Proof of Morrison-Kawamata cone conjecture for holomorphically symplectic manifolds

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The Kähler cone and its faces

This is **joint work with Ekaterina Amerik**.

DEFINITION: Let M be a compact, Kähler manifold, $\operatorname{Kah} \subset H^{1,1}(M, \mathbb{R})$ is Kähler cone (set of all Kähler classes), and $\overline{\operatorname{Kah}}$ its closure in $H^{1,1}(M, \mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\operatorname{Kah}}$ and a hyperplane $V \subset H^{1,1}(M, \mathbb{R})$ which has non-empry interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)
Let M be a Calabi-Yau manifold. Then the group $\operatorname{Aut}(M)$ of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

THEOREM: Morrison-Kawamata cone conjecture is true when M is holomorphically symplectic.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M,I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q **has signature** $(3, b_2-3)$. It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Monodromy group

DEFINITION: Monodromy group $\operatorname{Mon}(M)$ of a hyperkähler manifold (M,I) is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ generated by monodromy of Gauss-Manin connections for all families of deformations of (M,I). The **Hodge monodromy group** $\operatorname{Mon}(M,I)$ is a subgroup of $\operatorname{Mon}(M)$ preserving the Hodge decomposition.

REMARK: Define **pseudo-isomorphism** $M \longrightarrow M'$ as a birational map which is an isomorphism outside of codimension $\geqslant 2$ subsets of M, M'.

For any pseudo-isomorphic manifolds M, M', one has $H^2(M) = H^2(M')$.

DEFINITION: Let (M,I') be a holomorphic symplectic manifold pseudo-isomorphic to (M,I). A **Kähler chamber** of (M,I) is an image of the Kähler cone of (M,I') under the action of $\operatorname{Mon}(M,I)$.

CLAIM: $\operatorname{Mon}(M,I)$ acts on $H^{1,1}(M,I)$ mapping Kähler chambers to Kähler chambers.

CLAIM: The group of automorphisms $\operatorname{Aut}(M,I)$ is a group of all elements of $\operatorname{Mon}(M,I)$ preserving the Kähler cone.

Ample cone and Morrison-Kawamata cone conjecture

DEFINITION: Let P be the set of all real vectors in $H^{1,1}(M,I)$ satisfying $q(v,v) > 0$, where q is the Bogomolov-Beauville-Fujiki form on $H^2(M)$. The **positive cone** $\operatorname{Pos}(M,I)$ as a connected component of P containing a Kähler form. Then $\mathbb{P}\operatorname{Pos}(M,I)$ is a hyperbolic space, and $\operatorname{Aut}(M,I)$ acts on $\mathbb{P}\operatorname{Pos}(M,I)$ by hyperbolic isometries.

DEFINITION: Let $H^{1,1}(M,\mathbb{Q})$ be the set of all rational (1,1)-classes on (M,I), and $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ the set of all Kähler classes in $H^{1,1}(M,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}$. Then $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ is called **ample cone** of M.

REMARK: From global Torelli theorem it follows that $\operatorname{Mon}(M,I)$ is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$. Therefore, $\operatorname{Mon}(M,I)$ **acts on** $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I) := \mathbb{P}(\operatorname{Pos}(M,I)\cap H^{1,1}(M,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R})$ **with finite covolume;** in other words, the quotient $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$ is a finite volume hyperbolic orbifold.

THEOREM: (cone conjecture for hyperkähler manifolds)
The quotient $\operatorname{Kah}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$ is a finite hyperbolic polyhedron in $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$.

REMARK: In other words, **the action of $\operatorname{Aut}(M,I)$ on $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ has a finite polyhedral fundamental domain.**

MBM classes

DEFINITION: **Negative class** on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

DEFINITION: Let (M,I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M,I)$ is called **minimal** if for any $\mathbb{Q}$-effective homology classes $z_1, z_2 \in H_{1,1}(M,I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. A negative rational homology class $z \in H_{1,1}(M,I)$ is called **monodromy birationally minimal** (MBM) if $\gamma(z)$ is minimal and $\mathbb{Q}$-effective for one of birational models (M,I') of (M,I), where $\gamma \in O(H^2(M))$ is an element of the monodromy group of (M,I).

This property is **deformationally invariant**.

THEOREM: Let $z \in H^2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type (1,1) with respect to I and I'. Then **η is MBM in (M,I) $\Leftrightarrow$ it is MBM in (M,I')**.

DEFINITION: Let $z \in H^2(M,\mathbb{Z})$ be a negative class on a hyperkähler manifold (M,I). It is called **an MBM class** if for any complex structure I' in the same deformation class satisfying $z \in H^{1,1}(M,I')$, z is an MBM class.

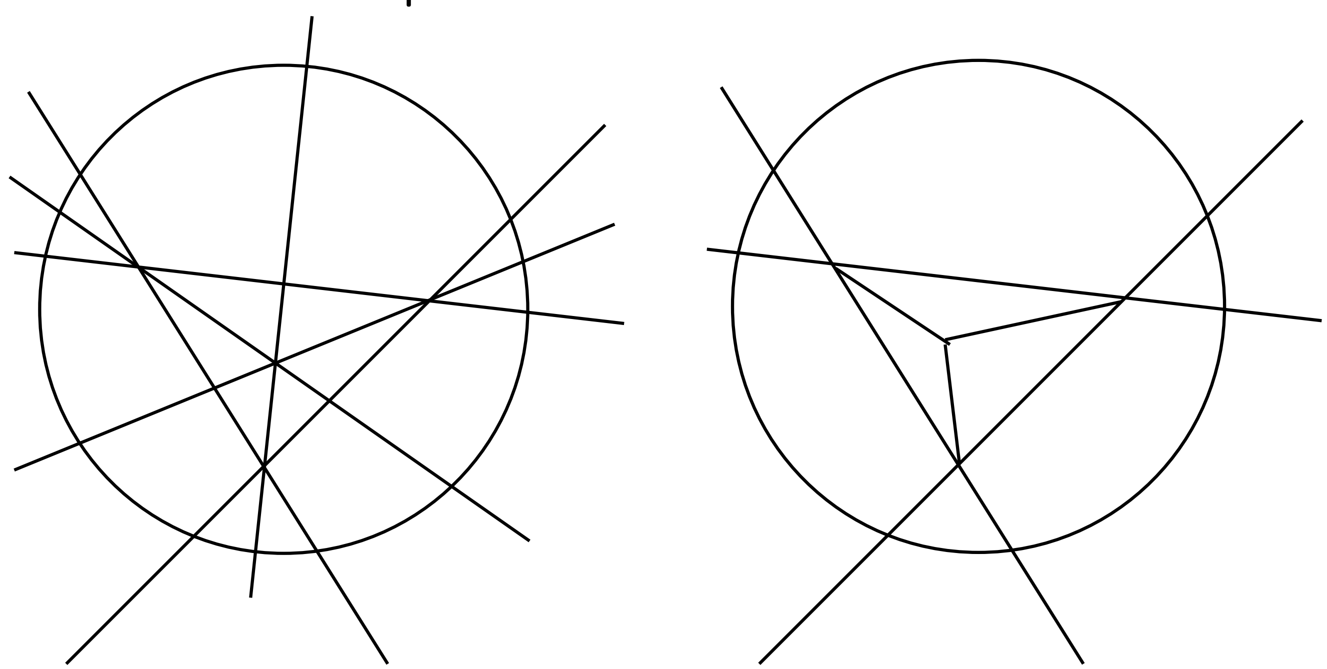
MBM classes and the Kähler cone

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \mid \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

MBM classes and the Kähler cone: the picture

REMARK: This implies that $z^\perp \cap \operatorname{Pos}(M,I)$ **either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM);** that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition **Prohibited partition**

MBM classes and cone conjecture

PROPOSITION: Suppose that $\operatorname{Mon}(M,I)$ acts on the set of MBM classes in $H^{1,1}(M,I)$ with finitely many orbits. **Then cone conjecture is true for (M,I).**

Proof: MBM classes are the faces of the Kähler cone. ■

THEOREM: Let X be a complete Riemannian orbifold of dimension at least three, constant negative curvature and finite volume, and $\{S_i\}$ an infinite set of complete, locally geodesic hypersurfaces. **Then the union of S_i is dense in X.**

COROLLARY: Let M be a simple hyperkähler manifold with $b_2(M) \geqslant 6$. **Then the group of automorphisms $\operatorname{Aut}(M)$ acts with finitely many orbits on the set of faces of the Kähler cone $\operatorname{Kah}(M)$.**

Proof: Consider a hyperbolic orbifold $X := \operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$, let $\tilde{S}_i \subset \operatorname{Pos}_{\mathbb{Q}}(M,I)$ the hyperplanes $s_i^{\perp}$, for all MBM classes $s_i \in H^{1,1}(M,I)$, and S_i their images in X. Since the ample cone is a connected component of $\operatorname{Pos}_{\mathbb{Q}}(M,I)\backslash \bigcup \tilde{S}_i$, the union of S_i cannot be dense in X. Therefore, $\operatorname{Mon}(M,I)$ acts on the faces $\{\tilde{S}_i\}$ with finitely many orbits. ■

Ratner's orbit closure theorem

DEFINITION: Let G be a Lie group, and $\Gamma \subset G$ a discrete subgroup. We say that Γ **has finite covolume** if the Haar measure of G/Γ is finite. In this case Γ is called **a lattice subgroup**.

REMARK: Borel and Harish-Chandra proved that an arithmetic subgroup of a reductive group G is a lattice whenever G has no non-trivial characters over $\mathbb{Q}$. In particular, **all arithmetic subgroups of a semi-simple group are lattices.**

DEFINITION: Let G be a Lie group, and $g \in G$ any element. We say that g is **unipotent** if $g = e^h$ for a nilpotent element h in its Lie algebra. A group G is **generated by unipotents** if G is multiplicatively generated by unipotent one-parameter subgroups.

THEOREM: (Ratner orbit closure theorem)
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. **Then the closure of any H-orbit Hx in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap x\Gamma x^{-1} \subset S$ is a lattice in S.**

Ratner's measure classification theorem

DEFINITION: Let (M, μ) be a space with a measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

REMARK: Ergodic measures are extremal rays in the cone of all G-invariant measures.

REMARK: By Choquet's theorem, **any G-invariant measure on M is expressed as an average of a certain set of ergodic measures.**

DEFINITION: Let G be a Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider an orbit $S \cdot x \subset G$ of a closed subgroup $S \subset G$, put the Haar measure on $S \cdot x$, and assume that its image in G/Γ is closed. A measure on G/Γ is called **algebraic** if it is proportional to the pushforward of the Haar measure on $S \cdot x/\Gamma$ to G/Γ.

THEOREM: (Ratner's measure classification theorem)
Let G be a connected Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider a finite measure μ on G/Γ. Assume that μ is invariant and ergodic with respect to an action of a subgroup $H \subset G$ generated by unipotents. **Then μ is algebraic.**

Mozes-Shah and Dani-Margulis

THEOREM: (Mozes-Shah)
A limit of algebraic measures is again an algebraic measure.

Proof: Follows from Ratner's measure classification theorem. ■

THEOREM: (a corollary of Mozes-Shah and Dani-Margulis theorem)
Let G be a connected Lie group, Γ a lattice, $\mathcal{P}(X)$ be the space of all finite measures on $X = G/\Gamma$, and $\mathcal{Q}(X) \subset \mathcal{P}(X)$ the space of all algebraic measures associated with subgroups $H \subset G$ generated by unipotents (as in Ratner theorems). **Then $\mathcal{Q}(X)$ is closed in $\mathcal{P}$.**

THEOREM: Let X be a complete Riemannian orbifold of dimension at least three, constant negative curvature and finite volume, and $\{S_i\}$ a set of complete, locally geodesic hypersurfaces. **Then the union of S_i is dense in X,** unless there are only finitely many of S_i.

Proof: Denote by μ_i the algebraic measure supported in S_i. Since the space of probabilistic measures is compact, μ_i converge to an algebraic measure on X. However, any orbit of a subgroup strictly containing S_i must coincide with X. **Therefore, there is either finitely many of S_i or their union is dense.**
■