Moduli of symplectic structures

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Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$ as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. **Then all ω_t are diffeomorphic.**

The proof of Moser's theorem

THEOREM: (Moser)
The **Teichmüler space** Teich$_s$ **is a manifold** (possibly, non-Hausdorff), and the **period map** Per : Teich$_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

Proof. Step 1: We can locally find a section S for the Diff$_0$-action on Symp, producing a local decomposition Symp $= O \times S$, where O is a Diff$_0$-orbit. Here O and S are both Frechet manifolds.

Step 2: The period map $P: U \longrightarrow H^2(M,\mathbb{R})$ is a smooth submersion. By Theorem 1, the fibers of P are 0-dimensional. Therefore, P is locally a diffeomorphism. ∎

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\mathrm{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: A Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: **This is equivalent to $\nabla\omega = 0$,** where ∇ is Levi-Civita connection.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection $Sp(n)$** if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: **(Bogomolov decomposition)**
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: In this talk, all holomorphically symplectic manifolds are assumed to be Kähler and compact.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

CLAIM: In these assumptions, $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphic symplectic on (M,I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold of maximal holonomy.**

EXAMPLES.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then** $T^{[2]}$ **is naturally isomorphic to the Kummer surface** $\widetilde{T/\pm 1}$**.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification)
Let M be a compact complex surface which is hyperkähler. **Then** M **is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3** is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the two series:** Hilbert schemes of K3, and generalized Kummer.

Main result

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω. A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: Let M be a maximal holonomy hyperkähler manifold. Then the period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes v such that $q(\omega,\omega) > 0$,** where q is a quadratic form on cohomology defined below.

Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2-3,3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

MBM classes

DEFINITION: Kähler cone of a Kähler manifold is the set of all cohomology classes $\omega \in H^{1,1}(M)$

DEFINITION: Face of a Kähler cone K is a subset $V \cap \partial K$ containing an open subset of V, for some hyperplane $V \subset H^{1,1}(M)$.

DEFINITION: Let M be a hyperkähler manifold. A homology class $z \in H_2(M, \mathbb{Q})$ is called **an MBM class** (monodromy birational minimal) if for some complex structure in the same deformation class, the annihilator $z^\perp$ contains a face of its Kähler cone.

DEFINITION: A cohomology class $z \in H^2(M, \mathbb{Q})$ is called **MBM class** if it becomes MBM after an identification $H^2(M, \mathbb{Q}) \cong H_2(M, \mathbb{Q})$ provided by the Bogomolov-Beauville-Fujiki form.

Properties of MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

THEOREM: Let (M,I) be a hyperkähler manifold, $\operatorname{rk} H^{1,1}(M,\mathbb{Z}) = 1$, and $z \in H_{1,1}(M,I)$ a non-zero negative class. **Then z is MBM if and only if $\pm z$ is $\mathbb{Q}$-effective,** that is, λz is represented by a complex curve. ■

DEFINITION: Positive cone $\operatorname{Pos}(M)$ on a Kähler surface is the one of the two components of

$$\{v \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \eta \wedge \eta > 0\}$$

which contains a Kähler form.

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \ | \ \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). ■

Teichmüller space of hyperkähler structures

DEFINITION: Consider the infinite-dimensional space Hyp of all quaternionic triples I, J, K on M which are induced by some hyperkähler structure, with C^∞-topology, and let $\operatorname{Teich}_h :$ Hyp / Diff_0 be the corresponding Teichmüller space, called **Teichmüller space of hyperkähler structures**.

DEFINITION: Consider the space $\mathbb{P}\mathrm{er}_h$ of all triples x, y, z satisfying $x^2 + y^2 + z^2 > 0$. Let M be a hyperkähler manifold of maximal holonomy, and $\operatorname{Per} : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ the map associating to a hyperkähler structure (M, I, J, K, g) the truple $\omega_I, \omega_J, \omega_K$. This map called **the period map for the Teichmüller space of hyperkähler structures**, and $\mathbb{P}\mathrm{er}_h$ **the period space of hyperkähler structures**.

THEOREM: Let M be a hyperkähler manifold of maximal holonomy, and $\operatorname{Per} : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ the period map for the Teichmüller space of hyperkähler structures. Then **the period map $\operatorname{Per} : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ is an open embedding for each connected component.** Moreover, **its image is the set of all spaces $W \in \mathbb{P}\mathrm{er}_h$ such that the orthogonal complement $W^\perp$ contains no MBM classes.**

Sketch of a proof: Follows immediately from Calabi-Yau theorem, global Torelli theorem for hyperkähler manifolds, and the description of the Kähler cone in terms of the MBM classes. ■

Torelli theorem for symplectic structures

THEOREM: Let M be a maximal holonomy hyperkähler manifold. Then the period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes v such that $q(v,v) > 0$.**

Proof. Step 1: Let $P: \operatorname{Teich}_h \longrightarrow \operatorname{Teich}_s$ be the forgetful map putting $\omega_I, \omega_J, \omega_K$ to ω_I. Calabi-Yau implies that P is surjective. Indeed, any Kähler form can be deformed to a Ricci-flat Kähler form in the same cohomology class.

Step 2: From Torelli theorem for hyperkähler structures it follows that the fiber $P^{-1}(\omega)$ of P is the space of pairs $x, y \in H^2(M)$ satisfying $x^2 = y^2 = \omega^2 > 0$, such that the space $\langle \omega, x, y\rangle^\perp$ contains no MBM classes.

Step 3: Since the fibers of P are complements to subsets of codimension 2, they are connected. By Moser's theorem, for each $(M, \omega_I, \omega_J, \omega_K) \in P^{-1}(\omega)$ the symplectic forms ω_I are diffeomorphic.

Torelli theorem for symplectic structures (2)

THEOREM: Let M be a maximal holonomy hyperkähler manifold. Then the period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes v such that $q(v,v) > 0$.**

Step 4: Consider the diagram

$$\begin{array}{ccc}
\operatorname{Teich}_h & \stackrel{P}{\longrightarrow} & \operatorname{Teich}_s \\
\Big\downarrow{\scriptstyle \operatorname{Per}_h} & & {\scriptstyle \operatorname{Per}_s}\Big\downarrow \\
\begin{array}{c}\{x,y,z \in H^2(M) | x^2 = y^2 = z^2 > 0,\\ \langle x,y,z\rangle^\perp \text{ contains no MBM classes}\}\end{array} & \longrightarrow & \{x \in H^2(M) | x^2 > 0\}
\end{array}$$

Step 3 implies that **the arrow Per_s on the above diagram is injective.** The rest of the arrows are surjective as shown, **hence Per_s is also surjective.** ■

Ergodicity of mapping class group action

THEOREM: (V., 2009)
Let M be a maximal holonomy hyperkähler manifold. **Then the image of the mapping class group Γ in $O(H^2(M,\mathbb{Z}))$ has finite index.**

COROLLARY: Γ acts on Teich_s with dense orbits.

Proof: We use a theorem of Calvin Moore:

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of Γ on G/H is ergodic.**

Applying this theorem to Γ inside $G = SO(H^2(M,\mathbb{R}), q)$ and H the stabilizer of $\omega \in H^2(M,\mathbb{R})$, we obtain that action of Γ on $\operatorname{Teich}_s \subset H^2(M,\mathbb{R})$ **is ergodic, hence has dense orbits.** ■

QUESTION: The Teichmüller space of standard symplectic structures on K3 is Hausdorff, as shown above. Are there any non-Hausdorff non-standard symplectic structures in the same connected component of Teich_s?