Deformation of curves on surfaces

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- Method: Direct calculation of obstructions $\cdots$ Possible for *singular* curves

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Solved affirmatively by Harris (1986).

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- Non-rational curves in very ample classes on K3 surfaces can be deformed to immersion (Dedieu-Sernesi)

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- $S \subset \mathbb{P}^3$: general surface, $n \leq \dim(|\mathcal{O}_S(m)|)$, Then $V_{m,n}$ has at least one component of expected dimension (Chiantini-Ciliberto)

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- Defect: there is no control on the geometry of deformed curves

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Here we call φ semiregular if the natural map $H^0(S, K_S) \to H^0(C, \varphi^* K_S)$ is surjective

Ideas and main theorem

Cohomological pairings as residues

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- These form a Čech 1-cocycle associated with the cover $\{U_i\}$, and φ deforms if and only if the corresponding cohomology class in $H^1(C, N_\varphi)$ vanishes

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- We can apply the residue calculation to them

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- In general, the obstruction to deforming φ_N does not vanish
- It means that we cannot deform φ_N no matter how hard we try

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- Eventually, we can construct a formal deformation of φ

More details

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At p_i, the pull back of coordinates on S can be written in the form

$$(z_i, w_i) = (s^a,\ s^b + s^{b+1} g_0(s))$$

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$a - 1$ is the multiplicity of the singularity p_i, that is, the coefficient of p_i of the divisor $Z = (d\varphi)$

Its deformation can be written as

$$(z_i, w_i) = \left(s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} t^j c_{a-i,j} s^i,\ \ s^b + s^{b+1} g_0(s) + \sum_{j=1}^{k} t^j g_j(s)\right)$$

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It is convenient to consider deformations of the form

$$(z_i, w_i) = (s^a + \sum_{i=0}^{a-2} c_{a-i} s^i,\ \ s^b + s^{b+1} g_0(s) + \sum_{j=1}^{k} t^j g_j(s)),$$

where $c_{a-i} \in t^{a-i}\mathbb{C}[[t]]$

Among such deformations, we consider those of the form

$$(z_i, w_i) = (S^a,\ S^b + S^{b+1} g_0(S)),$$

where $S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} (\frac{1}{a} - j) \frac{1}{i!} (\sum_{k=2}^{a} \frac{c_k}{s^k})^i)$

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- S is a reparameterization of C on a *punctured* neighborhood of p_i
- It gives the same image as the original

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- At some order t^N, $S^b + S^{b+1}g_0(S)$ acquires singular terms, and it produces the obstruction
- We modify the value of c_i so that the obstruction vanishes (this is where we use the transversality assumption of the polynomial system)

- If the obstruction vanishes, then we can modify the curve in the form $(z_i, w_i) = (S^a,\ S^b + S^{b+1} g_0(S) + H_i(s))$, and continue the deformation
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- As we mentioned earlier, this changes the map at the order lower than t^N
- We can check that the new map can be deformed beyond the order t^N

What is the system of polynomial equations?

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Substituting $S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} (\frac{1}{a} - j) \frac{1}{i!} (\sum_{k=2}^{a} \frac{c_k}{s^k})^i)$ to $S^b + S^{b+1} g_0(S)$, we have

$$S^b + S^{b+1} g_0(S) = s^b(1 + \sum_{i=1}^{\infty} f_i^{(b)}(c_2, \ldots, c_a) \frac{1}{s^i}) + \text{(higher order terms)}$$

- $f_i^{(b)}$ is given by

$$f_{b+j}^{(b)}(c_2, \ldots, c_a) = \sum_{\lambda \in \mathcal{P}(b+j;[2,a])} \binom{\frac{b}{a}}{\lambda(2) \; \cdots \; \lambda(a)} c_2^{\lambda(2)} \cdots c_a^{\lambda(a)}$$

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- $\mathcal{P}(b+j;[2,a])$ is the set of partitions of $b+j$ using only the integers in the interval $[2,a]$

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$$f_{b+j}^{(b)}(c_2, \ldots, c_a) = \sum_{\lambda \in \mathcal{P}(b+j;[2,a])} \binom{\frac{b}{a}}{\lambda(2) \ \cdots \ \lambda(a)} c_2^{\lambda(2)} \cdots c_a^{\lambda(a)}$$

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- $$\binom{\alpha}{\beta_1 \ \cdots \ \beta_k} = \frac{\prod_{i=0}^{\beta_1+\cdots+\beta_k-1}(\alpha - i)}{\beta_1! \cdots \beta_k!}$$

$$S^b + S^{b+1} g_0(S) = \text{(regular part)} + \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} + \text{(higher order terms)}$$

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- If $\eta \in H^0(C, \varphi^* K_S(Z))$, then the pairing between the obstruction class and η has contribution from the singular point p_i, given by the residue of

$$\langle \eta, \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} \rangle$$

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(G) The varieties defined by

$$\begin{cases} \bar{f}_{b+j}^{(b)} = 0, & j \in [1, a-1] \setminus \{k\}, \\ \bar{f}_{b+k}^{(b)} = C \neq 0, \end{cases}$$

have transverse intersection at some point for each k
$\bar{f}_{b+j}^{(b)}$ are modified version of $f_{b+j}^{(b)}$

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A singular point p_i of φ satisfies the condition $\mathbf{(D)}$ if the inequality

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The singularity has $a_i - 1$ parameters $c_2, \dots, c_{a_i}$ of deformations
So, roughly this condition says the expected dimension of local deformation is positive

Main theorem

Assume φ is semiregular. If the conditions $\mathbf{(G)}$ and $\mathbf{(D)}$ are satisfied at each $p_i \in \{p_1 \ldots, p_e\}$, then there is a non-trivial deformation of φ.

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- After checking the condition $\mathbf{(D)}$, one can completely forget curves and surfaces
- The problem reduces to studying a system of polynomial equations which depends only on two positive integers a and b

Table:

a=3	(G) holds for $\mathbf{4 \leq b \leq 30}$
4	(G) holds for $\mathbf{5 \leq b \leq 30}$ except $\mathbf{b = 6}$
5	(G) holds for $\mathbf{6 \leq b \leq 30}$
6	(G) holds for $\mathbf{7 \leq b \leq 30}$
7	(G) holds for $\mathbf{8 \leq b \leq 20}$
8	(G) holds for $\mathbf{9 \leq b \leq 20}$
9	(G) holds for $\mathbf{10 \leq b \leq 20}$
10	(G) holds for $\mathbf{11 \leq b \leq 15}$

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Let $\varphi : C \to S$ be a semiregular map whose singularities $p_1, \ldots, p_l$ satisfy $a = 2$.

Theorem

φ deforms if and only if at least one of the following conditions holds.

- There is at least one p_i such that there is no section of $H^0(C, \varphi^*\omega_X(p_i)) \setminus H^0(C, \varphi^*\omega_X)$.
- The set $H^0(C, \bar{N}_\varphi)$ is not zero.

For $a = 3$, condition (G) is reduced to following.

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For $b = 6k + 1$, set

$$f(X) = \binom{2k + \frac{1}{3}}{3k+1 \quad 0} X^k + \binom{2k + \frac{1}{3}}{3k-2 \quad 2} X^{k-1} + \cdots + \binom{2k + \frac{1}{3}}{1 \quad 2k}$$

$$g(X) = \binom{2k + \frac{1}{3}}{3k \quad 1} X^k + \binom{2k + \frac{1}{3}}{3k-3 \quad 3} X^{k-1} + \cdots + \binom{2k + \frac{1}{3}}{0 \quad 2k+1}$$

$$\binom{\alpha}{\beta_1 \quad \beta_2} = \frac{\prod_{i=0}^{\beta_1+\beta_2}(\alpha - i)}{\beta_1 ! \beta_2 !}$$

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- the same holds when we exchange f and g.