Generalized complex geometry and three-manifolds

Roberto Rubio

Universitat de Barcelona

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M manifold (smooth category)

$TM + T^*M$

ω presymplectic

$(\omega \in \Omega^2(M),\ d\omega = 0)$

$graph(\omega) \subset TM{+}T^*M$

P Poisson

$(P \in \mathfrak{X}^2(M),\ [P,P] = 0)$

$graph(P) \subset TM{+}T^*M$

J complex

$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$

ω symplectic

$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$\mathcal{J} \in \mathrm{End}(TM{+}T^*M),\ \mathcal{J}^2 = -\mathrm{Id}$

Pairing $\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\alpha(Y)+\beta(X))$

Maximally isotropic

Skew-symmetric, $\mathcal{J}^* + \mathcal{J} = 0$

The Dorfman **bracket** on $\Gamma(TM+T^*M)$

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

ω presymplectic
($\omega \in \Omega^2(M)$, $d\omega = 0$)
$graph(\omega) \subset TM+T^*M$

P Poisson
($P \in \mathfrak{X}^2(M)$, $[P,P] = 0$)
$graph(P) \subset TM+T^*M$

Maximally isotropic
Involutive (Dorfman)

Dirac structures
Courant, Weinstein...

J complex
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$

ω symplectic
$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$\mathcal{J} \in \mathrm{End}(TM+T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$

Skew-symmetric, $\mathcal{J}^* + \mathcal{J} = 0$
$+i$-eigenbundle involutive

Generalized complex geometry
Hitchin, Gualtieri, Cavalcanti...

The Dorfman **bracket**??

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha$$

$$\begin{aligned}[X + \alpha, X + \alpha] &= [X, X] + \mathcal{L}_X \alpha - \imath_X d\alpha \\ &= d\imath_X \alpha + \imath_X d\alpha - \imath_X d\alpha \\ &= d\imath_X \alpha = d\langle X + \alpha, X + \alpha \rangle\end{aligned}$$

It is not skew-symmetric, but satisfies, for $e, u, v \in \Gamma(TM + T^*M)$,

$$[e, [u, v]] = [[e, u], v] + [u, [e, v]]$$

$$\pi_{TM}(e)\langle u, v \rangle = \langle [e, u], v \rangle + \langle u, [e, v] \rangle$$

Actually, this structure has a name...

The Courant algebroid $(TM+T^*M, \langle\,,\rangle, [\,,], \pi_{TM})$

Definition (Liu-Weinstein-Xu)

A Courant algebroid over M is a tuple $(E, \langle\,,\rangle, [\,,], \pi)$ consisting of

- a vector bundle $E \to M$,
- a nondegenerate symmetric pairing $\langle\,,\rangle$,
- a bilinear bracket $[\,,]$ on $\Gamma(E)$,
- a bundle map $\pi : E \to TM$ covering id_M,

such that, for any $e \in E$,

- the map $[e, \cdot]$ is a derivation of both the bracket and the pairing,
- we have $[e, e] = d\langle e, e\rangle$.

Example

For $H \in \Omega^3_{cl}$, define the H-twisted bracket

$$[X + \alpha, Y + \beta]_H = [X, Y] + \mathcal{L}_X\beta - \imath_Y d\alpha + \imath_X\imath_Y H$$

The tuple $(TM+T^*M, \langle\,,\rangle, [\,,]_H, \pi_{TM})$ is a Courant algebroid

Automorphisms of Courant algebroids

Definition

The automorphism group $\mathrm{Aut}\, E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \mathrm{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

- $\langle Fu, Fv \rangle = f_* \langle u, v \rangle$,
- $[Fu, Fv] = f_*[u, v]$,
- $\pi_{TM} \circ F = f_* \circ \pi_{TM}$

Example

On $TM+T^*M$, for any $f \in \mathrm{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$,

$$\left.\begin{array}{ll} f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, & X + \alpha \mapsto f_* X + f_* \alpha \\ e^B = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}, & X + \alpha \mapsto X + \alpha + \imath_X B \end{array}\right. \in \mathrm{Aut}(TM+T^*M)$$

Actually, the so-called **generalized diffeomorphisms** are

$$\mathrm{Aut}(TM+T^*M) = \mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$$

Back to generalized complex structures

M generalized complex $\Longrightarrow$ M almost complex $\Longrightarrow$ $\dim M = n = 2m$

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$ $T_{1,0}$ involutive (Lie)	$\mathcal{J} \in \mathrm{End}(TM + T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$ L involutive (Dorfman) $\mathcal{J} + \mathcal{J}^* = 0$
$L \subset T_{\mathbb{C}}M$, $L \cap \overline{L} = \{0\}$ L involutive (Lie)	$L \subset (TM + T^*M)_{\mathbb{C}}$, $L \cap \overline{L} = \{0\}$ L involutive (Dorfman) L maximally isotropic
Idea: $\varphi = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$ $L = \mathrm{Ann}(\varphi) = \{X : \imath_X \varphi = 0\}$ (locally)	What is φ? What is Ann here? What extra property?

Idea: $\varphi = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\varphi) = \{X : \imath_X \varphi = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X+\alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X+\alpha, X+\alpha \rangle \varphi$.
So $(X+\alpha) \cdot \varphi = 0$ implies $\langle X+\alpha, X+\alpha \rangle = 0 \implies \mathrm{Ann}(\varphi)$ isotropic.

Chevalley pairing ($\wedge^{top} T^*_{\mathbb{C}} M$-valued): $(\varphi, \psi) = [\varphi^T \wedge \psi]_{top}$.

$$L = \mathrm{Ann}(\varphi)$$

$L \cap \overline{L} = \{0\}$	L involutive (Dorfman)	L maximally isotropic
$(\varphi, \overline{\varphi}) \neq 0$	$d\varphi = (X+\alpha) \cdot \varphi$	$\varphi \sim e^{B + i\omega} \theta_1 \wedge \ldots \wedge \theta_r$
(nowhere	for some $X + \alpha$	for $B, \omega \in \wedge^2 T^* M$
vanishing)	in $(TM + T^*M)_{\mathbb{C}}$	$\theta_j \in T^*_{\mathbb{C}} M$, ($\varphi$ **pure**)

Examples, recall: pure $\varphi \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$	$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
$\varphi = dz_1 \wedge \ldots \wedge dz_m$	$\varphi = e^{i\omega}$
$(\varphi, \overline{\varphi}) \sim \varphi \wedge \overline{\varphi} \neq 0$	$(\varphi, \overline{\varphi}) \sim \omega^m \neq 0$
$d\varphi = 0$	$d\varphi = 0$
pure ($B = \omega = 0$)	pure ($B = 0, r = 0$)

For pure φ, φ': $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\varphi') \iff \varphi = f\varphi'$ for non-vanishing f.

Definition

For $\varphi = \varphi_0 + \ldots + \varphi_n$, define **type**: least index j with $\varphi_j \neq 0$ (function).

Complex: type m; symplectic: type 0 (type ranges between 0 and m).

Type change example

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\varphi = z + dz \wedge dw$$

We check all the conditions:

$$(\varphi, \overline{\varphi}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\varphi = dz = (-\tfrac{\partial}{\partial w} + 0) \cdot \varphi$$

Pure: $z \neq 0$, $\varphi \sim 1 + \frac{dz \wedge dw}{z} = e^{\frac{dz \wedge dw}{z}}$, pure and type 0
$z = 0$, $\varphi = dz \wedge dw$, pure and type 2

Type change: generically B-transform of symplectic, blows up along $z = 0$

Why generalized geometry

Unifying framework

for instance, complex and symplectic structures

Suitable language

generalized Kähler revived bihermitian geometry

New interesting structures...

Let's look at this.

{complex}⊆{generalized complex}⊆{almost complex}

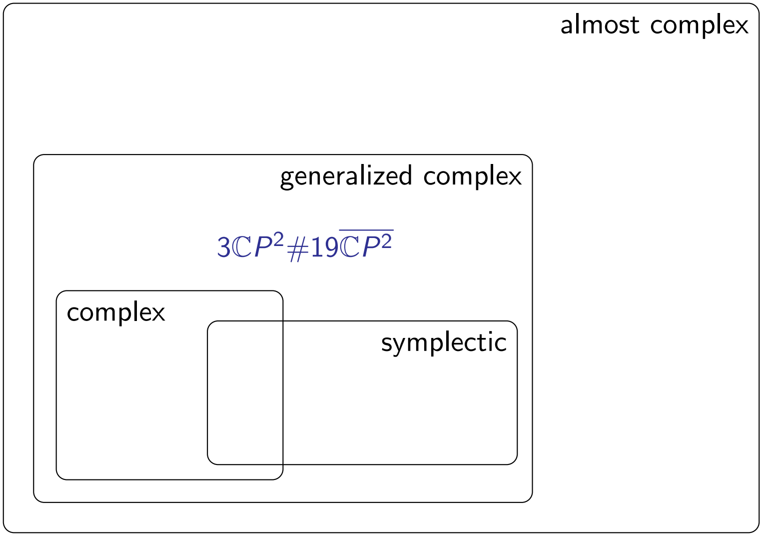
Remember $\varphi = z + dz \wedge dw$ on $\mathbb{C}^2$?
Invariant by translation on w, we can define it on

$$D \times T^2 \subset \mathbb{C} \times T^2 \subset \mathbb{C} \times \frac{\mathbb{C}}{\mathbb{Z}^2}.$$

It is possible to do surgery on certain symplectic 4-manifold, by removing a normal neighbourhood of a torus and obtain the following:

Theorem (Gualtieri,Cavalcanti'2006)

The neither complex nor symplectic compact manifold $3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2}$ is a generalized complex manifold.


almost complex
generalized complex
$3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$
complex
symplectic

Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

$$\begin{aligned} \pi(X + f + \alpha) &= X \\ \langle X + f + \alpha, X + f + \alpha \rangle &= \imath_X \alpha + f^2 \\ [X + f + \alpha, Y + g + \beta\rangle] &= [X, Y] + X(g) - Y(f) \\ &\quad + \mathcal{L}_X \beta - \imath_Y d\alpha + 2g df. \end{aligned}$$

is a Courant algebroid, and moreover an $\mathrm{O}(n+1, n)$-bundle.
As $\mathrm{O}(n+1, n)$ is a real form of $\mathrm{O}(2n+1, \mathbb{C})$, of Lie type B_n:

Generalized geometry of type B_n

Main features of B_n-generalized geometry

More generalized diffeomorphisms: A-fields, $A \in \Omega^1_{cl}(M)$ acting by

$$X + f + \alpha \mapsto X + f + \imath_X A + \alpha - (2f + \imath_X A)A.$$

With notation $1 \to \Omega^2_{cl}(M) \to \Omega^{2+1}_{cl}(M) \to \Omega^1_{cl}(M) \to 1$, they give

$$\mathrm{GDiff}(M) = \mathrm{Diff}(M) \ltimes \Omega^{2+1}_{cl}(M).$$

Definition

A B_n-generalized complex structure is a subbundle

$$L \subset (TM + 1 + T^*M)_{\mathbb{C}}$$

that is maximally isotropic, involutive and satifies $L \cap \overline{L} = 0$.

Generalized complex structures are B_{2m}-generalized complex structures.
Also exist for $\dim M$ odd (normal almost contact + cosymplectic).

Pure spinors are of the form $ce^{A+i\sigma} \cdot e^{B+\omega}\theta_1 \wedge \ldots \wedge \theta_r$.

Type change

For $\varphi = \varphi_0 + \ldots + \varphi_n$, recall **type**: least index j with $\varphi_j \neq 0$ (function).

Definition

Type-change locus: $\{x \in M : \varphi_0(x) = 0\}$.

Hypotheses from now on: compact manifold + stable structure
Stable: generically $\varphi_0(x) \neq 0$, whenever $\varphi_0(x) = 0$, $d\varphi_0(x) \neq 0$.
Type-change locus corresponds to $\varphi_0 = 0$: codimension 2 submanifold.

We saw a torus for a compact four-manifold.
(Parity + $(\varphi, \overline{\varphi}) \neq 0 \implies 4$ is first type-change dimension)

We look at **three-manifolds**: type-change locus is a union of circles.

Type-change locus

Theorem ((Hitchin),R.)

The type-change locus cannot be a single circle.

Lemma (R.)

Around a type-change circle C, we can find coordinates (z, ψ) such that:

$$\varphi = z + \lambda dz + \mu dz \wedge d\bar{z} + \nu dz \wedge d\psi,$$

with $\lambda, \nu \in \mathbb{C}^*$, $\mu \in \{0, 1\}$, and C corresponding to $z = 0$.

Proof: Assume C is the only type-change: φ_2/φ_0 well defined on $M \setminus C$
Tubular neighbourhood N_C, $\iota : N_C \setminus C \to M \setminus C$,

$$\int_{\partial N_C} \iota^*(\varphi_2/\varphi_0) = \int_{\partial N_C} \iota^*(\nu \frac{dz}{z} \wedge d\psi) = \int_0^{2\pi} \int_0^{2\pi} \nu d\theta \wedge d\psi = 4\pi^2 \nu \neq 0.$$

On the open manifold $M \setminus N_C$, Stokes' theorem says

$$\int_{\partial N_C} \iota^*(\varphi_2/\varphi_0) = \int_{M \setminus N_C} d(\varphi_2/\varphi_0) = \int_{M \setminus N_C} 0 = 0.$$

Contradiction. □

Example of type-change locus

Analogue of Marsden-Weinstein symplectic reduction:

Proposition (R.)

The reduction of an S^1 or $\mathbb{R}^+$-invariant generalized complex structure on $M \times \mathrm{S}^1$ or $M \times \mathbb{R}^+$ is a B_n-generalized complex structure on M.

Example

Example: $zw + dz \wedge dw$ on $\mathbb{C}^2 \setminus \{0\} \cong \mathrm{S}^3 \times \mathbb{R}^+$ reduces to a

B_3-generalized complex structure on S^3
($\mathrm{S}^3 \subset \mathbb{C}^2$ corresponds to $|z|^2 + |w|^2 = 1$)

Type change on $\mathbb{C}^2 \setminus \{0\}$ gives type-change locus corresponding to

$z = 0$ and $w = 0$: the Hopf link!

S^3 to open book decompositions

The Hopf link gives an open book decomposition of S^3:

Open book decomposition of M: link L called binding and fibration $M \setminus L \to \mathrm{S}^1$ whose fibres are diffeomorphic to a surface Σ with boundary L.

Equivalently, mapping torus of a manifold with boundary, with boundary identified (binding)

S^3 is the mapping torus of a cylinder with boundary by a Dehn twist.

Proposition (Porti, R.)

There is a B_3-generalized complex structure on the mapping torus of the cylinder with boundary by an n-Dehn twist (S^3, lens space?)

Idea of the proof: Unravel open book of S^3 in coordinates, change the structure so that it is compatible with the n-Dehn twist, glue again. □

Thurston's geometries

Sol	H^3

$+$

Seifert	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	E^3	$H^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2\mathbb{R}}$

Unlike for cosymplectic or normal almost contact...

Observation (Porti, R.)

For each Thurston geometry there is a geometric manifold admitting a B_3-generalized complex structure.

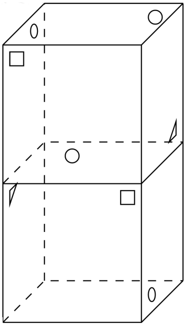
Thurston's geometries

Lemma (Porti, R.)

A geometric manifold that is neither cosymplectic nor normal almost contact has to be Sol or hyperbolic (not fibering over the circle), or the only euclidean manifold not fibering over the circle.

Looking now for a neither cosymplectic nor normal almost contact B_3-generalized complex manifold.

Candidate:
Hantzsche-Wendt manifold.

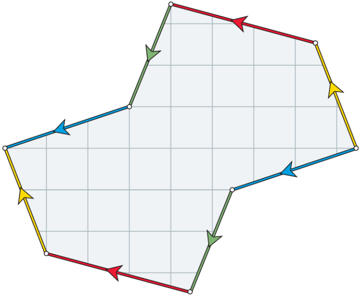


B_n interesting also in even dimensions

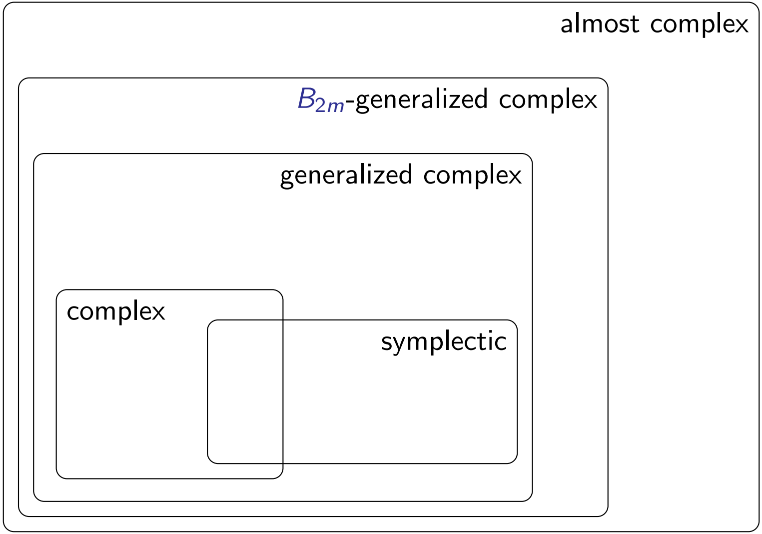
Type change for B_n-geometry is already possible for surfaces:

$$\varphi_0 + \varphi_1 + \varphi_2$$

Relation to meromorphic forms (φ_1/φ_0) and translational surfaces



B_n interesting also in even dimensions



גיאומטריה מורכבת מוכללת ושלוש-יריעות

רוברטו רוביו

UNIVERSITAT DE BARCELONA

סמינר גיאומטריה ממשית ומורכבת
אוניברסיטת תל אביב

יב בטבת 5782