Deformation of curves on surfaces

Takeo Nishinou

Rikkyo University

2023.11
1 Overview

2 Background

3 Ideas and main theorem
Object to study: Deformation theoretic properties of algebraic curves on surfaces
Overview

- Object to study: Deformation theoretic properties of algebraic curves on surfaces
- Problem: In general, deformation theory with possibly nontrivial obstruction classes is difficult
Deformation of curves on surfaces

Overview

Object to study: Deformation theoretic properties of algebraic curves on surfaces

Problem: In general, deformation theory with possibly nontrivial obstruction classes is difficult

Method: Direct calculation of obstructions
Object to study: Deformation theoretic properties of algebraic curves on surfaces

Problem: In general, deformation theory with possibly nontrivial obstruction classes is difficult

Method: Direct calculation of obstructions

... Possible for singular curves
The study of curves on surfaces has long history.
The study of curves on surfaces has long history. From deformation theoretic viewpoint, one of famous problems is the Severi’s problem:
The study of curves on surfaces has long history. From the deformation theoretic viewpoint, one of the famous problems is the Severi’s problem:

- Is the moduli space of nodal plane curves of given degree and genus irreducible?
The study of curves on surfaces has long history. From deformation theoretic view point, one of famous problems is the Severi’s problem:

- Is the moduli space of nodal plane curves of given degree and genus irreducible?

Solved affirmatively by Harris (1986).
Related problem:
Related problem:

- Given a singular curve on a surface, is it possible to deform it to a nodal or immersed curve without changing geometric genus?
Related problem:

- Given a singular curve on a surface, is it possible to deform it to a nodal or immersed curve without changing geometric genus?

Several positive answers are known:
Deformation of curves on surfaces

Background

Related problem:

- Given a singular curve on a surface, is it possible to deform it to a nodal or immersed curve without changing geometric genus?

Several positive answers are known:

- Any integral curve on Hirzebruch surfaces can be deformed to nodal (Harris)
Related problem:

- Given a singular curve on a surface, is it possible to deform it to a nodal or immersed curve without changing geometric genus?

Several positive answers are known:

- Any integral curve on Hirzebruch surfaces can be deformed to nodal (Harris)
- Curves in multiple of anti-canonical classes on del Pezzo surfaces can be deformed to nodal (Harris)
Related problem:

- Given a singular curve on a surface, is it possible to deform it to a nodal or immersed curve without changing geometric genus?

Several positive answers are known:

- Any integral curve on Hirzebruch surfaces can be deformed to nodal (Harris)
- Curves in multiple of anti-canonical classes on del Pezzo surfaces can be deformed to nodal (Harris)
- Non-rational curves in very ample classes on K3 surfaces can be deformed to immersion (Dedieu-Sernesi)
In this direction, few results are known for surfaces of general type.
In this direction, few results are known for surfaces of general type. Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes.
In this direction, few results are known for surfaces of general type. Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes. Some known results:
In this direction, few results are known for surfaces of general type.
Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes.
Some known results:

- $S$: surface, $K_S$: ample, $C \in |pK_S|$, $p \geq 2$. If $C$ is nodal and $n$ is small, then $V_{C,n}$ is smooth of expected dimension at $C$ (Chiantini-Sernesi)
In this direction, few results are known for surfaces of general type.
Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes

Some known results:

- $S$: surface, $K_S$: ample, $C \in |pK_S|$, $p \geq 2$. If $C$ is nodal and $n$ is small, then $V_{C,n}$ is smooth of expected dimension at $C$ (Chiantini-Sernesi)
In this direction, few results are known for surfaces of general type. Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes.

Some known results:

- $S$: surface, $K_S$: ample, $C \in |pK_S|$, $p \geq 2$. If $C$ is nodal and $n$ is small, then $V_{C,n}$ is smooth of expected dimension at $C$ (Chiantini-Sernesi).
- $S \subset \mathbb{P}^3$: general surface, $n \leq \dim(|O_S(m)|)$,
In this direction, few results are known for surfaces of general type. Known results mainly concern properties of the Severi variety $V_{C,n}$, the moduli of nodal curves in given linear equivalence class and with $n$ nodes.

Some known results:

- $S$: surface, $K_S$: ample, $C \in \mid pK_S \mid$, $p \geq 2$. If $C$ is nodal and $n$ is small, then $V_{C,n}$ is smooth of expected dimension at $C$ (Chiantini-Sernesi).

- $S \subset \mathbb{P}^3$: general surface, $n \leq \dim(\mid O_S(m) \mid)$, Then $V_{m,n}$ has at least one component of expected dimension (Chiantini-Ciliberto).
In general, study of curves on surfaces of general type is very hard.
In general, study of curves on surfaces of general type is very hard
One exception: semiregularity
In general, study of curves on surfaces of general type is very hard

One exception: semiregularity

- A curve \( i : C \hookrightarrow S \) is semiregular iff the map \( H^0(S, K_S) \to H^0(C, i^* K_S) \) is surjective
In general, study of curves on surfaces of general type is very hard

One exception: semiregularity

- A curve $i: C \hookrightarrow S$ is semiregular iff the map $H^0(S, K_S) \rightarrow H^0(C, i^*K_S)$ is surjective
- If $C$ is semiregular, then it is unobstructed in the sense that any first order deformation can be extended to arbitrary high order (Severi, Kodaira-Spencer, Bloch)
In general, study of curves on surfaces of general type is very hard

One exception: semiregularity

- A curve $i : C \hookrightarrow S$ is semiregular iff the map $H^0(S, K_S) \to H^0(C, i^* K_S)$ is surjective
- If $C$ is semiregular, then it is unobstructed in the sense that any first order deformation can be extended to arbitrary high order (Severi, Kodaira-Spencer, Bloch)
- Defect: there is no control on the geometry of deformed curves
Rough statement of our result:
Rough statement of our result: We consider a semiregular map $\varphi : C \to S$ from a smooth curve to a surface birational to the image.
Rough statement of our result: We consider a semiregular map $\varphi: C \rightarrow S$ from a smooth curve to a surface birational to the image.

- Reduce the deformation problem of the map to some system of polynomial equations.
Rough statement of our result: We consider a semiregular map $\varphi: C \to S$ from a smooth curve to a surface birational to the image

- Reduce the deformation problem of the map to some system of polynomial equations
- Under some transversality assumption on this system, we can solve it
Rough statement of our result: We consider a semiregular map \( \varphi: C \rightarrow S \) from a smooth curve to a surface birational to the image

- Reduce the deformation problem of the map to some system of polynomial equations
- Under some transversality assumption on this system, we can solve it
- As a result, we will see that if \( \varphi \) is semiregular, it will have good deformation theoretic property almost as optimal as possible
Rough statement of our result: We consider a semiregular map $\varphi : C \to S$ from a smooth curve to a surface birational to the image

- Reduce the deformation problem of the map to some system of polynomial equations
- Under some transversality assumption on this system, we can solve it
- As a result, we will see that if $\varphi$ is semiregular, it will have good deformation theoretic property almost as optimal as possible

Here we call $\varphi$ semiregular if the natural map $H^0(S, K_S) \to H^0(C, \varphi^* K_S)$ is surjective
Ideas and main theorem
Cohomological pairings as residues

Given \( \varphi: C \to S \), its obstruction class to deforming is represented by a Čech 1-cocycle:
Cohomological pairings as residues

Given $\varphi: C \to S$, its obstruction class to deforming is represented by a Čech 1-cocycle: 

$\{U_i\}$: open cover of $C$
Given $\varphi : C \to S$, its obstruction class to deforming is represented by a Čech 1-cocycle:

- $\{U_i\}$: open cover of $C$
- $\tilde{\varphi}_i$: local deformation of $\varphi|_{U_i}$
Cohomological pairings as residues

Given $\varphi: C \to S$, its obstruction class to deforming is represented by a Čech 1-cocycle:

$\{U_i\}$: open cover of $C$

$\tilde{\varphi}_i$: local deformation of $\varphi|_{U_i}$

- The difference between $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ naturally gives a section of the normal sheaf $N_{\varphi}$ on $U_i \cap U_j$
Cohomological pairings as residues

Given $\varphi : C \to S$, its obstruction class to deforming is represented by a Čech 1-cocycle:

$\{U_i\}$: open cover of $C$

$\tilde{\varphi}_i$: local deformation of $\varphi|_{U_i}$

- The difference between $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ naturally gives a section of the normal sheaf $N_\varphi$ on $U_i \cap U_j$

- These form a Čech 1-cocycle associated with the cover $\{U_i\}$, and $\varphi$ deforms if and only if the corresponding cohomology class in $H^1(C, N_\varphi)$ vanishes.
\( \mathcal{L} \): line bundle on \( C \)
\( \{\xi_{ij}\} \): \( \mathcal{L} \)-valued Čech 1-cocycle assoc. with \( \{U_i\} \)
\( L \): line bundle on \( C \)
{\xi_{ij}}: \( L \)-valued Čech 1-cocycle assoc. with \( \{U_i\} \)
{\xi_i}: \( L \)-valued meromorphic sections on \( \{U_i\} \) such that \( \xi_i - \xi_j = \xi_{ij} \) on \( U_i \cap U_j \)
$\mathcal{L}$: line bundle on $C$

$\{\xi_{ij}\}$: $\mathcal{L}$-valued Čech 1-cocycle assoc. with $\{U_i\}$

$\{\xi_i\}$: $\mathcal{L}$-valued meromorphic sections on $\{U_i\}$ such that $\xi_i - \xi_j = \xi_{ij}$ on $U_i \cap U_j$

$H^0(C, \mathcal{L}^\vee \otimes K_C) = H^1(C, \mathcal{L})^\vee$ by the Serre duality
\( \mathcal{L} \): line bundle on \( C \)
\( \{\xi_{ij}\} \): \( \mathcal{L} \)-valued Čech 1-cocycle assoc. with \( \{U_i\} \)
\( \{\xi_i\} \): \( \mathcal{L} \)-valued meromorphic sections on \( \{U_i\} \) such that \( \xi_i - \xi_j = \xi_{ij} \) on \( U_i \cap U_j \)

\[ H^0(C, \mathcal{L}^\vee \otimes K_C) = H^1(C, \mathcal{L})^\vee \]
by the Serre duality

- For \( \eta \in H^0(C, \mathcal{L}^\vee \otimes K_C) \), the fiberwise pairing \( \langle \eta, \xi_i \rangle \) gives a meromorphic section of \( K_C|_{U_i} \)
\( \mathcal{L} \): line bundle on \( C \)

\( \{\xi_{ij}\} \): \( \mathcal{L} \)-valued Čech 1-cocycle assoc. with \( \{U_i\} \)

\( \{\xi_i\} \): \( \mathcal{L} \)-valued meromorphic sections on \( \{U_i\} \) such that \( \xi_i - \xi_j = \xi_{ij} \) on \( U_i \cap U_j \)

\[ H^0(C, \mathcal{L}^\vee \otimes \mathcal{K}_C) = H^1(C, \mathcal{L})^\vee \] by the Serre duality

- For \( \eta \in H^0(C, \mathcal{L}^\vee \otimes \mathcal{K}_C) \), the fiberwise pairing \( \langle \eta, \xi_i \rangle \) gives a meromorphic section of \( \mathcal{K}_C|_{U_i} \)
- Let \( \{p_\lambda\} \) be the set of poles of these local sections and \( r_{p_\lambda} \) the residues of them at \( p_\lambda \)
$\mathcal{L}$: line bundle on $C$

$\{\xi_{ij}\}$: $\mathcal{L}$-valued Čech 1-cocycle assoc. with $\{U_i\}$

$\{\xi_i\}$: $\mathcal{L}$-valued meromorphic sections on $\{U_i\}$ such that $\xi_i - \xi_j = \xi_{ij}$ on $U_i \cap U_j$

$H^0(C, \mathcal{L}^\vee \otimes K_C) = H^1(C, \mathcal{L})^\vee$ by the Serre duality

- For $\eta \in H^0(C, \mathcal{L}^\vee \otimes K_C)$, the fiberwise pairing $\langle \eta, \xi_i \rangle$ gives a meromorphic section of $K_C|_{U_i}$

- Let $\{p_\lambda\}$ be the set of poles of these local sections and $r_{p_\lambda}$ the residues of them at $p_\lambda$

- The pairing $\langle \eta, \{\xi_{ij}\} \rangle$ is given by

$$\sum_\lambda r_{p_\lambda}$$
\( \varphi: C \to S \): map from a smooth curve to a surface
\( \varphi: C \to S \): map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)
\( \varphi : C \to S \): map from a smooth curve to a surface
\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)
\( Z = (d\varphi) \): ramification divisor of \( \varphi \)
\( \varphi : C \to S \): map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)

\( Z = (d\varphi) \): ramification divisor of \( \varphi \)

The normal sheaf \( \mathcal{N}_\varphi \) lies in the exact sequence

\[
0 \to \mathcal{H}_\varphi \to \mathcal{N}_\varphi \to \bar{\mathcal{N}}_\varphi \to 0
\]

\( \mathcal{H}_\varphi \) is a torsion sheaf supported at \( \{p_1, \ldots, p_e\} \)

\( \bar{\mathcal{N}}_\varphi \) is locally free
\( \varphi : C \to S \): map from a smooth curve to a surface
\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)
\( Z = (d\varphi) \): ramification divisor of \( \varphi \)
The normal sheaf \( \mathcal{N}_\varphi \) lies in the exact sequence
\[
0 \to \mathcal{H}_\varphi \to \mathcal{N}_\varphi \to \tilde{\mathcal{N}}_\varphi \to 0
\]
\( \mathcal{H}_\varphi \) is a torsion sheaf supported at \( \{p_1, \ldots, p_e\} \)
\( \tilde{\mathcal{N}}_\varphi \) is locally free
- The obstruction to deforming \( \varphi \) lies in \( H^1(C, \tilde{\mathcal{N}}_\varphi) \)
\( \varphi : C \to S \): map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)

\( Z = (d\varphi) \): ramification divisor of \( \varphi \)

The normal sheaf \( \mathcal{N}_\varphi \) lies in the exact sequence

\[
0 \to \mathcal{H}_\varphi \to \mathcal{N}_\varphi \to \tilde{\mathcal{N}}_\varphi \to 0
\]

\( \mathcal{H}_\varphi \) is a torsion sheaf supported at \( \{p_1, \ldots, p_e\} \)

\( \tilde{\mathcal{N}}_\varphi \) is locally free

- The obstruction to deforming \( \varphi \) lies in \( H^1(C, \tilde{\mathcal{N}}_\varphi) \)
- Its dual space is \( H^0(C, \varphi^* K_S(Z)) \)
\( \varphi : C \to S \): map from a smooth curve to a surface
\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \), i.e., \( d\varphi_{p_i} = 0 \)
\( Z = (d\varphi) \): ramification divisor of \( \varphi \)
The normal sheaf \( \mathcal{N}_\varphi \) lies in the exact sequence

\[
0 \to \mathcal{H}_\varphi \to \mathcal{N}_\varphi \to \tilde{\mathcal{N}}_\varphi \to 0
\]

\( \mathcal{H}_\varphi \) is a torsion sheaf supported at \( \{p_1, \ldots, p_e\} \)
\( \tilde{\mathcal{N}}_\varphi \) is locally free

- The obstruction to deforming \( \varphi \) lies in \( H^1(C, \tilde{\mathcal{N}}_\varphi) \)
- Its dual space is \( H^0(C, \varphi^*K_S(Z)) \)
- We can apply the residue calculation to them
Deformation of curves on surfaces

Ideas and main theorem

Rough outline

\( \varphi : C \to S \): a map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \)
Rough outline

\[ \varphi : C \to S : \text{a map from a smooth curve to a surface} \]
\[ \{p_1, \ldots, p_e\} : \text{Singular points of } \varphi \]
Assume we have constructed an \( N \)-th order deformation \( \varphi_N \) of \( \varphi \)
Rough outline

\( \varphi : C \to S \): a map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \)

Assume we have constructed an \( N \)-th order deformation \( \varphi_N \) of \( \varphi \)

- In general, the obstruction to deforming \( \varphi_N \)
  does not vanish
Rough outline

\( \varphi : C \rightarrow S \): a map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \)

Assume we have constructed an \( N \)-th order deformation \( \varphi_N \) of \( \varphi \)

- In general, the obstruction to deforming \( \varphi_N \) does not vanish
- It means that we cannot deform \( \varphi_N \) no matter how hard we try
To construct deformations of higher order, we need to go back to some $\varphi_{N'}$, $N' < N$, and try to deform it in a different way from $\varphi_N$. 
To construct deformations of higher order, we need to go back to some \( \varphi_{N'} \), \( N' < N \), and try to deform it \textit{in a different way from} \( \varphi_N \).

\[ \varphi \rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_{N'} \rightarrow \cdots \rightarrow \varphi_N \rightarrow \times \]
To construct deformations of higher order, we need to go back to some $\varphi_{N'}$, $N' < N$, and try to deform it in a different way from $\varphi_N$. 

\[
\varphi \rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_{N'} \rightarrow \cdots \rightarrow \varphi_N \rightarrow \times
\]

\[
\bar{\varphi}_{N'+1} \rightarrow \bar{\varphi}_{N'+2} \rightarrow \cdots \rightarrow \times
\]
To construct deformations of higher order, we need to go back to some $\varphi_{N'}$, $N' < N$, and try to deform it \textit{in a different way from} $\varphi_N$.
We can show this is possible when the system of polynomial equations has a solution
We can show this is possible when the system of polynomial equations has a solution.

Moreover, as $N \to \infty$, $N' \to \infty$, too.
We can show this is possible when the system of polynomial equations has a solution.

Moreover, as $N \to \infty$, $N' \to \infty$, too.

Eventually, we can construct a formal deformation of $\varphi$. 
More details

\( \varphi: C \to S: \) a map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\}: \) Singular points of \( \varphi \)
More details

\( \varphi: C \rightarrow S \): a map from a smooth curve to a surface

\( \{p_1, \ldots, p_e\} \): Singular points of \( \varphi \)

At \( p_i \), the pull back of coordinates on \( S \) can be written in the form

\[
(z_i, w_i) = (s^a, s^b + s^{b+1} g_0(s))
\]

\( s \): a parameter on \( C \) around \( p_i \)

\( g_0 \): a holomorphic function around \( p_i \)

\( a < b \), assume \( a \nmid b \) for simplicity
More details

\( \varphi : C \to S \): a map from a smooth curve to a surface

\{p_1, \ldots, p_e\}: Singular points of \( \varphi \)

At \( p_i \), the pull back of coordinates on \( S \) can be written in the form

\[
(z_i, w_i) = (s^a, s^b + s^{b+1} g_0(s))
\]

\( s \): a parameter on \( C \) around \( p_i \)

\( g_0 \): a holomorphic function around \( p_i \)

\( a < b \), assume \( a \nmid b \) for simplicity

\( a - 1 \) is the multiplicity of the singularity \( p_i \), that is, the coefficient of \( p_i \) of the divisor \( Z = (d\varphi) \)
Its deformation can be written as

\[(z_i, w_i) = (s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} t^j c_{a-i,j}s^i, s^b + s^{b+1} g_0(s) + \sum_{j=1}^{k} t^j g_j(s))\]
Its deformation can be written as

\[(z_i, w_i) = (s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} t^j c_{a-i,j}s^i, s^b + s^{b+1} g_0(s) + \sum_{j=1}^{k} t^j g_j(s))\]

It is convenient to consider deformations of the form

\[(z_i, w_i) = (s^a + \sum_{i=0}^{a-2} c_{a-i}s^i, s^b + s^{b+1} g_0(s) + \sum_{j=1}^{k} t^j g_j(s)),\]

where \(c_{a-i} \in t^{a-i} \mathbb{C}[[t]]\).
Among such deformations, we consider those of the form

\[(z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S)),\]

where \(S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} (\frac{1}{a} - j) \frac{1}{i!} (\sum_{k=2}^{a} \frac{c_k}{s^k})^i)\)
Among such deformations, we consider those of the form

\[(z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S)),\]

where 

\[S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \left(\frac{1}{a} - j\right) \frac{1}{i!} (\sum_{k=2}^{a} \frac{c_k}{s^k})^i)\]

Note 

\[S^a = s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} c_{a-i} s^i\]
Among such deformations, we consider those of the form

\[(z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S)),\]

where \(S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1}(\frac{1}{a} - j)\frac{1}{i!}(\sum_{k=2}^{a} \frac{c_k}{s^k})^i)\)

Note \(S^a = s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} c_{a-i}S^i\)

\(S\) is a reparameterization of \(C\) on a punctured neighborhood of \(p_i\)
Among such deformations, we consider those of the form

\[(z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S)),\]

where \(S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1}(\frac{1}{a} - j)\frac{1}{i!}(\sum_{k=2}^{a} \frac{c_k}{s^k})^i)\)

Note \(S^a = s^a + \sum_{j=1}^{k} \sum_{i=0}^{a-2} c_{a-i}s^i\)

- \(S\) is a reparameterization of \(C\) on a punctured neighborhood of \(p_i\)
- It gives the same image as the original

\[(z_i, w_i) = (s^a, s^b + s^{b+1}g_0(s))\]
$(z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S))$ is a reparameterization of the original curve on the punctured disk around $p_i$. 
\( (z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S)) \) is a reparameterization of the original curve on the punctured disk around \( p_i \),

and extendable to \( p_i \) so long as \( S^b + S^{b+1} g_0(S) \) does not have singular terms.
(z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S)) is a reparameterization of the original curve on the punctured disk around p_i,

and extendable to p_i so long as S^b + S^{b+1}g_0(S) does not have singular terms

At some order t^N, S^b + S^{b+1}g_0(S) acquires singular terms, and it produces the obstruction
(z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S)) is a reparameterization of the original curve on the punctured disk around \( p_i \),

and extendable to \( p_i \) so long as \( S^b + S^{b+1}g_0(S) \) does not have singular terms.

At some order \( t^N \), \( S^b + S^{b+1}g_0(S) \) acquires singular terms, and it produces the obstruction.

We modify the value of \( c_i \) so that the obstruction vanishes (this is where we use the transversality assumption of the polynomial system).
If the obstruction vanishes, then we can modify the curve in the form
\[(z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S) + H_i(s)),\]
and continue the deformation. 

\(H_i(s)\) is a meromorphic function around \(p_i\).
If the obstruction vanishes, then we can modify the curve in the form
\[(z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S) + H_i(s)),\] and continue the deformation
\[H_i(s)\] is a meromorphic function around \(p_i\).

Here, although we are at the order \(t^N\), in general we need to modify \(c_i\) in the order lower than \(t^N\).
If the obstruction vanishes, then we can modify the curve in the form 
\[(z_i, w_i) = (S^a, S^b + S^{b+1} g_0(S) + H_i(s)),\]
and continue the deformation 
\[H_i(s)\] is a meromorphic function around \(p_i\).

Here, although we are at the order \(t^N\), in general we need to modify \(c_i\) in the order lower than \(t^N\).

As we mentioned earlier, this changes the map at the order lower than \(t^N\).
If the obstruction vanishes, then we can modify the curve in the form
\((z_i, w_i) = (S^a, S^b + S^{b+1}g_0(S) + H_i(s))\), and continue the deformation
\(H_i(s)\) is a meromorphic function around \(p_i\)

Here, although we are at the order \(t^N\), in general we need to modify \(c_i\) in the order lower than \(t^N\)

As we mentioned earlier, this changes the map at the order lower than \(t^N\)

We can check that the new map can be deformed beyond the order \(t^N\)
What is the system of polynomial equations?
What is the system of polynomial equations? Substituting $S = s(1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{1}{a - j} \frac{1}{i!} (\sum_{k=2}^{a} \frac{c_k}{s^k})^i)$ to $S^b + S^{b+1} g_0(S)$, we have:

$$S^b + S^{b+1} g_0(S) = s^b (1 + \sum_{i=1}^{\infty} f_i^{(b)}(c_2, \ldots, c_a) \frac{1}{s^i}) + (\text{higher order terms})$$
\( f_i^{(b)} \) is given by

\[
 f_i^{(b)}(c_2, \ldots, c_a) = \\
 \sum_{\lambda \in \mathcal{P}(b+j;[2,a])} \left( \begin{array}{c} \frac{b}{a} \\ \lambda(2) \end{array} \right) \cdots \left( \begin{array}{c} \lambda(a) \end{array} \right) \\
 c_2^{\lambda(2)} \cdots c_a^{\lambda(a)}
\]
$f_i^{(b)}$ is given by

$$f_{b+j}^{(b)}(c_2, \ldots, c_a) = \sum_{\lambda \in \mathcal{P}(b+j;[2,a])} \left( \begin{array}{c} \frac{b}{a} \\ \lambda(2) & \cdots & \lambda(a) \end{array} \right) c_2^{\lambda(2)} \cdots c_a^{\lambda(a)}$$

$\mathcal{P}(b + j; [2, a])$ is the set of partitions of $b + j$ using only the integers in the interval $[2, a]$
Deformation of curves on surfaces

Ideas and main theorem

\( f_i^{(b)} \) is given by

\[
f_i^{(b)}(c_2, \ldots, c_a) = \sum_{\lambda \in \mathcal{P}(b+j;[2,a])} \lambda(2) \cdots \lambda(a) c_2^{\lambda(2)} \cdots c_a^{\lambda(a)}
\]

\( \mathcal{P}(b+j;[2,a]) \) is the set of partitions of \( b+j \) using only the integers in the interval \([2,a]\)

\[
\begin{pmatrix} \alpha \\ \beta_1 & \cdots & \beta_k \end{pmatrix} = \frac{\prod_{i=0}^{\beta_1+\cdots+\beta_k-1} (\alpha - i)}{\beta_1! \cdots \beta_k!}
\]
Deformation of curves on surfaces
Ideas and main theorem

\[ S^b + S^{b+1}g_0(S) = (\text{regular part}) + \]
\[ \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} + (\text{higher order terms}) \]
Deformation of curves on surfaces

Ideas and main theorem

\[ S^b + S^{b+1} g_0(S) = (\text{regular part}) + \]
\[ \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} + (\text{higher order terms}) \]

- Roughly, the part \( \frac{f^{(i)}_{b+1}}{s} + \cdots + \frac{f^{(i)}_{b+a-1}}{s^{a-1}} \) controls the obstruction
Deformation of curves on surfaces

Ideas and main theorem

\[ S^b + S^{b+1}g_0(S) = (\text{regular part}) + \]

\[ \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} + (\text{higher order terms}) \]

- Roughly, the part \( \frac{f^{(i)}_{b+1}}{s} + \cdots + \frac{f^{(i)}_{b+a-1}}{s^{a-1}} \) controls the obstruction
- If \( \eta \in H^0(C, \varphi^*K_S(Z)) \), then the pairing between the obstruction class and \( \eta \) has contribution from the singular point \( p_i \), given by the residue of

\[ \langle \eta, \frac{f^{(b)}_{b+1}}{s} + \cdots + \frac{f^{(b)}_{b+a-1}}{s^{a-1}} \rangle \]
So, if we can control the values of $f_{b+j}^{(b)}$, then we can set their value to cancel the obstruction.
So, if we can control the values of $f_{b+j}$, then we can set their value to cancel the obstruction. Explicitly, the following condition will suffice:
So, if we can control the values of $f^{(b)}_{b+j}$, then we can set their value to cancel the obstruction. Explicitly, the following condition will suffice:

\[(G) \text{ The varieties defined by} \]

\[
\begin{align*}
\left\{ & \bar{f}^{(b)} = 0, \quad j \in [1, a - 1] \setminus \{k\}, \\
& \bar{f}^{(b)}_{b+k} = C \neq 0, \\
\right. \\
\end{align*}
\]

have transverse intersection at some point for each $k$

$f^{(b)}_{b+j}$ are modified version of $f^{(b)}_{b+j}$
On the side of the dual space $H^0(C, \varphi^* K_S(Z))$, we introduce the following condition:
On the side of the dual space \( H^0(C, \varphi^* K_S(Z)) \), we introduce the following condition:
A singular point \( p_i \) of \( \varphi \) satisfies the condition (D) if the inequality
\[
\text{dim } H^0(C, \varphi^* \omega_X((a_i-1)p)) < \text{dim } H^0(C, \varphi^* \omega_X) + a_i - 1
\]
holds, where \( a_i \) is the coefficient of \( p_i \) in \( Z = (d\varphi) \).
On the side of the dual space $H^0(C, \varphi^* K_S(Z))$, we introduce the following condition:
A singular point $p_i$ of $\varphi$ satisfies the condition (D) if the inequality

$$\dim H^0(C, \varphi^* \omega_X((a_i-1)p)) < \dim H^0(C, \varphi^* \omega_X)+a_i-1$$

holds, where $a_i$ is the coefficient of $p_i$ in $Z = (d\varphi)$.

The singularity has $a_i - 1$ parameters $c_2, \ldots, c_{a_i}$ of deformations.
On the side of the dual space $H^0(C, \varphi^* K_S(Z))$, we introduce the following condition:
A singular point $p_i$ of $\varphi$ satisfies the condition (D) if the inequality

$$\dim H^0(C, \varphi^* \omega_X((a_i-1)p)) < \dim H^0(C, \varphi^* \omega_X) + a_i - 1$$

holds, where $a_i$ is the coefficient of $p_i$ in $Z = (d\varphi)$
The singularity has $a_i - 1$ parameters $c_2, \ldots, c_{a_i}$ of deformations
So, roughly this condition says the expected dimension of local deformation is positive
Main theorem

Assume $\varphi$ is semiregular. If the conditions (G) and (D) are satisfied at each $p_i \in \{p_1, \ldots, p_e\}$, then there is a non-trivial deformation of $\varphi$. 
Main theorem

Assume $\varphi$ is semiregular. If the conditions (G) and (D) are satisfied at each $p_i \in \{p_1 \ldots, p_e\}$, then there is a non-trivial deformation of $\varphi$.

- After checking the condition (D), one can completely forget curves and surfaces.
Main theorem

Assume $\varphi$ is semiregular. If the conditions (G) and (D) are satisfied at each $p_i \in \{p_1, \ldots, p_e\}$, then there is a non-trivial deformation of $\varphi$.

- After checking the condition (D), one can completely forget curves and surfaces.
- The problem reduces to studying a system of polynomial equations which depends only on two positive integers $a$ and $b$. 
### Table:

<table>
<thead>
<tr>
<th>$a$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(G) holds for $4 \leq b \leq 30$</td>
</tr>
<tr>
<td>4</td>
<td>(G) holds for $5 \leq b \leq 30$ except $b = 6$</td>
</tr>
<tr>
<td>5</td>
<td>(G) holds for $6 \leq b \leq 30$</td>
</tr>
<tr>
<td>6</td>
<td>(G) holds for $7 \leq b \leq 30$</td>
</tr>
<tr>
<td>7</td>
<td>(G) holds for $8 \leq b \leq 20$</td>
</tr>
<tr>
<td>8</td>
<td>(G) holds for $9 \leq b \leq 20$</td>
</tr>
<tr>
<td>9</td>
<td>(G) holds for $10 \leq b \leq 20$</td>
</tr>
<tr>
<td>10</td>
<td>(G) holds for $11 \leq b \leq 15$</td>
</tr>
</tbody>
</table>
When $a = 2$ (double point), a stronger assertion holds due to the simple form $f^{(b)}_{b+1} = c \frac{b+1}{2}$.
When $a = 2$ (double point), a stronger assertion holds due to the simple form $f^{(b)}_{b+1} = \frac{b+1}{2}$.

Let $\varphi: C \to S$ be a semiregular map whose singularities $p_1, \ldots, p_l$ satisfy $a = 2$. 
When \( a = 2 \) (double point), a stronger assertion holds due to the simple form
\[
f^{(b)}_{b+1} = \frac{b+1}{2}
\]
Let \( \varphi : C \to S \) be a semiregular map whose singularities \( p_1, \ldots, p_l \) satisfy \( a = 2 \).

**Theorem**

\( \varphi \) deforms if and only if at least one of the following conditions holds.

- There is at least one \( p_i \) such that there is no section of \( H^0(C, \varphi^* \omega_X(p_i)) \setminus H^0(C, \varphi^* \omega_X) \).
- The set \( H^0(C, \tilde{N}_\varphi) \) is not zero.
For $\alpha = 3$, condition (G) is reduced to following.
For $a = 3$, condition (G) is reduced to following.

For $b = 6k + 1$, set

$$f(X) = \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 3k + 1 & 0 \end{array} \right) X^k + \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 3k - 2 & 2 \end{array} \right) X^{k-1} + \cdots + \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 1 & 2k \end{array} \right)$$

$$g(X) = \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 3k & 1 \end{array} \right) X^k + \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 3k - 3 & 3 \end{array} \right) X^{k-1} + \cdots + \left( \begin{array}{cc} 2k + \frac{1}{3} & 0 \\ 0 & 2k + 1 \end{array} \right)$$

$$\begin{pmatrix} \alpha \\ \beta_1 & \beta_2 \end{pmatrix} = \prod_{i=0}^{\beta_1+\beta_2}(\alpha - i) \quad \frac{\beta_1! \beta_2!}{\beta_1! \beta_2!}$$
Then, (G) is equivalent to
Then, (G) is equivalent to

- there is a simple zero of $f$ which is not a multiple zero of $g$, and
Then, (G) is equivalent to

- there is a simple zero of $f$ which is not a multiple zero of $g$, and
- the same holds when we exchange $f$ and $g$. 