# Germs of maps, group actions and large modules inside group orbits.

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## Abstract

A map $(\mathbb{k}^n, o) \to (\mathbb{k}^p, o)$ with no critical point at the origin can be rectified to a linear map. Maps with critical points have complicated/rich structure. They are studied up to the groups of right/left-right/contact equivalence. The group orbits are complicated and are traditionally studied via their tangent spaces. This transition is classically done by vector fields integration, thus binding the theory to the real/complex case.

I will present the new approach to this subject. One studies the maps of germs of Noetherian schemes, in any characteristic. The corresponding groups of equivalence admit 'good' tangent spaces. The submodules of the tangent spaces lead to submodules of the group orbits. This allows to bring these maps to 'convenient' forms. For example, we get the (relative) finite determinacy, and accordingly the (relative) algebraization of maps/ideals/modules.

Based on arXiv:1212.6894 (jointly with G. Belitski), arXiv:1808.06185 (jointly with A.-F. Boix, G.-M. Greuel), and my own recent work.

# Prologue

Let $\mathbb{k} \in \mathbb{R}, \mathbb{C}$ and $(\mathbb{k}^n, o) \xrightarrow{f} (\mathbb{k}^1, o)$ be $\mathbb{k}$-analytic or $C^\infty$.

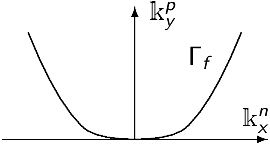
1. Suppose $f'|_o \neq 0$, then in some local coordinates: $f(x) = x_1$. (IFT)
2. $(n = 1)$ Take $(\mathbb{k}^1, o) \xrightarrow{f} (\mathbb{k}^1, o)$, with $ord(f) = d < \infty$. Then in some local coordinates $f(x) = \underset{\mathbb{k}=\mathbb{R}}{(\pm)} x^d$.

Question: How to extend this to the case: $n > 1$ and $f'|_o$ is degenerate?
(This was one of the starting points of Singularity Theory in 19'th century)

# The three main equivalences

Let $\Bbbk \in \mathbb{R}, \mathbb{C}$, consider $Maps := \{(\Bbbk^n, o) \xrightarrow{f} (\Bbbk^p, o)\}$. ($\Bbbk$-analytic or $C^\infty$)
Always assume $f'|_o$ is degenerate, i.e. $rank[f'|_o] < min(p, n)$.

**Right equivalence**




$\Phi_X \circlearrowleft (\Bbbk^n_x, o)$
$f \rightsquigarrow \Phi^*_X(f) := f \circ \Phi^{-1}_X$
$\mathscr{R} \circlearrowleft Maps((\Bbbk^n, o), (\Bbbk^p, o))$

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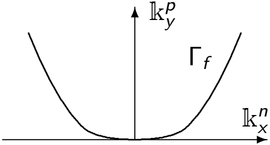
$\mathscr{R}$-action is $\Bbbk$-linear
$\Phi^*_X(a \cdot f + b \cdot g) = a \cdot \Phi^*_X(f) + b \cdot \Phi^*_X(g)$

**Contact equivalence**




$\Phi_X \circlearrowleft (\Bbbk^n_x, o)$
$(x, y) \to (\Phi_X(x), \Psi(x, y))$ s.t.
$\Psi(x, o) = o$, $\Psi(o, y)$ invertible.
$f \rightsquigarrow \Psi(x, f \circ \Phi^{-1}_X)$
$\mathscr{K} \circlearrowleft Maps((\Bbbk^n, o), (\Bbbk^p, o))$

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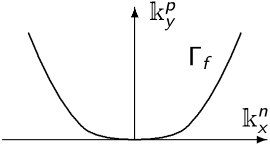
$\mathscr{K}$-action is not $\Bbbk$-linear
but it is "linearizable".
$\mathscr{K}_{Lin} := GL(\mathcal{O}_{(\Bbbk^p, o)}, p) \rtimes \mathscr{R}$
$f(x) \rightsquigarrow U(x) \cdot f(\Phi^{-1}_X(x))$
$\mathscr{K}_{Lin}$-action is $\Bbbk$-linear.
**Fact:** $\mathscr{K} f = \mathscr{K}_{Lin} f$, $\forall f$.
Thus often use $\mathscr{K}_{Lin}$ instead of $\mathscr{K}$.

**Left-Right equivalence**




$\Phi_X \circlearrowleft (\Bbbk^n_x, o)$, $\Phi_Y \circlearrowleft (\Bbbk^p_y, o)$
$f \rightsquigarrow \Phi_Y \circ f \circ \Phi^{-1}_X$
$\mathscr{A} \circlearrowleft Maps((\Bbbk^n, o), (\Bbbk^p, o))$

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$\mathscr{A}$-action is not $\Bbbk$-linear
$\Phi_Y(a \cdot f + b \cdot g) \neq a \cdot \Phi_Y(f) + b \cdot \Phi_Y(g)$

It is not known whether/how
$\mathscr{A}$-action can be "linearized".
(Despite numerous attempts)

## Tangent spaces for group actions $\mathscr{G} \circlearrowright Maps((\Bbbk^n,o),(\Bbbk^p,o)) \cong \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$

**Right action, $\mathscr{R} \circlearrowright \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$**

$f(x) \to f(\Phi_X^{-1}(x))$

$T_{\mathscr{R}}$ =germs of vector fields =derivations $=Der_{\Bbbk}(\mathcal{O}_{(\Bbbk^n,o)})$ $= \{\sum a_i(x)\frac{\partial}{\partial x_i}\}$

$T_{\mathscr{R}} f = \mathcal{O}_{(\Bbbk^n,o)}\langle \frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n}\rangle \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$.

An $\mathcal{O}_{(\Bbbk^n,o)}$-submodule.

**Contact action**

$\mathscr{K}, \mathscr{K}_{Lin} \circlearrowright \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$

$f(x) \to U(x)\cdot f(\Phi_X^{-1}(x))$

$T_{\mathscr{K}} f = Mat_{p\times p}(\mathcal{O}_{(\Bbbk^n,o)})f + T_{\mathscr{R}} f \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$

An $\mathcal{O}_{(\Bbbk^n,o)}$-submodule.

**Left-right action, $\mathscr{A} \circlearrowright \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$**

$f \to \Phi_Y \circ f \circ \Phi_X^{-1}$

$T_{\mathscr{A}} = T_{\mathscr{L}} \oplus T_{\mathscr{R}} = Der_{\Bbbk}(\mathcal{O}_{(\Bbbk^p,o)}) \oplus Der_{\Bbbk}(\mathcal{O}_{(\Bbbk^n,o)})$

$T_{\mathscr{A}} f = T_{\mathscr{R}} f + f^{-1}(\mathcal{O}_{(\Bbbk^p,o)})^{\oplus p} \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$

$(\xi_Y,\xi_X)(f) = \xi_Y(y)|_f + \xi_X(f)$

Here: $f^{-1}(\mathcal{O}_{(\Bbbk^p,o)})$, not $f^*(\mathcal{O}_{(\Bbbk^p,o)})$.

$T_{\mathscr{A}} f$ is not an $\mathcal{O}_{(\Bbbk^n,o)}$-module.

**Example:** $(\Bbbk^n,o) \xrightarrow{f} (\Bbbk^1,o)$, i.e. $p=1$. Then $T_{\mathscr{R}}(f) = Jac(f) \subseteq \mathcal{O}_{(\Bbbk^n,o)}$, the Jacobian ideal.
$T_{\mathscr{K}}(f) = Jac(f) + (f) \subseteq \mathcal{O}_{(\Bbbk^n,o)}$, the Tjurina ideal.
$T_{\mathscr{A}}(f) = Jac(f) + Span_{\Bbbk}(1,f,f^2,\dots) \subseteq \mathcal{O}_{(\Bbbk^n,o)}$. This is a $\Bbbk$-vector subspace, not an ideal!
Observe: $T_{\mathscr{R}} f \subseteq T_{\mathscr{K}}$ and $T_{\mathscr{A}} f \subseteq T_{\mathscr{K}} f + \Bbbk$.

Altogether, we have: $T_{\mathscr{R}} f \subseteq T_{\mathscr{K}} f \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$, fin.gen. submodules. But $T_{\mathscr{A}} f \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ is not an $\mathcal{O}_{(\Bbbk^n,o)}$-module. It is only an $\mathcal{O}_{(\Bbbk^p,o)}$-module. And not finitely-generated! (see the example)

**History:** In 50's-60's J. Mather studied the $\mathscr{R},\mathscr{A}$-orbits. He had to introduce $\mathscr{K}$-equivalence because of the pathologies of $\mathscr{A}$-equivalence. The usual approach: prove smth for $\mathscr{R}$ and $\mathscr{K}$, then try to descend to $\mathscr{A}$. Here one uses: $T_{\mathscr{A}} f \subseteq T_{\mathscr{K}} f + \Bbbk^{\oplus p}$.

How large are the orbits $\mathscr{G}f$? For $\mathscr{G} \in \mathscr{R}, \mathscr{K}, \mathscr{A}$.

Our goal: to study the orbits $\mathscr{G}f$ vs $T_{\mathscr{G}}f$.

**Example:** Take the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{(\Bbbk^n,o)}$. How to ensure $\mathscr{G}f \supseteq \{f\} + \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$? The meaning: all the terms in $f$ of order$\geq d$ are irrelevant, i.e. can be removed by $\mathscr{G}$-action. (In this case $f$ is called "$(d-1)$-determined".) In particular, one can replace $f$ by a polynomial.

**Finite Determinacy theorem.** *$f \in \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ is finitely determined iff $T_{\mathscr{G}}f \subseteq \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ is of finite codimension.* *i.e.* $dim_{\Bbbk} \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}/T_{\mathscr{G}}f < \infty$.

More precisely, for $\mathscr{G} \in \mathscr{R}, \mathscr{K}$: $\mathscr{G}f \supseteq \{f\} + \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ iff $\mathfrak{m}^2 \cdot T_{\mathscr{G}}f \supseteq \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$.

For $\mathscr{G} = \mathscr{A}$:

- If $\mathfrak{m}^2 \cdot T_{\mathscr{A}}f \supseteq \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ then $\mathscr{A}f \supseteq \{f\} + \mathfrak{m}^{2d-1} \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$.
- If $\mathscr{A}f \supseteq \{f\} + \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$ then $\mathfrak{m}^2 \cdot T_{\mathscr{A}}f \supseteq \mathfrak{m}^d \cdot \mathcal{O}^{\oplus p}_{(\Bbbk^n,o)}$.

**History.** The $\mathscr{R}, \mathscr{K}$-cases are "relatively simple", e.g. [Tougeron.68]. (Maybe was known before?) The first result for $\mathscr{A}$ was [Mather.68]. Hard/long proofs, poor estimates. These were gradually improved, [Gaffney.79], [du Plessis.80], [Wall.81],..., [Wall.95]. The standard proofs use: unfoldings, vector fields integrations (ODE's), and/or affine unipotent algebraic groups.

**Problem/example.** Let $f(x,y) = x^p + y^p$, $p \geq 4$. $T_{\mathscr{R}}f = Jac(f) = (x^{p-1}, y^{p-1}) \subset \mathcal{O}_{(\Bbbk^2,o)}$. Thus $\mathfrak{m}^2 \cdot T_{\mathscr{R}}f \supseteq \mathfrak{m}^{2p-3}$. Thus $\mathscr{R}f \supseteq \{f\} + \mathfrak{m}^{2p-3}$. But in fact: $\mathscr{R}f \supseteq \{f\} + \mathfrak{m}^{2p-3} + \mathfrak{m} \cdot (x)^p + \mathfrak{m} \cdot (y)^p + \cdots$. The orbit is larger, even asymptotically!

**Intermezzo.** In 2004-2005 I have asked E.Shustin "How does one prove the finite determinacy?" His answer: "You just kill the higher order terms, one-by-one. Nothing difficult."
I read the actual proofs (for $\mathscr{R}, \mathscr{K}$ cases) only in 2010-2012. And I did not like them.
Removing higher order terms is a simple algebraic procedure. One does not need vector fields integration /the theory of affine unipotent algebraic groups (over $\mathbb{R}, \mathbb{C}$).

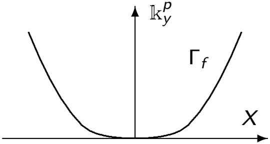
Let's begin the talk.

Let $(R_X, \mathfrak{m})$ be a local Noetherian $\Bbbk$-algebra. Thus $X := Spec(R_X)$ is a local Noetherian scheme.
**Examples.** • $R_X = \Bbbk[[x]]/_I$, i.e. $X = V(I) \subseteq (\Bbbk^n, o)$ is a formal scheme-germ over $\Bbbk$. Here $\Bbbk$ is a(ny) field, or a complete local ring. (Local rings are needed for families/deformations)
• $R_X = \Bbbk\{x\}/_I$, i.e. $X \subseteq (\Bbbk^n, o)$ is an analytic germ. Here $\Bbbk$ is a normed field (or a normed local ring), complete w.r.t. its norm. (e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \dots$)
• $R_X = \Bbbk\langle x\rangle/_I$, i.e. $X \subseteq (\Bbbk^n, o)$ is a henselian germ. Here $\Bbbk$ is a(ny) field or a henselian local ring. (Henselian germs are important in algebraic geometry, as étale neighborhoods.)

Consider $Maps := \{X \stackrel{f}{\to} Y\}$, here $Y = (\Bbbk^p, o)$ is a regular germ, e.g. $R_Y = \Bbbk[[y]]$, $\Bbbk\{y\}$, $\Bbbk\langle y\rangle$.
Fix some coordinates $y = (y_1, \dots, y_p)$ on $(\Bbbk^p, o)$, then each $f$ is presentable as $(f_1, \dots, f_p) \in \mathfrak{m} \cdot R_X^{\oplus p}$. Accordingly $Maps(X, (\Bbbk^p, o)) \cong \mathfrak{m} \cdot R_X^{\oplus p}$.

## Group actions on maps of germs of Noetherian schemes, $\mathscr{G} \circlearrowright Maps(X,(\mathbb{k}^p,o)) \cong \mathfrak{m} \cdot R_X^{\oplus p}$.

### Right equivalence




$\Phi_X \circlearrowright X$, i.e. $\Phi_X^* \in Aut_{\mathbb{k}}(R_X)$

$f \rightsquigarrow \Phi_X^*(f) := f \circ \Phi_X^{-1}$

$\mathscr{R} = Aut_{\mathbb{k}}(R_X) \circlearrowright Maps(X,(\mathbb{k}^p,o))$

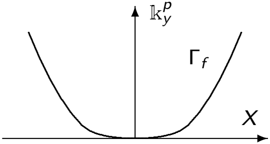
$\mathscr{R}$-action is $\mathbb{k}$-linear.

$T_{\mathscr{R}} = Der_{\mathbb{k}}(R_X) \circlearrowright R_X^{\oplus p}$

$(\xi_X, f) \to \xi_X(f)$

$T_{\mathscr{R}} f \subseteq R_X^{\oplus p}$ is an $R_X$-submodule

### Contact equivalence




$(x,y) \overset{\mathscr{K}}{\to} (\Phi_X(x), \Psi(x,y))$, $\quad \Psi(x,o) = o$

$\psi(o,y) \in Aut_{\mathbb{k}}(R_Y)$, $\quad f \rightsquigarrow \Psi(x, f \circ \Phi_X^{-1})$

$\mathscr{K} \circlearrowright Maps(X,(\mathbb{k}^p,o))$

not $\mathbb{k}$-linear, but "linearizable".

$\mathscr{K}_{Lin} := GL(R_X, p) \rtimes \mathscr{R}$

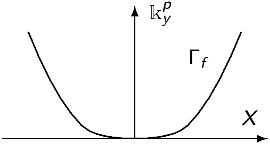
$f \rightsquigarrow U(x) \cdot f(\Phi_X^{-1}(x))$

$\mathscr{K}_{Lin}$-action is $\mathbb{k}$-linear

**Fact:** $\mathscr{K} f = \mathscr{K}_{Lin} f$, $\forall\ f$.

$T_{\mathscr{K}_{Lin}} = T_{\mathscr{R}} \oplus Mat_{p \times p}(R_X)$

### Left-Right equivalence




$\Phi_X \circlearrowright X$, $\quad \Phi_Y \circlearrowright (\mathbb{k}^p_y, o)$

$f \rightsquigarrow \Phi_Y \circ f \circ \Phi_X^{-1}$

$\mathscr{A} = Aut_{\mathbb{k}}(R_X) \times Aut_{\mathbb{k}}(R_Y)$

not $\mathbb{k}$-linear $\quad \circlearrowright Maps(X,(\mathbb{k}^p,o))$

$T_{\mathscr{A}} = Der_{\mathbb{k}}(R_Y) \oplus Der_{\mathbb{k}}(R_X) \circlearrowright R_X^{\oplus p}$

$(\xi_Y, \xi_X) f = \xi_Y(y)|_f + \xi_X(f)$.

$T_{\mathscr{A}} f = T_{\mathscr{R}} f + f^{-1}(R_Y)^{\oplus p}$

$T_{\mathscr{A}} f \subseteq R_X^{\oplus p}$ is not an $R_X -$module

$T_{\mathscr{A}} f$ is an $R_Y -$module. Not f.g.

**Goal:** Relate the properties of $\mathscr{G} f$ to those of $T_{\mathscr{G}} f$.
e.g. $\mathscr{G} f \supseteq \{f\} + J \cdot R_X^{\oplus p}$ vs $T_{\mathscr{G}} f \supseteq J \cdot R_X^{\oplus p}$. (for the largest possible $J$)

# The first criterion

Let $f \in Maps(X, (\mathbb{k}^p, o))$. The annihilator ideal, $\mathfrak{a}_{\mathscr{G}} := Ann\, R^{\oplus x}/T_{\mathscr{G}} f \subset R_X$.

**Theorem (K. 2021).** Let $R_X \in \mathbb{k}[[x]], \mathbb{k}\{x\}, \mathbb{k}\langle x\rangle$. Take ideals $\mathfrak{a} \subseteq I \subseteq (x) \subset R_X$ and a map $f \in I \cdot R_X^{\oplus p}$.

- **($\mathscr{R}$)** Suppose $(f) \subseteq (x)^2$. If $\mathfrak{a}^2 \cdot I^{ord(f)-2} \subseteq (x) \cdot \mathfrak{a} \cdot \mathfrak{a}_{\mathscr{R}}$ then $\{f\} + \mathfrak{a} \cdot T_{\mathscr{R}} f \subseteq \mathscr{R} f$.
- **($\mathscr{K}$)** Suppose $(f) \subseteq (x)^2$ and $p > 1$. If $\mathfrak{a}^2 \cdot I^{ord(f)-2} \subseteq \mathfrak{a} \cdot \mathfrak{a}_{\mathscr{K}}$ then $\{f\} + \Big(\mathfrak{a} \cdot \mathfrak{a}_{\mathscr{K}} + \mathfrak{m} \cdot (f)\Big) \cdot R_X^{\oplus p} \subseteq \mathscr{K} f$.
- **($\mathscr{A}$)** Suppose $\mathfrak{a}$ satisfies: $\mathfrak{a}^2 \cdot I^{ord(f)-2} \cdot R_X^{\oplus p} \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot T_{\mathscr{R}} f + f^{-1}(y^2) \cdot T_{\mathscr{L}} f$.
  Then $\mathscr{A} f \supseteq \{f\} + \mathfrak{a}^2 \cdot I^{ord(f)-2} \cdot R_X^{\oplus p} + f^{-1}(y^2) \cdot T_{\mathscr{L}} f$.

**Example.** Let $\mathscr{G} = \mathscr{R}$, $p = 1$, $\mathbb{k}$-any field. Then $T_{\mathscr{R}} f = Jac(f)$ and $\mathfrak{a}_{\mathscr{R}} = Jac(f)$.

1. Take $\mathfrak{a} = \mathfrak{m} \cdot Jac(f)$. Then $\mathfrak{a}^2 \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot Jac(f)$. Thus $\mathscr{R} f \supseteq \{f\} + \mathfrak{m} \cdot Jac(f)^2$.
2. Suppose $ord(f) \geq 3$. Take $\mathfrak{a} = Jac(f) \subseteq I := \mathfrak{m}$. Then $\mathfrak{a}^2 \cdot I^{ord(f)-2} \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot Jac(f)$. Thus $\mathscr{R} f \supseteq \{f\} + Jac(f)^2$.
3. Compare the theorem to the standard bound: "If $\mathfrak{m}^2 \cdot Jac(f) \supseteq \mathfrak{m}^d$ then $\mathscr{R} f \supseteq \{f\} + \mathfrak{m}^d$." e.g. let $f$ be an ordinary multiple point. One can show: $d = n(ord(f) - 2) - 1$. And our bound gives: $\mathscr{R} f \supseteq \{f\} + \mathfrak{m}^{\lceil \frac{d-\mathbf{1}-ord(f)}{\mathbf{2}} \rceil} \cdot Jac(f)$. Here $\mathfrak{m}^{\lceil \frac{d-\mathbf{1}-ord(f)}{\mathbf{2}} \rceil} \cdot Jac(f) \supset \mathfrak{m}^d$ and is essentially larger. (Even asymptotically.)

Let $R_X$ be local Noetherian. Fix an ideal $I\subseteq\mathfrak{m}\subset R_X$.
Take the filtration on the space of maps, $Maps(X,(\Bbbk^p,o))\cong\mathfrak{m}\cdot R_X^{\oplus p}\supset I^\bullet\cdot R_X^{\oplus p}$.
For an(y) action $\mathscr{G}\circlearrowright R_X^{\oplus p}$ we get the subgroup $\mathscr{G}\geq\mathscr{G}^{(0)}=\{g\in\mathscr{G}|\ g(I^d\cdot R_X^{\oplus p})=I^d\cdot R_X^{\oplus p}\}$.
Moreover, we define $\mathscr{G}^{(0)}\triangleright\mathscr{G}^{(1)}=\{g\in\mathscr{G}^{(0)}|\ g|_{I^d/I^{d+1}}=Id|_{I^d/I^{d+1}}\}$. More generally,
$\mathscr{G}^{(0)}\triangleright\mathscr{G}^{(j)}=\{g\in\mathscr{G}^{(0)}|\ g|_{I^d/I^{d+j}}=Id|_{I^d/I^{d+j}}\}$, for every $j\geq 1$.
Thus we get the group-filtration: $\mathscr{G}\geq\mathscr{G}^{(0)}\triangleright\mathscr{G}^{(1)}\dots$. (unipotent filtration)

Similarly, for a tangent space $T_{\mathscr{G}}\circlearrowright R_X^{\oplus p}$ we get
$T_{\mathscr{G}}\supset T_{\mathscr{G}^{(0)}}=\{\xi\in T_{\mathscr{G}}|\ \xi(I^d\cdot R_X^{\oplus p})\subseteq I^d\cdot R_X^{\oplus p}\}$. Moreover, we consider
$T_{\mathscr{G}}\supset T_{\mathscr{G}^{(j)}}=\{\xi\in T_{\mathscr{G}}|\ \xi(I^d\cdot R_X^{\oplus p})\subseteq I^{d+j}\cdot R_X^{\oplus p}\}$, for every $j\geq 1$. Thus we get the tangent space filtration: $T_{\mathscr{G}}\supset T_{\mathscr{G}^{(0)}}\supseteq\cdots$. (nilpotent filtration)

**Example.** $\mathscr{G}=\mathscr{R}$, $I=\mathfrak{m}$. Then $\mathscr{R}=\mathscr{R}^{(0)}$, as local coordinate changes preserve the origin, i.e. the autos of a local ring preserve $\mathfrak{m}$. $\mathscr{R}^{(j)}=\{x\to x+\phi(x)|\phi(x)\in(x)^{j+1}\}$.
$T_{\mathscr{R}}=Der_{\Bbbk}(R_X)\supset T_{\mathscr{R}^{(0)}}=\{\xi|\ \xi(\mathfrak{m})\subseteq\mathfrak{m}\}\supset T_{\mathscr{R}^{(j)}}=\{\xi|\ \xi(\mathfrak{m})\subseteq\mathfrak{m}^{j+1}\}$.

The pairs $(T_{\mathscr{G}^{(j)}},\mathscr{G}^{(j)})$, for $j\geq 1$, are simpler than $(T_{\mathscr{G}},\mathscr{G})$, because the filtration induces topology. Then one can define the "convergence" and the maps $Exp[\dots]$, $Ln[\dots]$. In this way one can "integrate vector fields".

**Theorem (K.2021).** Let $\mathscr{G}\in\mathscr{R},\mathscr{K}$. Fix some integers $1\leq j<d$, suppose $2,\dots,\lceil\frac{d-ord(f)}{j}\rceil\in R^\times$. (And a technical condition.) Then:

$$\overline{\mathscr{G}^{(j)}f}\supseteq\{f\}+I^d\cdot R_X^{\oplus p}\qquad\text{if and only if}\qquad T_{\mathscr{G}^{(j)}}f\supseteq I^d\cdot R_X^{\oplus p}.$$

**Theorem (K.2021).** Let $\mathscr{G} \in \mathscr{R}, \mathscr{K}$. Fix some integers $1 \leq j < d$, suppose $2, \ldots, \lceil \frac{d - ord(f)}{j} \rceil \in R^\times$. (And a technical condition.) Then:

$$\overline{\mathscr{G}^{(j)} f} \supseteq \{f\} + I^d \cdot R_X^{\oplus p} \qquad \text{if and only if} \qquad T_{\mathscr{G}^{(j)}} f \supseteq I^d \cdot R_X^{\oplus p}.$$

**Remarks**

- Here $\overline{\mathscr{G}^{(j)} f}$ is the closure in the filtration topology, $\overline{\mathscr{G}^{(j)} f} = \cap_\bullet \left(\mathscr{G}^{(j)} f + I^\bullet \cdot R_X^{\oplus p}\right)$. Thus $\overline{\mathscr{G}^{(j)} f} \supseteq \{f\} + I^d \cdot R_X^{\oplus p}$ means: $\{f\} + I^d \cdot R_X^{\oplus p} \overset{\mathscr{G}^{(j)}}{\rightsquigarrow} \{f\} + I^{d+1} \cdot R_X^{\oplus p} \overset{\mathscr{G}^{(j)}}{\rightsquigarrow} \{f\} + I^{d+2} \cdot R_X^{\oplus p} \cdots$.
  For many rings the orbit is closed, $\overline{\mathscr{G}^{(j)} f} = \mathscr{G}^{(j)} f$. e.g. for $R = \Bbbk[[x]]$, $\Bbbk\{x\}$, $\Bbbk\langle x \rangle$, $\Bbbk$ any field, also for many rings with $R \supseteq \mathbb{Q}$.
- *The technical condition:* given $\xi \in T_{\mathscr{G}^j}$ and $N \in \mathbb{N}$, take the truncated Exp: $Id + \xi + \cdots + \frac{\xi^N}{N!}$. This is a self-map of $R_X^{\oplus p}$, but is not necessarily an element of $\mathscr{G}$.
  We say " $jet_N(Exp)$ holds", if $Id + \xi + \cdots + \frac{\xi^N}{N!} + \phi \in \mathscr{G}$, where $\phi$ is of "high order". And similarly for $jet_N(Ln)$. This condition holds for many rings, also in positive characteristic, [BGK].
- For $\mathscr{G} = \mathscr{A}$ the statement is similar.
- This theorem generalizes the classical criteria over $\mathbb{R}, \mathbb{C}$. Without any ODE's, vector field integration. Yet, this result is restricted, e.g. one needs $2, \ldots, N \in R^\times$. (What happens in low characteristic?) And one would like more general fitrations, $M_\bullet \subset R_X^{\oplus p}$, rather than $I^\bullet \cdot R_X^{\oplus p}$.
  In [Belitski-K.] we have developed the machinery to address this when $\Bbbk \supseteq \mathbb{Q}$.
  Then we obtain the classical/expected statements.
  When $\Bbbk \not\supseteq \mathbb{Q}$, and the characteristic is low, the situation is more delicate. This was studied in [Boix-Greuel-K.] for $\mathscr{G} \in \mathscr{R}, \mathscr{K}$.

# An application: relative algebraization of maps

Let $f \in Maps(X, (\mathbb{k}^p, o)) \cong \mathfrak{m} \cdot R_X^{\oplus p}$. Suppose $f$ is finitely $\mathscr{G}$-determined, i.e. $\mathscr{G}f \supseteq \{f\} + \mathfrak{m}^d \cdot R_X^{\oplus p}$, for $d \gg 1$. Then in particular: $f \overset{\mathscr{G}}{\sim}$(a polynomial map).

What to do when $f$ is not finitely $\mathscr{G}$-determined? (i.e. its critical/singular/instability locus is non-isolated)

**Example (Whitney):** $f(x, y, z) = xy(x + y)(x - zy)(x - e^z y) \in \mathbb{C}\{x, y, z\}$ is not $\mathscr{K}$-equivalent to a polynomial. Here $Sing(V(f)) = V(x, y) \subset \mathbb{C}^3$. But at least $f$ is a polynomial "in the direction transverse to the $\hat{z}$-axis".

The natural generalization:

**Proposition.** Let $R_X = \mathbb{k}[[x]]$, $\mathbb{k}$ a(ny) field, and $f \in \mathfrak{m}^2 \cdot R_X^{\oplus p}$. Let $\mathscr{G} \in \mathscr{R}, \mathscr{K}, \mathscr{A}$. Suppose the annihilator ideal $\mathfrak{a}_{\mathscr{G}} \subset R_X$ is of height $c$. Then $f$ is $\mathscr{G}$-equivalent to an element of $R_{n-c}[x_1, \ldots, x_c]^{\oplus p}$, here $R_{n-c}$ is $\mathbb{k}[[x_{c+1}, \ldots, x_n]]$.

*Thanks for your attention!*