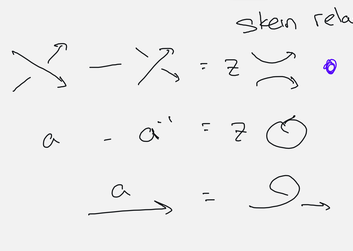
$$Sk(M^3) = \frac{\mathbb{Z}[a^{\pm}, z^{\pm}]\langle \text{framed links}\rangle}{\text{skein relations}}$$



$Sk(S^3) \simeq \mathbb{Z}[a^{\pm}, z^{\pm}]$

$\emptyset \longleftarrow 1$

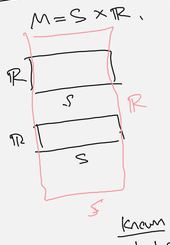
(existence of HOMELYPT invariant)

Skein trace

finite cover $M \longrightarrow N$

is there some relationship between $Sk(M)$ & $Sk(N)$?

$M = S \times \mathbb{R}$, $N = T \times \mathbb{R}$



after various specialization of parameters, $Sk(S \times \mathbb{R})$ dequantizes to ring of fns on the GL_n character variety for S. (Turaev)

$z = q^{1/2} - q^{-1/2}$, $a = q^n$

Fock conjectured $\exists$ maps

GL_n skein (T)

$\downarrow$

GL_1 skein (S)

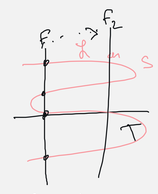
Bonahon-Wong constructed one for $n=2$.

known. Character varieties admit cluster charts associated to certain covers of surfaces (Fock-Goncharov)

$T^*T \supset S$ Lagrangian ($n:1$)

$Fuk(T^*T) \simeq Loc(T) \longleftarrow Loc(S)$ — objects on S

degree n local system $\longleftarrow$ degree 1 local system



$Hom(f_1, S) \to Hom(f_2, S)$

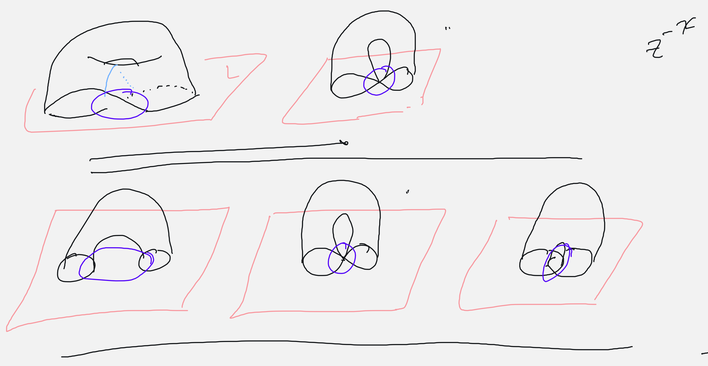
Skein-valued curve counting

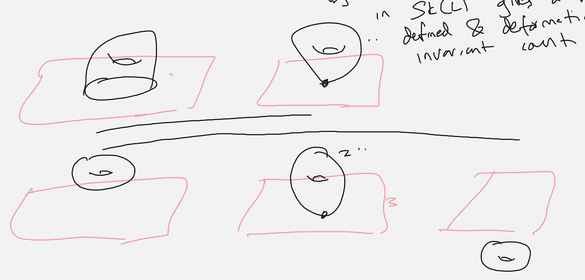
(X, L) — want to count hol. curves in X ending on L.

X: Calabi-Yau 3-fold; L: Maslov-0 Lagrangian

Good: moduli has virtual dim $= 0$.

Bad: moduli has cod-1 boundaries so cannot expect invariance.



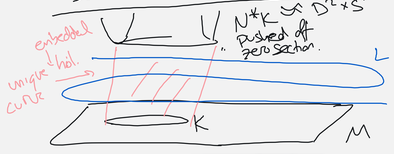


Thm. (Ekholm-S) Counting hol. curves in (X, L) by class of their boundaries in $Sk(L)$ gives a well defined & deformation invariant count

$T^*M^3 \supset L^3$ Lagrangian.

Want: $Sk(M^3) \xrightarrow{f} Sk(L)$

ie, for any link $K \subset M$ need to produce $f(K) \in Sk(L)$ such that f annihilates the skein relations.



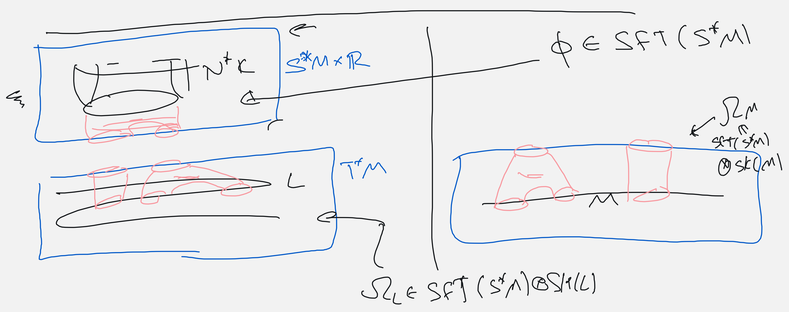
$f(K) :=$ count of curves from N^*K to L going once around the S^1 of N^*K. $\in Sk(L)$

for $L = M$, f is the identity.

Expectation. "Skein-valued Abbondandolo-Schwarz"

$SFT_0(S^*M^3) \simeq Sk(M^3)$

orbit $\longmapsto$ skein-valued count of curves in T^*M^3 ending on orbit.



$\phi \in SFT(S^*M)$

$\Omega_M \in SFT(S^*M) \otimes Sk(M)$

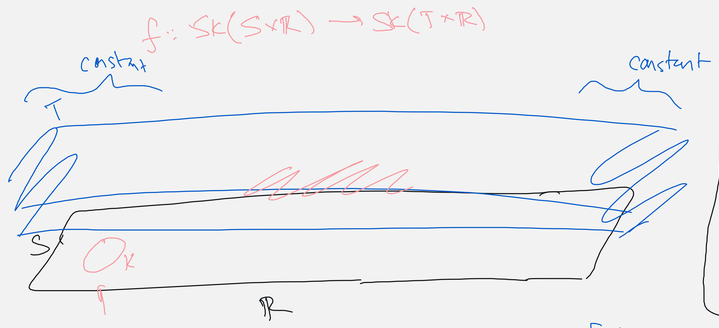
$\Omega_L \in SFT(S^*M) \otimes Sk(L)$

$\phi \circ \Omega_M = K \in Sk(M) \implies \phi = \Omega_M^{-1}(K)$

$\phi \circ \Omega_L = f(K)$

$\implies f(K) = \Omega_L \circ \Omega_M^{-1}(K)$

$f : Sk(S \times \mathbb{R}) \longrightarrow Sk(T \times \mathbb{R})$



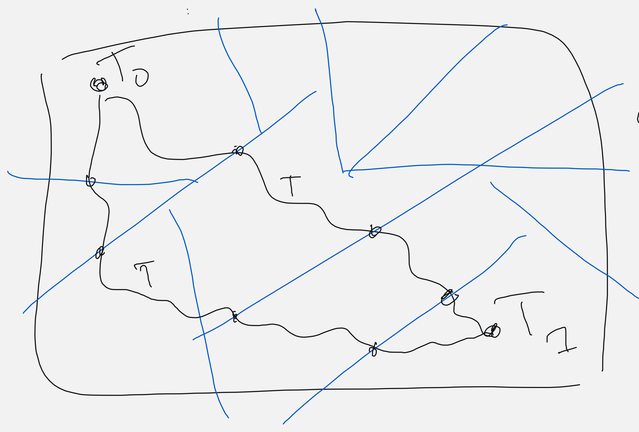
$f_L(K) \cdot \boxed{\Omega_T}$
$\parallel$
$f(K)$
$\parallel$
$\boxed{\Omega_T} \cdot f_R(K)$

$T_t \subset T^*S$

e.g. T_t a family of spectral curves generically bounding no hol. curves but bounding a disk at $t = 1$

Fact: in this case, Ω_T is a (skein lift of) dilogarithm

$Sk(T) \ni \Omega_T$ count of hol. curves ending on T.



choices of $T \subset T^*S$

$\Omega_{T'} = \Omega_T$