, Q, D \in \mathbb{C}[x]$, $D(x)$ is square-free.

Ex. $D(x) = x^2 - 1 \Rightarrow \infty$ solutions: Chebyshev polynomials

Thm. *[Abel] THAE for* $D(x)$

1. Pell-Abel equation has a solution. (Then inf. many)

2. $\sqrt{D(x)}$ *can be repr. as a periodic continued fraction.*

3. $\int \frac{p(x)}{\sqrt{D(x)}} \mathrm{d}x$ *for some* p *with* $\deg p \leq \deg D - 2$ *can be computed in elementary functions.*

T is a plane tree, d_i are coordinates of the true shape of its vertices of odd valencies (regardless colors). Then $D(x) = \prod(x - d_i)$ satisfies Abel theorem.

Other applications:

1. Mathematical physics

   a. String theory.

   b. Matrix models and matrix integrals.

2. Compactification of Hurwitz and moduli spaces.

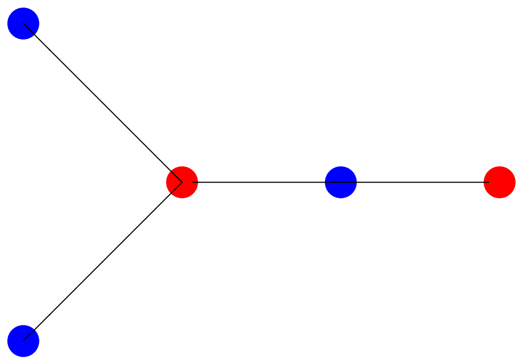
3. Stratification of Hurwitz and moduli spaces.

4. Quantum computations. 5. Etc.

How to find Shabat polynomials?

Examples:

1. A very simple example:



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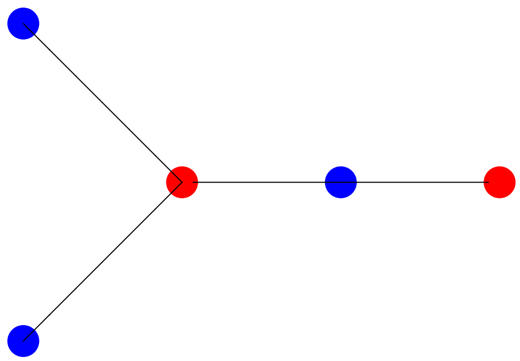

Put $c_0 = 0$.

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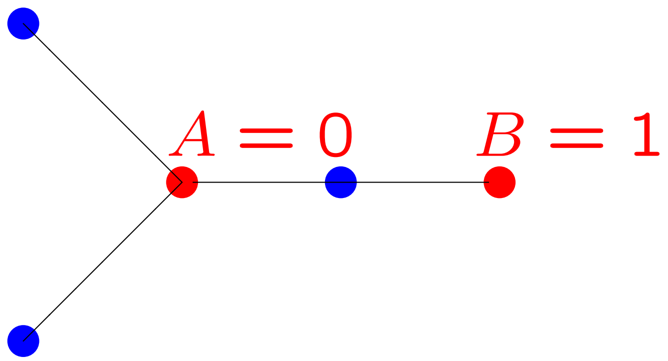

Put $c_0 = 0, A = 0, B = 1$.

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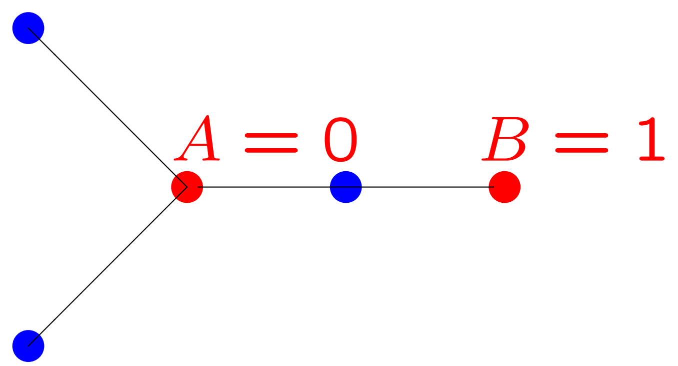

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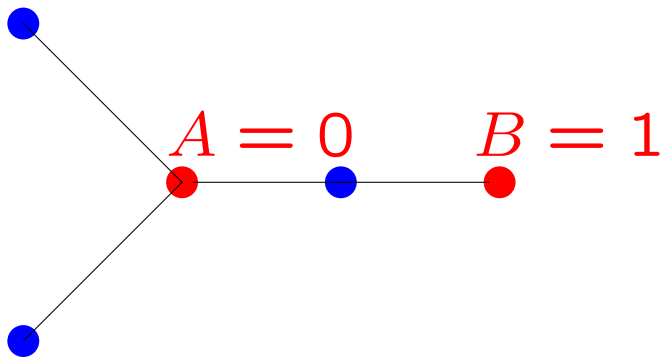

Put $c_0 = 0, A = 0, B = 1$. Then

$P(z) = z^3(z-1) \Rightarrow P'(z) = z^2(4z-3)$, $P(3/4) = -\frac{27}{256}$.

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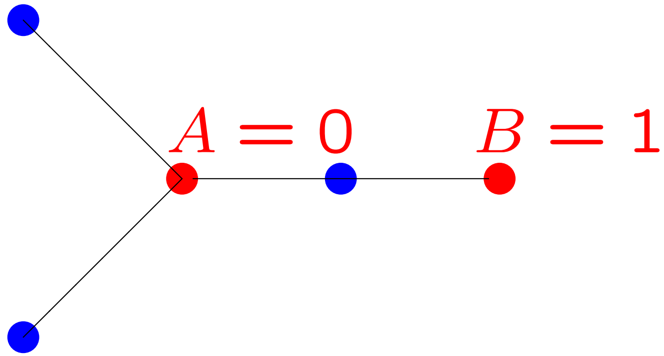
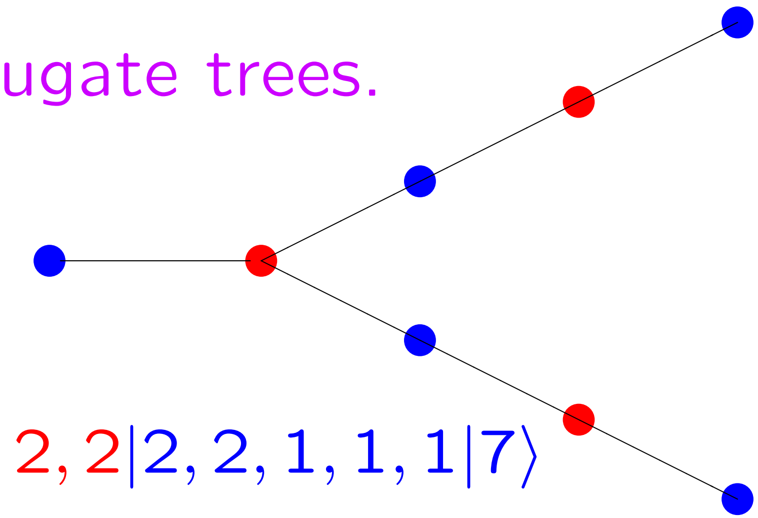


Put $c_0 = 0, A = 0, B = 1$. Then

$P(z) = z^3(z-1) \Rightarrow P'(z) = z^2(4z-3),\ P(3/4) = -\frac{27}{256}$,

$P(z) = -27/256 \Rightarrow z = 3/4,\ z = (-1 \pm \sqrt{-2})/4$.

2. First example of conjugate trees.



$$\langle 3, 2, 2 | 2, 2, 1, 1, 1 | 7 \rangle$$

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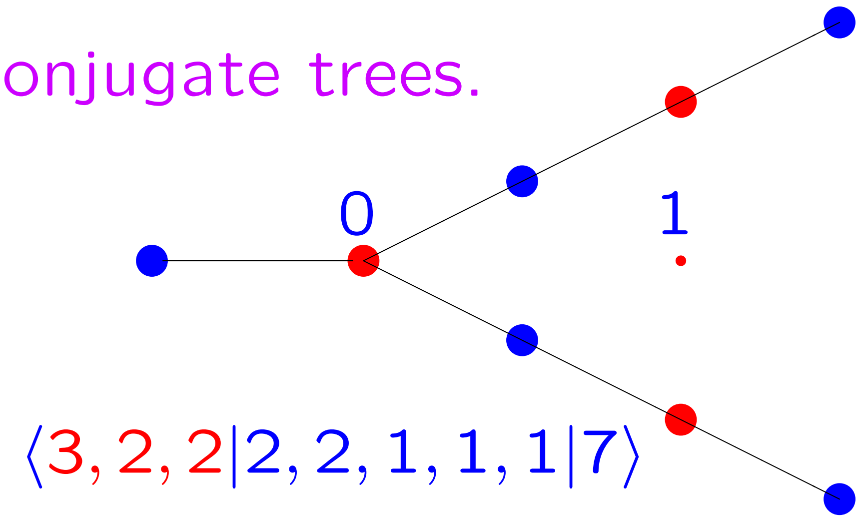

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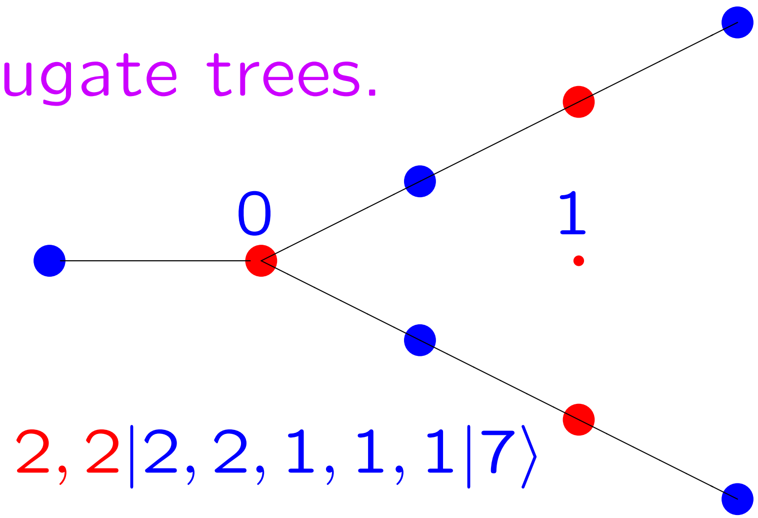

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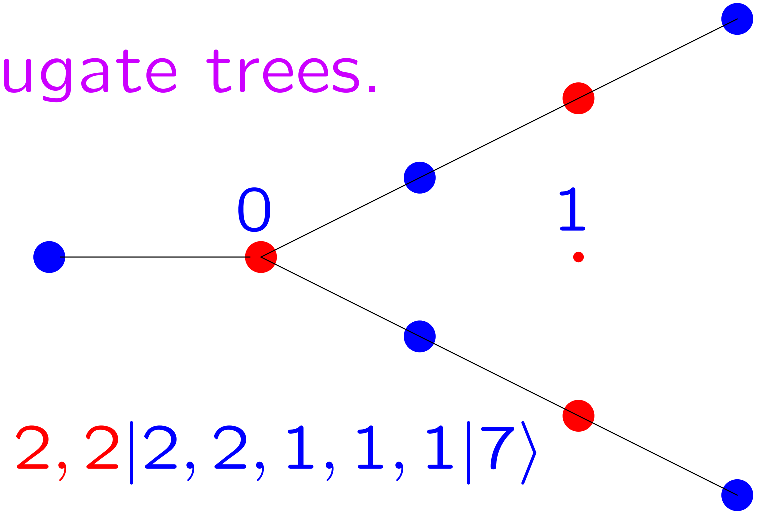


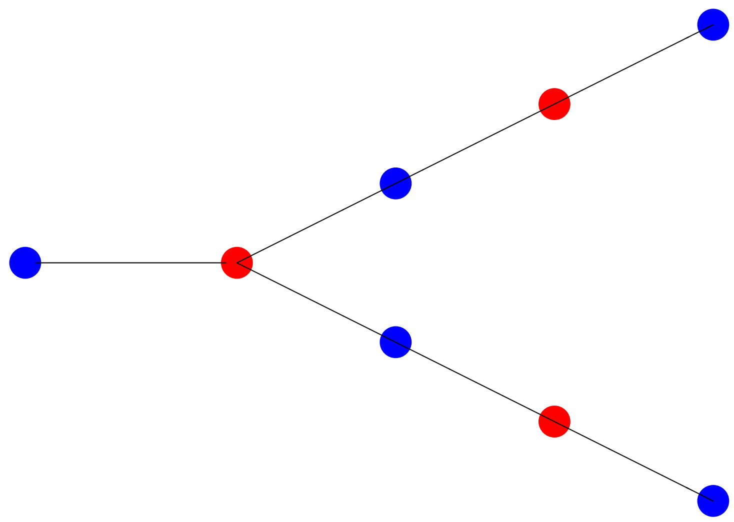
$$\langle 3,2,2|2,2,1,1,1|7\rangle$$

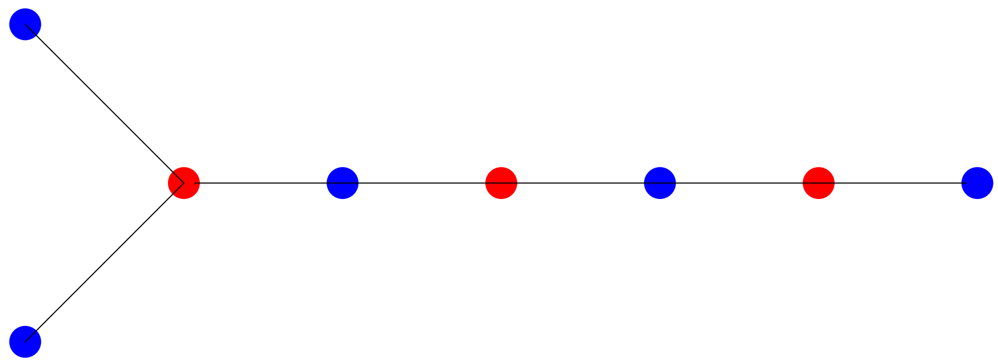
$$P(z) = z^3(z^2 - 2z + a)^2$$

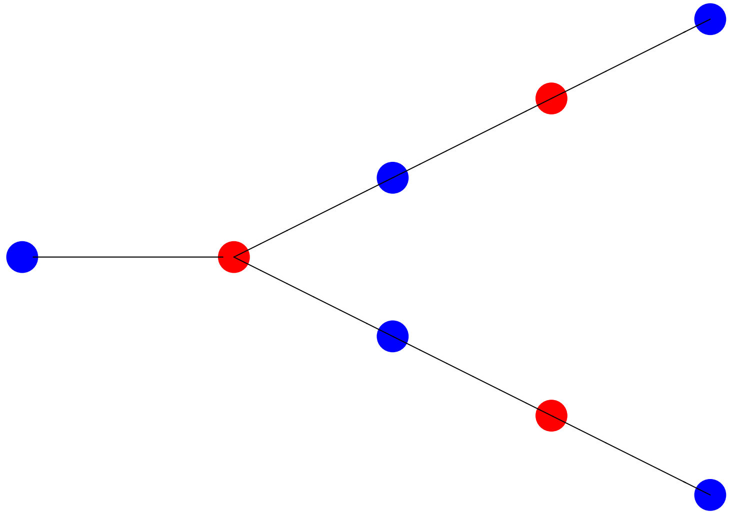
$$P'(z) = z^2(z^2-2z+a)(7z^2-10z+3a) = z^2(z^2-2z+a)Q(z)$$

The values of $P(z)$ at the roots of $Q(z)$ must be the same!

$$a = \frac{1}{7}(34 \pm 6\sqrt{21})$$







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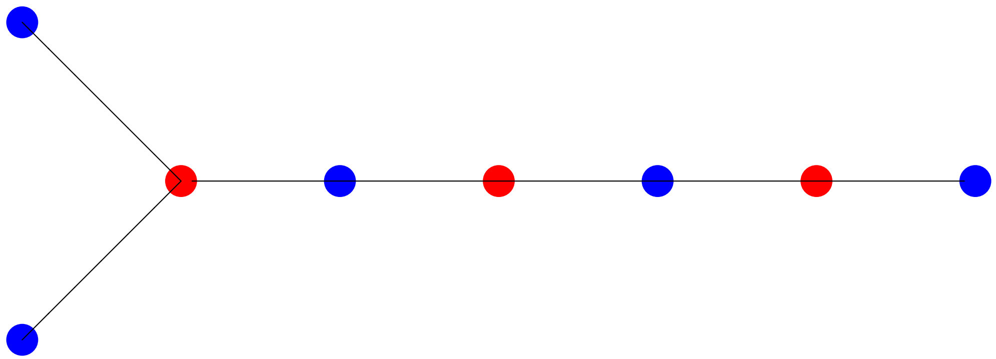

$$\mathcal{X} = \mathbb{CP}^1, \qquad \langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle,$$

$$crit(\beta) \subseteq \{0, 1, \infty\}$$

Let $A_1, \ldots, A_\alpha, W_1, \ldots, W_\omega, C_1, \ldots, C_\gamma$ be the corresponding complex coordinates. Then

$A_i \to 0, W_j \to 1, C_i \to \infty.$

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$A_i \to 0, W_j \to 1, C_i \to \infty.$ $\Downarrow$

$$K_1 \frac{(z - A_1)^{a_1} \cdots (z - A_\alpha)^{a_\alpha}}{(z - C_1)^{c_1} \cdots (z - C_\gamma)^{c_\gamma}}$$

$$\mathcal{X} = \mathbb{CP}^1, \qquad \langle a_1, \ldots, a_\alpha | w_1, \ldots, w_\omega | c_1, \ldots, c_\gamma \rangle$$

$$crit(\beta) \subseteq \{0, 1, \infty\}$$

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$$K_1 \frac{(z - A_1)^{a_1} \cdots (z - A_\alpha)^{a_\alpha}}{(z - C_1)^{c_1} \cdots (z - C_\gamma)^{c_\gamma}} - 1$$

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$$\mathcal{S}(Val): \quad \begin{aligned} &K_1(z - A_1)^{a_1} \cdots (z - A_\alpha)^{a_\alpha} - (z - C_1)^{c_1} \cdots (z - C_\gamma)^{c_\gamma} = \\ &= K_2(z - W_1)^{w_1} \cdots (z - W_\omega)^{w_\omega} \end{aligned}$$

Methods of computation:

1. Symmetries: to find, to factorize by, hence simplification of the structure

2. Approximate computations:

   a. Modular functions

   b. Machine learning methods

3. Matrix models and matrix integrals

4. Belyi pairs as singular points on Fried curves in the Hurwitz space.

5. Etc.

Directions of study in dessin d'enfant theory:

1. Geometric — true shape.

Directions and Applications:

1. Geometric — true shape.

2. Algebraic — Galois group action.

Directions and Applications:

1. Geometric — true shape.

2. Algebraic — Galois group action.

$\Gamma = Aut(\overline{\mathbb{Q}})$ is Galois group, $G \subseteq \Gamma$, $g \in G$.

Then $g(\chi, \beta)$ is a Belyi pair.

D is dessin then $g(D)$ is dessin with the same comb. type.

Thm. *Let $g \in \Gamma$, T be a bicolored plane tree. Then $g(T)$ is a plane tree of the same combinatorial type.*

Here $g(T)$ is def. by $T \to P(z) \to g(P(z)) \to g(T)$

Proof. If $(P_1, c_0, c_1) \sim (Q_1, d_0, d_1)$

then $\exists\ A, a, B, b,\ A, a \neq 0$:

$$Q_1(z) = AP_1(az + b) + B, \quad d_0 = Ac_0 + B, \quad d_1 = Ac_1 + B$$

Then for $P_2 = g(P_1), Q_2 = g(Q_1), d_i' = g(d_i), c_i' = g(c_i), i = 0, 1$:

$$Q_2(z) = g(A)P_2(g(a)z + g(b)) + g(B),$$

$$d_0 = g(A)c_0 + g(B), \quad d_1 = g(A)c_1 + g(B)$$

$g(A), g(a) \neq 0$ since g is a field automorphism.

Def. Field of definition of T is $K \subset \overline{\mathbb{Q}}$ which is the Galois extension of k corresponding to $G \subset \Gamma$ which fixes T.

Thm. *[Shabat, Couveignes] For any bicolored plane tree there is a Shabat polynomial whose coefficients belong to the field of definition of the tree.*

This is not the case for general dessins!

Thm. *[Lenstra, Schneps] The action of Γ on the set of trees is faithful.*

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This is not the case for general dessins!

Thm. *[Lenstra, Schneps] The action of Γ on the set of NOT trees BUT dessins with 1 cell is faithful.*

Corollary 2.

1. The class of fields of definition of the trees is not a specific class of fields: all number fields belong to it.

2. Philosophically speaking the world of bipartite plane trees is as rich as the world of algebraic numbers.

Directions and Applications:

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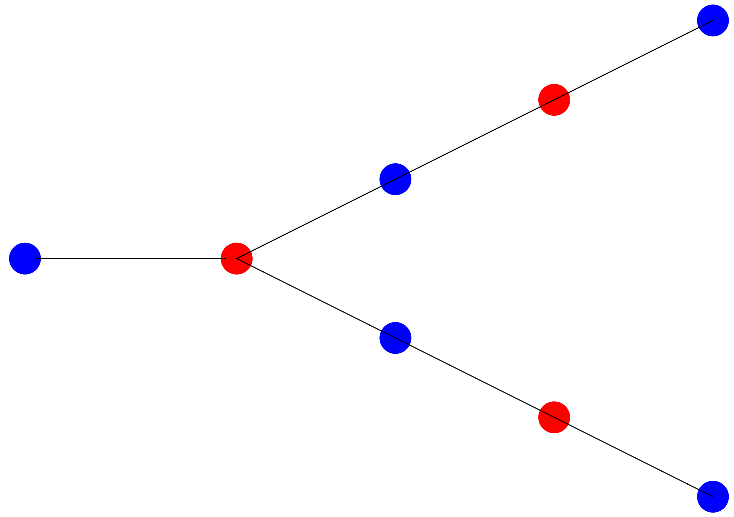
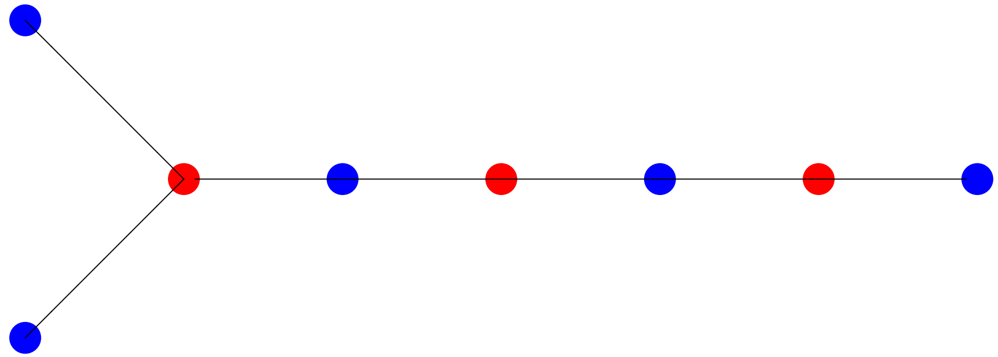
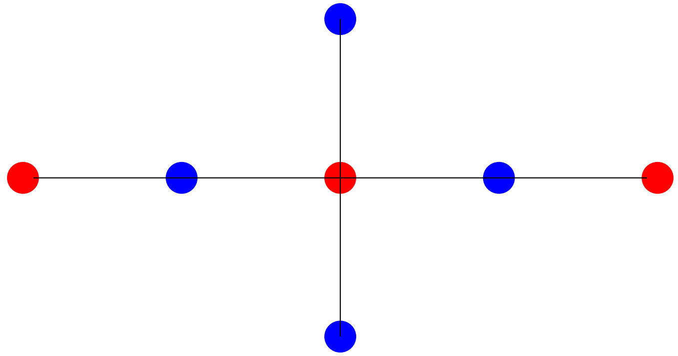
Other "geometric" Galois invariants = ???

Galois orbits



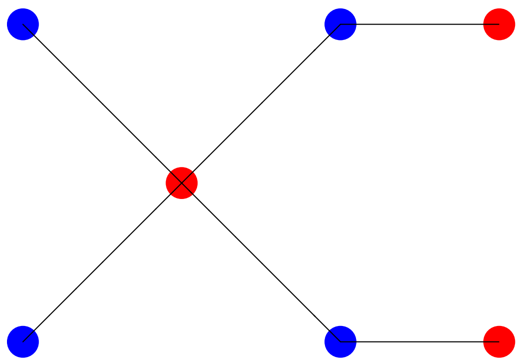
$$\langle 3,2,2|2,2,1,1,1|7\rangle,\ \mathbb{Q}(\sqrt{21})$$





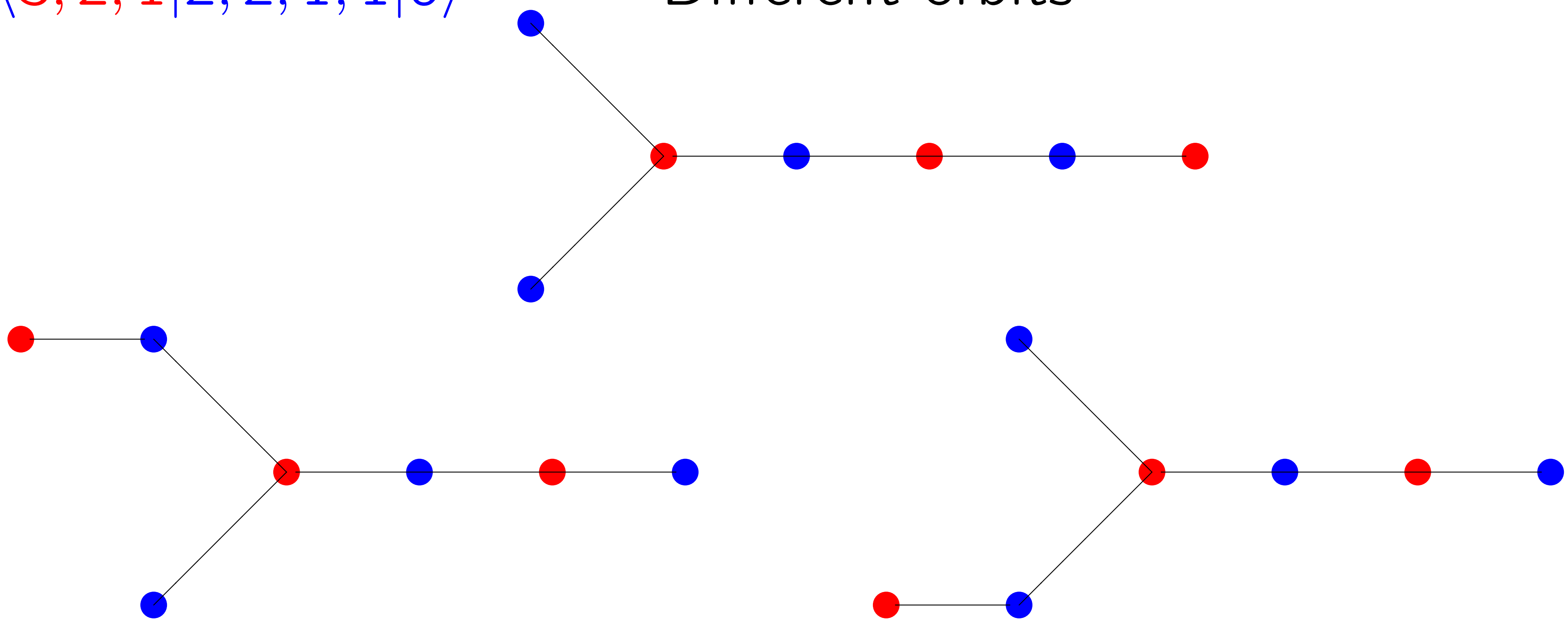
$$P(z) = z^4(z^2 - 1)$$

$$\langle 4, 1, 1 | 2, 2, 1, 1 | 6 \rangle, \ \mathbb{Q}$$



$$P(z) = z^4(z^2 - 2z + 25/9)$$

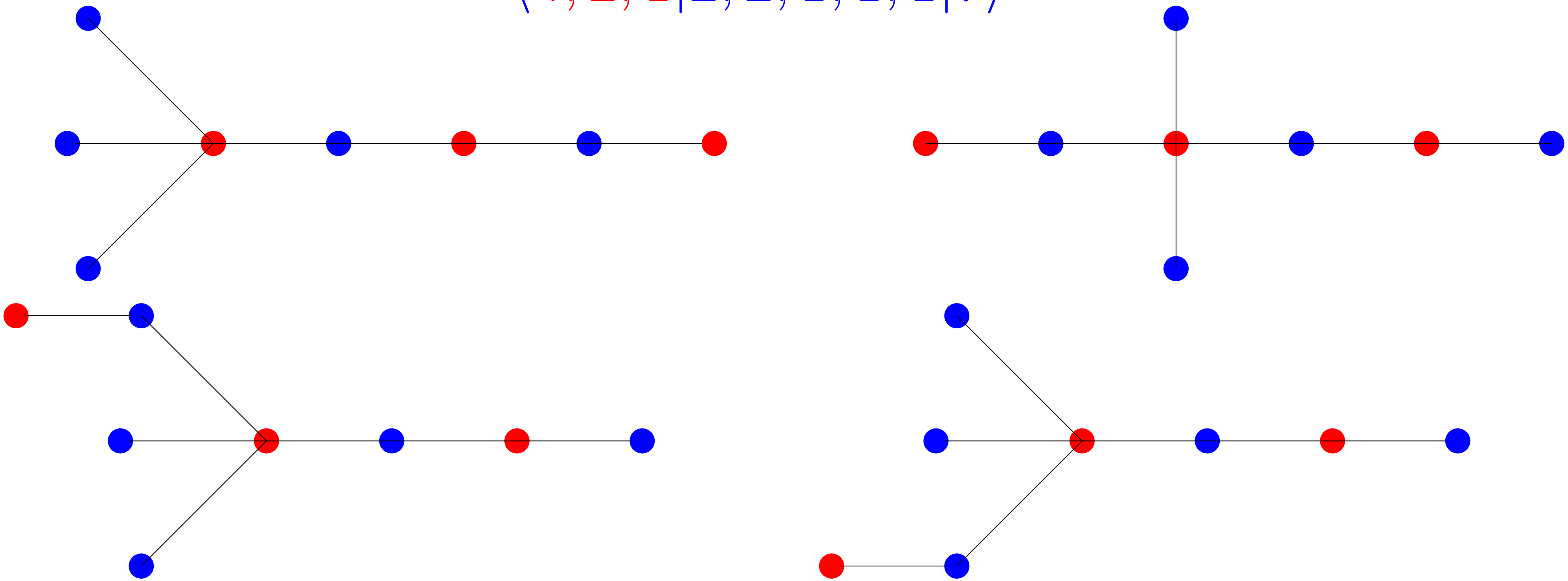
$\langle 3,2,1|2,2,1,1|6\rangle$ Different orbits



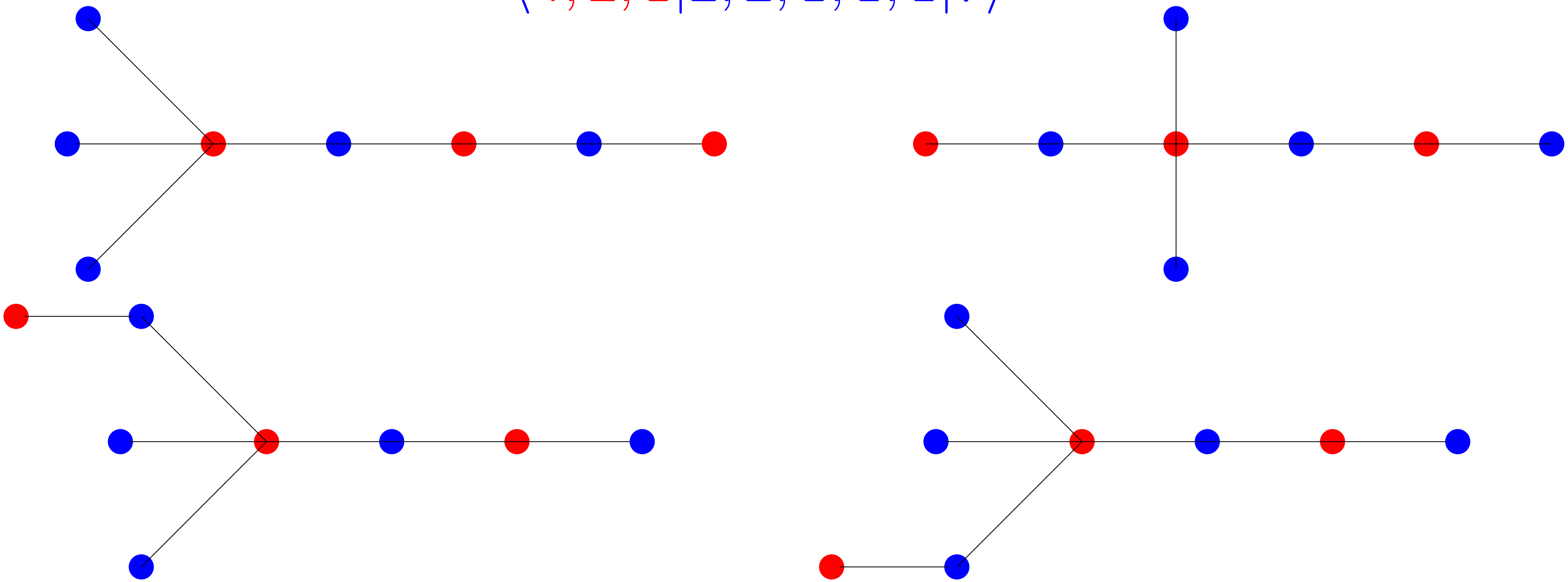
$P(z) = z^3(z-1)^2(z-a)$, $25a^3 - 12a^2 - 24a - 16 = 0$, $\mathbb{Q}(\sqrt[3]{2})$,

$G = S_3$

$$\langle 4, 2, 1 | 2, 2, 1, 1, 1 | 7 \rangle$$



$\langle 4, 2, 1 | 2, 2, 1, 1, 1 | 7 \rangle$

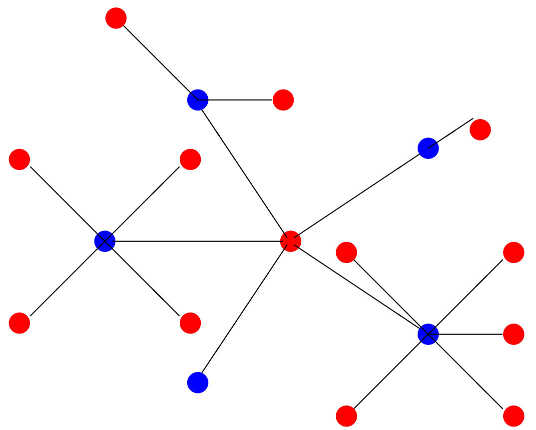


$\mathbb{Q}(\sqrt{21})$, $\mathbb{Q}(\sqrt{-7})$

Leila flower

Def. A diameter of a tree is the maximal number of edges in any chain of edges in this tree.

Trees of diameter four have well-defined central vertex:



$\langle 5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 | 1, 6, 2, 3, 5 | 12 \rangle$

The combinatorial type of a tree of diameter four is:

$$\langle \omega, \underbrace{1, \ldots, 1}_{\left(\sum\limits_{i=1}^{\omega} w_i\right) - \omega} | w_1, \ldots, w_\omega | \sum_{i=1}^{\omega} w_i \rangle$$

The Shabat polynomial is equal to $\prod\limits_{i=1}^{\omega} (z - B_i)^{w_i}$.

$$\begin{cases} w_1 x_1 + w_2 x_2 + \ldots + w_\omega x_\omega = 0 \\ w_1 x_1^2 + w_2 x_2^2 + \ldots + w_\omega x_\omega^2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_1 x_1^{\omega-1} + w_2 x_2^{\omega-1} + \ldots + w_\omega x_\omega^{\omega-1} = 0 \end{cases}$$

$x_i = \dfrac{1}{B_i}$ are unknowns.

Thm. [K.] *If* w_i *are pair-wise different, then Anti-Vandermonde system*

$$\left\{\begin{array}{lcccccc} w_1x_1 & + & w_2x_2 & + & \dots & + & w_\omega x_\omega & = & 0 \\ w_1x_1^2 & + & w_2x_2^2 & + & \dots & + & w_\omega x_\omega^2 & = & 0 \\ \multicolumn{9}{c}{\dotfill} \\ w_1x_1^{\omega-1} & + & w_2x_2^{\omega-1} & + & \dots & + & w_\omega x_\omega^{\omega-1} & = & 0 \end{array}\right.$$

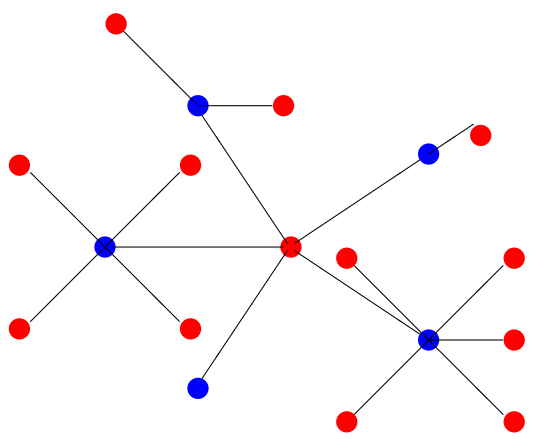
has exactly $\omega!$ *different solutions.*

Thm (Kochetkov, Shabat, Zapponi). $\omega = 5$*:*

There are two Galois orbits

$$\Updownarrow$$

$$w_1 \cdot w_2 \cdot w_3 \cdot w_4 \cdot w_5 \cdot (w_1 + w_2 + w_3 + w_4 + w_5)$$ *is a full square.*



$$\langle 5,1,1,1,1,1,1,1,1,1,1,1,1|1,6,2,3,5|12\rangle$$

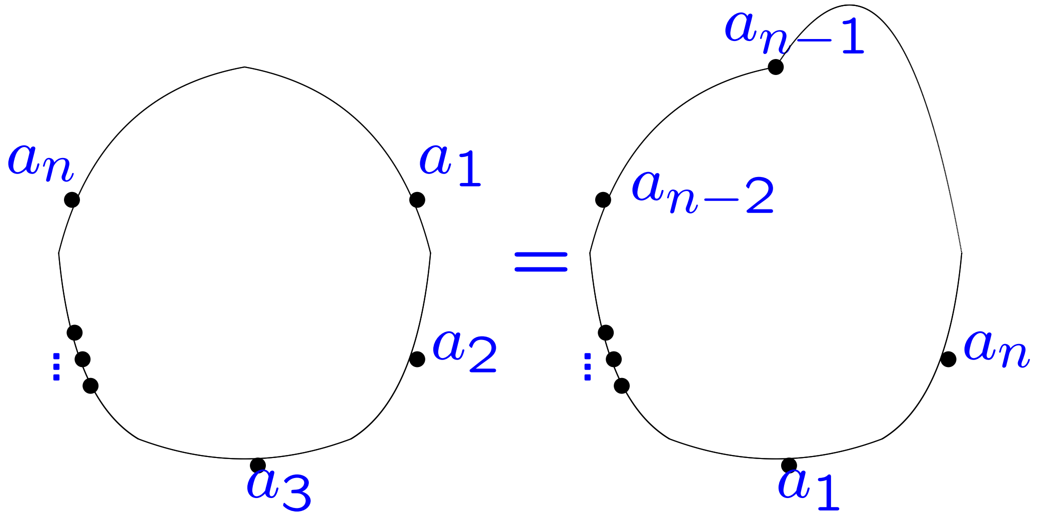
Directions and Applications:

1. Geometric — true shape.

2. Algebraic — Galois group action.

3. Numerical.

Belyi pair for $\mathcal{L}(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}})$

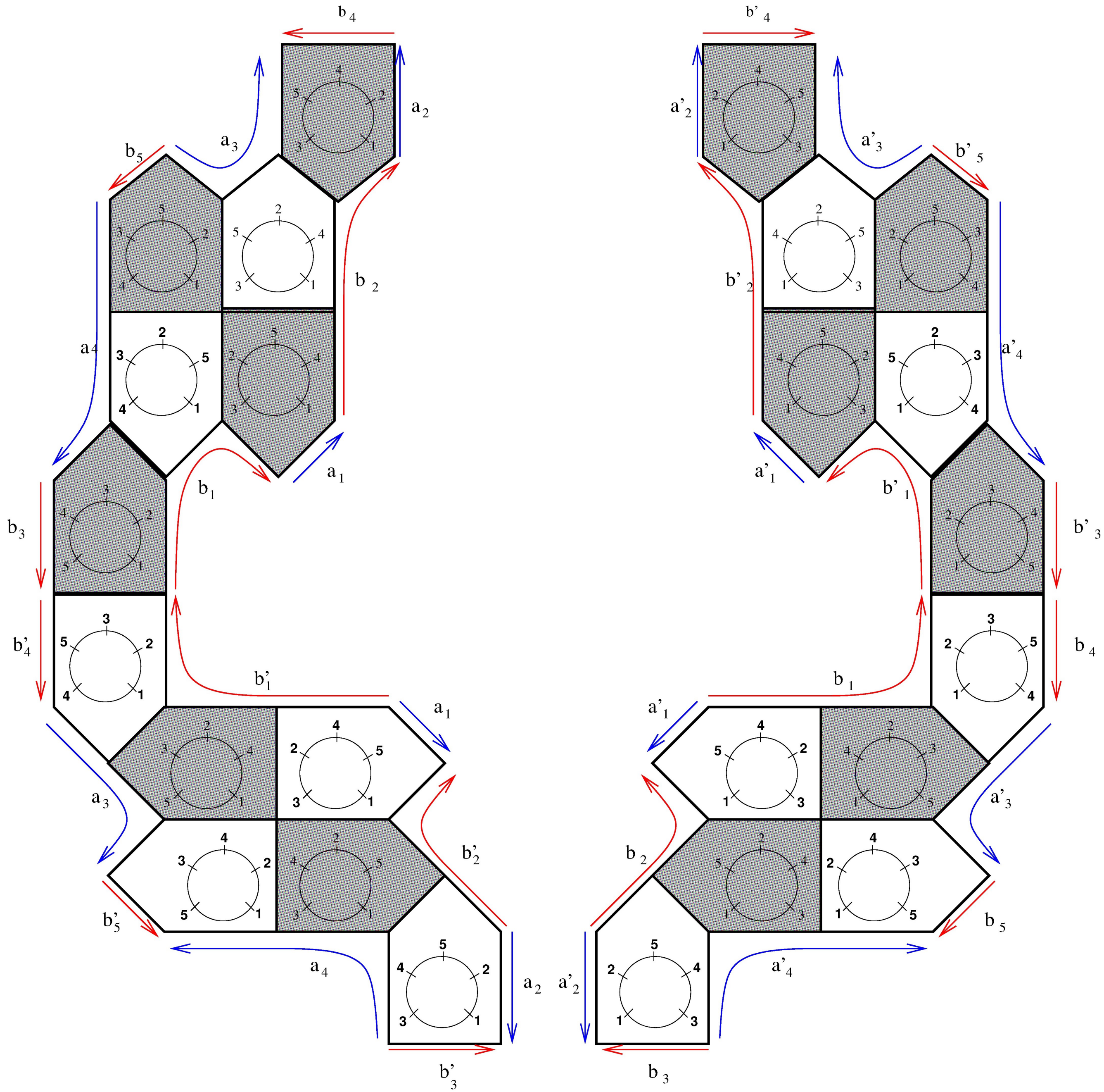
$\mathcal{M}^{\mathbb{R}}_{0,n} = \Bigg\{$  $\Bigg\} / \sim$

real alg. curves of genus 0 with n marked and numb. points

$\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ is the *Deligne-Mumford compactification* of $\mathcal{M}^{\mathbb{R}}_{0,n}$

$\mathcal{L}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$ is its orientation covering.

$\dim \overline{\mathcal{M}^{\mathbb{R}}_{0,n}} = 3g - 3 + n$, hence for $n = 5, g = 0$ it's a surface.

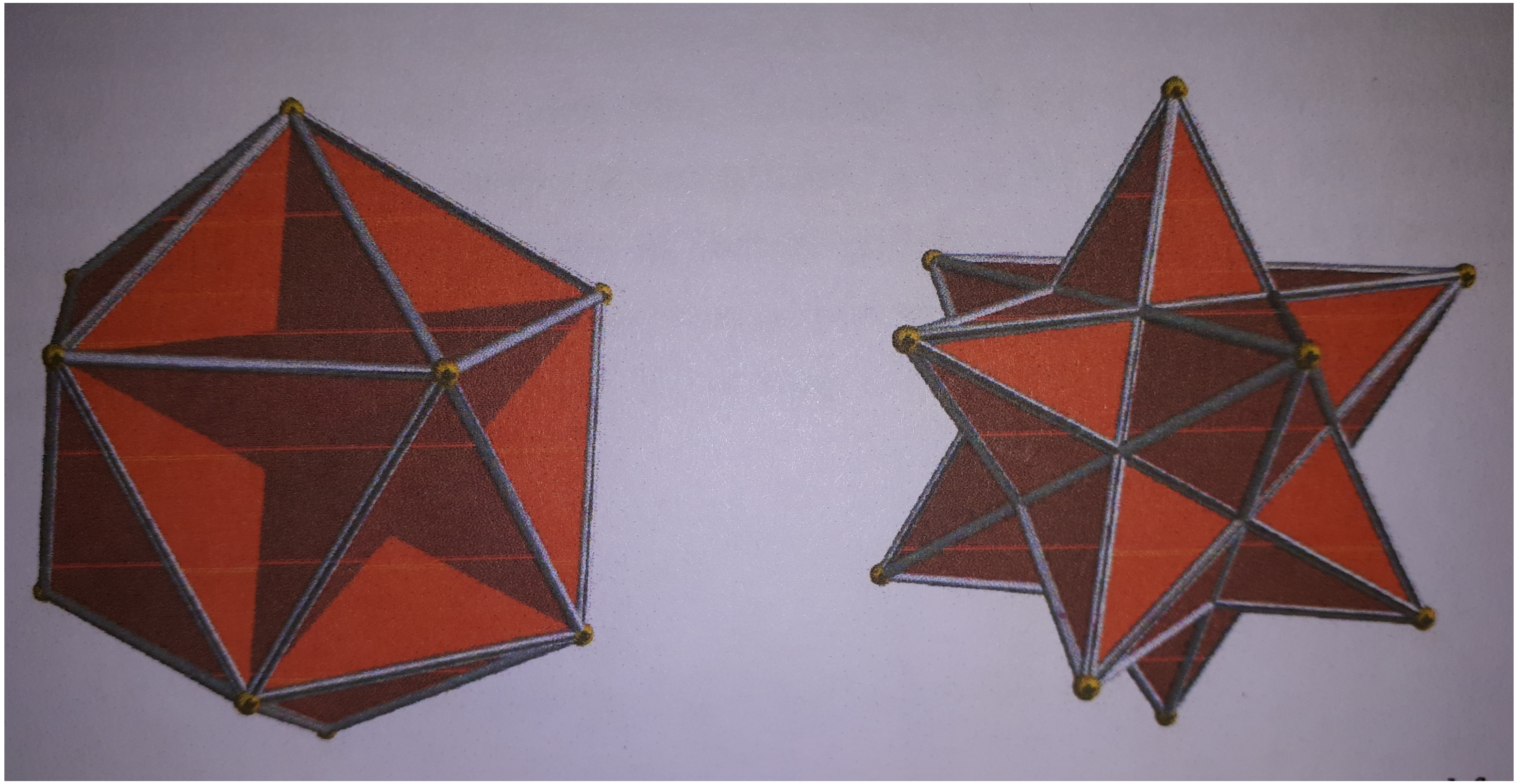

b_4
a_2
a_3
b_5
b_2
a_4
a_1
b_1
b_3
b'_4
b'_1
a_1
a_3
b'_2
b'_5
a_4
a_2
b'_3
b'_4
a'_2
a'_3
b'_5
b'_2
a'_4
a'_1
b'_1
b'_3
b_4
b_1
a'_1
a'_3
b_2
b_5
a'_2
a'_4
b_3

Thm. *[Amburg, K.] Belyi pair for* $\mathcal{L}(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}})$ *is the Bring curve — an algebraic curve in dim=4 complex projective space with coordinates* $x_1 : \ldots : x_5$ *defined by*

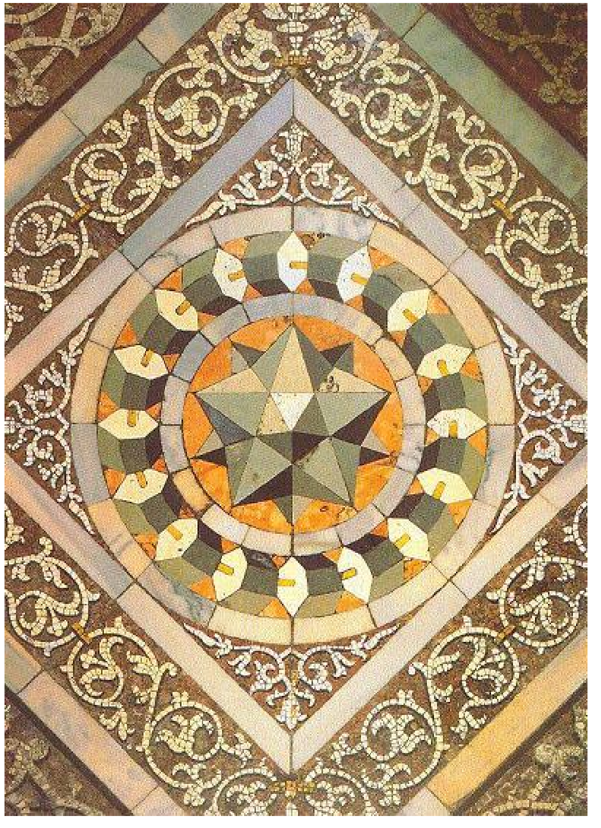
$$B_5 : \quad \begin{cases} \sum_{i=1}^5 x_i = 0 \\ \sum_{i=1}^5 x_i^2 = 0 \\ \sum_{i=1}^5 x_i^3 = 0 \end{cases}$$

and the function

$$\beta = \frac{3125(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5})^4}{256 x_1 x_2 x_3 x_4 x_5} \quad .$$



Icosahedron $g = 4$ appeared as a mosaic by Paolo Uccello on the floor of San Marco cathedral, Venice, 1430.



תודה רבה

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